Construction of the maximal solution of Backus' problem

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Abstract

Let Ω denote the domain exterior to the unit sphere S. The (simplified) Backus problem consists in finding a function $u \in C^1(\bar{\Omega})$, harmonic on Ω, such that u tends to zero at infinity and the norm of the gradient of u takes prescribed values on S. Apart from a change of sign, the solution is not unique in general. However, the solution is unique in the class of functions with the additional property that the radial component of the gradient of u on S is nonpositive, such as it is relevant in Geodesy. If this solution exists, then $\pm u$ are the maximal and the minimal solutions of the problem. In this paper we continue our previous research on this problem, but this time our purpose is to further a method of successive approximations studied by F. Sacerdote and F. Sansò (1989) that makes possible to construct the maximal solution of the Backus problem.

1 Introduction

Let Ω denote the domain exterior to the unit sphere S. Let $\mathcal{H}(\Omega)$ be the real space of functions which are harmonic in Ω and regular at infinity. We use the notation $\mathcal{H}^k(\bar{\Omega}) = \mathcal{H}(\Omega) \cap C^k(\bar{\Omega})$ where $k \in \{0, 1\}$. For $x \in \bar{\Omega}$ we write $r = |x|$ and $s = x/|x|$. Each function $u \in \mathcal{H}^0(\bar{\Omega})$ can be expanded in spherical harmonics

$$u = \sum_{n=0}^{\infty} u_n \,,$$

where

$$u_n(x) = r^{-(n+1)} Y_n(s) \,,$$

and, with $\langle \ , \ \rangle$ denoting the scalar product,

$$Y_n(s) = \frac{2n+1}{4\pi} \int_S u(s') P_n(\langle s, s' \rangle) \, ds' \,.$$

Here P_n is the Legendre polynomial of degree n. Since $P_0 = 1$, then

$$Y_0 = (4\pi)^{-1} \int_S u(s) \, ds \,.$$

Let $C_+(S)$ be the set of nonnegative continuous functions on S. We define the map $G : \mathcal{H}^1(\bar{\Omega}) \to C_+(S)$ by

$$G(u)(x) = |\nabla u(x)| \,, \ x \in S,$$

where ∇u is the gradient of u. For a given $g \in C_+(S)$, the Backus problem consists in solving the equation $G(u) = g$ (see [1]):

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ |\nabla u| = g & \text{on } S, \\ u(x) \to 0 & \text{as } x \to \infty. \end{cases} \tag{1.1}$$

Apart from a change of sign ($G(-u) = G(u)$), the solution of this boundary problem is not unique in general [2]. However, the solution is unique if u is subject to the condition $u' \leq 0$, where u' is the radial component of ∇u on the unit sphere. Let $K = \{u \in \mathcal{H}^1(\bar{\Omega}) : u' \leq 0\}$ and let $u \in K$ satisfy $G(u) = g$. Then, $|\, u^*| < u$ for any other solution $u^* \neq -u$ of the equation $G(u^*) = g$. In other words, $-u$ and u are the minimal and the maximal solutions of the Backus problem. For $g = 1$ the maximal solution is $u = 1/r$. We note that if $u \in K \setminus \{0\}$, then u is a non-constant positive function ($u > 0$) in $\bar{\Omega}$. For these results we refer to [5].

The Backus problem has application in geodesy and geomagnetism. Many geodesists have contributed to the study of the Backus problem: e.g. Koch and Pope [9], Bjerhammar and Svensson [3], Grafarend [6], Heck [7] , Sacerdote and Sansò [11], Holota [8], Čunderlík et al. [4] ... The major achievements are local solvability results and a deep knowledge of the linearized problem. In geodesy g is the length of the gravity vector. Since the earth's gravity points toward the interior, the restriction $u \in K$ is natural. The solution must be of the form

$$u = \frac{c}{r} + \sum_{n=1}^{\infty} u_n \,,$$

where $c = (4\pi)^{-1} \int_S u(s)\, ds > 0$ is a positive constant. Instead, in geomagnetism the solution must satisfy $c = 0$:

$$\int_S u(s)\, ds = 0\,.$$

Hence u' changes its sign on S. As an example, Fig. 1 displays the field of a magnetic dipole. It is observed that:

1. ∇u is tangential to the unit sphere along the equator E.
2. $\nabla u|_E$ is orthogonal to E.
3. u' changes its sign on S through E from plus to minus in the direction of the vector field $\nabla u|_E$.

Fig. 2 shows the inclination of the Earth magnetic field. In geomagnetism, the magnetic inclination is defined as the angle measured from the horizontal plane to the magnetic field vector where downward is positive. It is clearly seen that ∇u is tangential to the earth's surface along a closed curve called the *dip equator*.

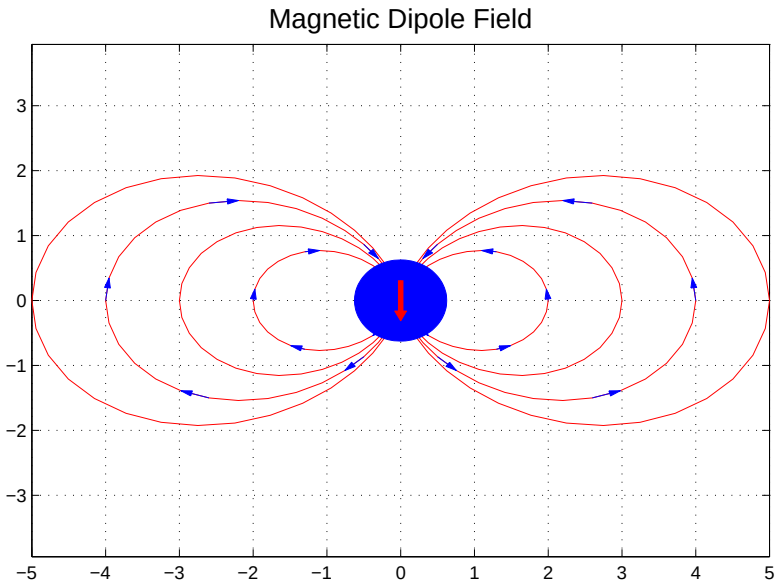

Figure 1. Magnetic dipole field using the matlab function **lforce2d** by A. Abokhodair: `http://www.mathworks.com/matlabcentral/fileexchange`.

In this paper we propose an algorithm to construct the maximal solution of the Backus problem and hence the kind of solution of geodetic interest. This construction is motivated by [11]. No attempt has been made here to study the convergence of this algorithm. However we illustrate our approach with a numerical example. Specifically, we find the maximal solution of the Backus problem (1.1) where g is the norm of the gradient of $u^* = z/r^3$. This can be considered as an explicit example of the non-uniqueness of solutions of the Backus problem complementing the results in [2].

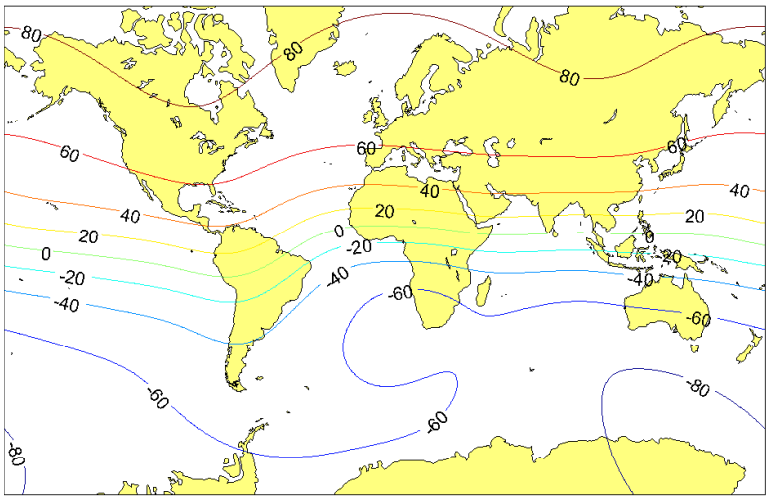

Figure 2. Inclination of the Earth magnetic field (epoch 2005) from the USA National Geophysical Data Center (NGDC): `http://www.ngdc.noaa.gov/geomag/data.shtml`

2 Algorithm for constructing the maximal solution

Here G denotes the map $G : \mathcal{H}^1(\bar{\Omega}) \to C_+(S)$ defined by

$$G(u)(x) = |\nabla u(x)|^2\,,\ x \in S.$$

Let $u \in K$ satisfy $G(u) = f$, where $f = g^2$. Since $u > 0$, there exists an unknown positive constant μ such that

$$\mu^{1/2}\, u = \frac{1}{r} + v\,, \tag{2.1}$$

where $v \in \mathcal{H}^1(\bar{\Omega})$ has not spherical harmonic of degree zero,

$$v = \sum_{n=1}^{\infty} \Big(\frac{1}{r}\Big)^{n+1} v_n\,.$$

This is equivalent to the property

$$\int_S v\, ds = 0\,. \tag{2.2}$$

In addition, $u' \le 0$ if and only if $v' \le 1$.

Since $G(\mu^{1/2}\, u) = \mu f$, by (2.1) it is easily seen that the function v in (2.1) satisfies the boundary condition

$$2v' = 1 + G(v) - \mu f\,. \tag{2.3}$$

By Green's second identity we have

$$\int_S (v' + v)\, ds = 0\,.$$

Consequently, v satisfies (2.2) if and only if

$$\int_S v'\, ds = 0\,.$$

From (2.3) we obtain

$$\mu = f_0^{-1}\Big[1 + \frac{1}{4\pi}\int_S G(v)\,ds\Big]\,, \tag{2.4}$$

where

$$f_0 = \frac{1}{4\pi}\int_S f\,ds > 0$$

is the spherical harmonic of degree zero of the function f.

According to (2.3) and (2.4), to solve the equation $G_K(u) = f$, where G_K denotes the restriction of G to K, we define the following sequence $\{v_n, \mu_n\} \in \mathcal{H}^1(\bar{\Omega}) \times \mathbb{R}^+$:

First approximation:

$$\mu_1 = f_0^{-1} \tag{2.5}$$

$$v_1 \in \mathcal{H}^1(\bar{\Omega}) : \boxed{2v_1' = 1 - \mu_1 f \quad \text{on } S.}$$

Successive approximations ($n \geq 1$):

$$\mu_{n+1} = \mu_1\Big[1 + \frac{1}{4\pi}\int_S G(v_n)\,ds\Big] \tag{2.6}$$

$$v_{n+1} \in \mathcal{H}^1(\bar{\Omega}) : \boxed{2v_{n+1}' = 1 + G(v_n) - \mu_{n+1} f \quad \text{on } S.}$$

Observe that each v_n satisfies

$$\int_S v_n\,ds = 0\,.$$

Let $w_1 \in \mathcal{H}^1(\bar{\Omega})$ be the solution of the exterior Neumann problem

$$\Delta w_1 = 0 \quad \text{in } \Omega, \quad w_1(x) \to 0 \quad \text{as } x \to \infty, \quad w_1' = f/2 \quad \text{on } S, \tag{2.7}$$

and let $w_{n+1} \in \mathcal{H}^1(\bar{\Omega})$ $(n \geq 1)$ be the solution of the exterior Neumann problem

$$\Delta w_{n+1} = 0 \quad \text{in } \Omega, \quad w_{n+1}(x) \to 0 \quad \text{as } x \to \infty, \quad w_{n+1}' = G(v_n)/2 \quad \text{on } S. \tag{2.8}$$

Then, we have

$$v_1 = -\frac{1}{2r} - \mu_1 w_1, \tag{2.9}$$

$$\begin{aligned} v_{n+1} &= -\frac{1}{2r} + w_{n+1} - \mu_{n+1} w_1 \\ &= v_1 - \alpha_n w_1 + w_{n+1}\,, \end{aligned} \tag{2.10}$$

where

$$\alpha_n = f_0^{-1}[G(v_n)]_0\,, \quad [G(v_n)]_0 = \frac{1}{4\pi}\int_S G(v_n)\,ds\,.$$

Finally, by (2.1), the successive approximations to construct the maximal solution of the Backus problem are

$$\begin{aligned} u_1 &= \mu_1^{-1/2}(r^{-1} + v_1)\,, \\ u_{n+1} &= \mu_{n+1}^{-1/2}(r^{-1} + v_{n+1}) \quad (n \geq 1)\,. \end{aligned}$$

Remark. Fix $\mu > 0$. In [11] the successive approximations $\{v_n\}_{n\geq 0} \subset \mathcal{H}^1(\bar{\Omega})$ satisfy the boundary conditions

$$2v'_{n+1} = G(v_n) + (1 - \mu f)\,, \tag{2.11}$$

where, for example, $v_0 = 0$. It is clear that $v_n = v_n(\mu)$ $(n \geq 1)$. Here, the constant μ is to be chosen later to guarantee the convergence of this sequence.

Let us assume the following estimate for functions $u \in \mathcal{H}^1(\bar{\Omega})$ with the property that $u' \in C^\alpha(S)$ for some $\alpha \in (0,1)$:

$$\| \, |\nabla_t u| \, \|_{\alpha,S} \leq C \|u'\|_{\alpha,S} \tag{2.12}$$

where $C > 0$ is a constant and $\nabla_t u$ the tangential component of ∇u on the unit sphere S. If $f \in C^\alpha(S)$, from (2.11) and (2.12) we have

$$2\|v'_{n+1}\|_\alpha \leq (1 + C^2)\|v'_n\|_\alpha^2 + \|1 - \mu f\|_\alpha\,.$$

From this inequality, Sacerdote and Sansò [11] prove the following.

Theorem 1. Let $k(\mu) = \big(1 + C^2\big)\|1 - \mu f\|_\alpha$. If there exists μ_0 such that $k(\mu_0) \leq 1$, then

$$\|v'_n(\mu_0)\|_\alpha \leq (1 + C^2)^{-1}\big[1 - \sqrt{1 - k(\mu_0)}\big] < 1$$

for all $n \geq 0$. If $k(\mu_0) < 1$, then the sequence $\{v'_n(\mu_0)\} \subset C^\alpha(S)$ is convergent. □

Unfortunately, the convergence condition

$$\exists \mu : \varphi(\mu) := \|1 - \mu f\|_\alpha < (1 + C^2)^{-1}\,?$$

is somewhat restrictive. To see this, we can assume that $\|f\|_\alpha = 1$, for if not, we replace the equation $G_K(u) = f$ by $G_K(\|f\|_\alpha^{-1/2} u) = f/\|f\|_\alpha$. Let $M = \max_S f \leq 1$, $m = \min_S f$ and $a = M + m \in (0, 2]$. Since the α-Hölder quotient $[f]_{\alpha,S}$ is equal to $1 - M$, a computation shows that

$$\varphi(\mu) = \begin{cases} 1 + (1 - a)\mu & \text{if} \quad \mu \leq 2/a, \\ \mu - 1 & \text{if} \quad \mu \geq 2/a. \end{cases}$$

We distinguish two cases.

(a) If $a \leq 1$, then $\varphi(\mu)$ is increasing and its minimum value is equal to 1. Therefore

$$k(\mu) = \big(1 + C^2\big)\varphi(\mu) \geq 1 + C^2 > 1\,,$$

for all μ, and the convergence condition fails.

(b) If $a > 1$, then

$$\min_{\mu>0} \varphi(\mu) = \varphi(2/a) = \frac{2}{a} - 1 < 1\,.$$

Hence

$$\begin{aligned} k(\mu) < 1 &\Leftrightarrow \varphi(\mu) < \frac{1}{1 + C^2} \\ &\Rightarrow \frac{2}{a} - 1 < \frac{1}{1 + C^2} \\ &\Leftrightarrow M + m > 1 + \frac{C^2}{2 + C^2} := b\,. \end{aligned}$$

Conversely, if

$$M + m > b\,, \tag{2.13}$$

then $k(\mu) < 1$ if and only if

$$\mu \in \left[2(a-1)^{-1}(1-b^{-1}), 2b^{-1}\right].$$

Since $m \leq M \leq 1$, from (2.13) we have $M \in \big(b/2, 1\big]$. For each of these admissible values of M, m must be in the interval $\big(b-M, M\big]$. Therefore, given $M \in \big(b/2, 1\big]$ the maximum value of the oscillation $M - m$ of the function f is equal to $\delta(M) := 2M - b$.

Example. If $C = 2$ in (2.12), then the convergence condition is satisfied if and only if $M + m > 5/3 \approx 1.67$. Thus $1 \geq M > 5/6 \approx 0.83$. Figure 3 displays the maximum values of the oscillation $\delta(M)$ of the function f for this case. For example, if $M = 0.9$ the oscillation of the function f can be at most equal to 0.13 approximately.

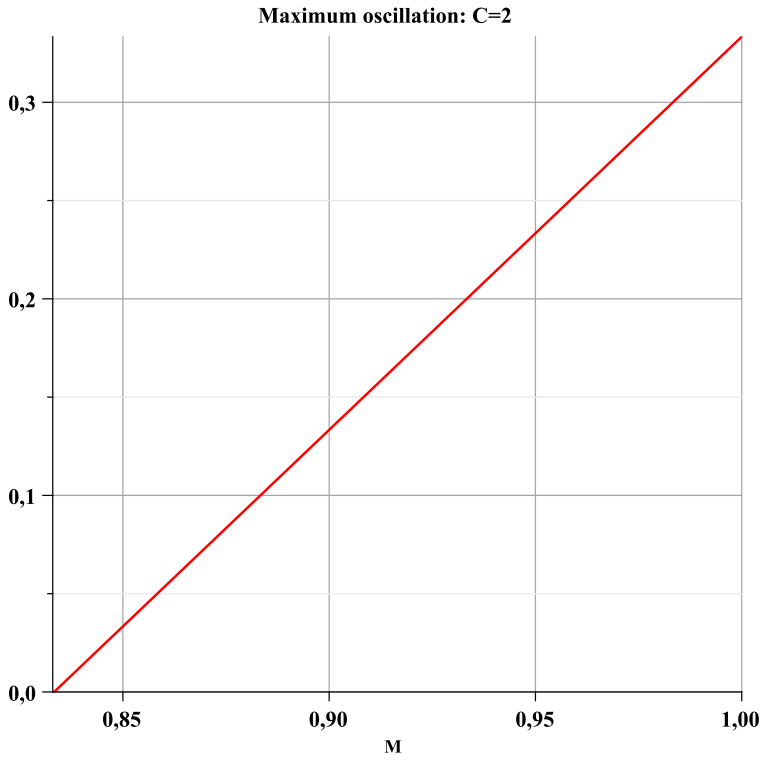

Figure 3. Function $\delta(M) = 2M - b(2)$.

The value of the constant C in the basic inequality (2.12) is not yet completely known. In [11] the authors estimate the α-Hölder quotient of the gradient in terms of the α-norm of its radial component on S. To estimate the maximum norm of the modulus of the tangential gradient we refer to [10]. □

3 Numerical example

We now construct the maximal solution of the Backus problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad u(x) \to 0 \quad \text{as } x \to \infty, \quad |\nabla u|^2 = 1 + 3\cos^2\theta \quad \text{on } S, \tag{3.1}$$

where $\theta \in [0, \pi]$ is the colatitude. A solution of this problem is the dipole potential $u^* = z/r^3$ where $z = r\cos\theta$. This function is not the maximal solution of (3.1) because $\partial u^*/\partial r \,|_{r=1} = -2\cos\theta$ changes its sign on S vanishing on the equator $\theta = \pi/2$.

The successive approximations to the maximal solution of (3.1) are given by

$$u_1 = \mu_1^{-1/2}(r^{-1} + v_1)\,, \quad u_{n+1} = \mu_{n+1}^{-1/2}(r^{-1} + v_{n+1}) \quad (n \geq 1)\,,$$

where (μ_1, v_1) and (μ_{n+1}, v_{n+1}) are defined by (2.5)-(2.9) and (2.6)-(2.10), respectively. We get

$$\mu_1 = \frac{1}{2}\,, \quad v_1 = \frac{1}{6}\frac{P_2(\cos\theta)}{r^3}\,.$$

For $n \geq 2$ we have:

1. Sequence μ_n $(n = 2, \ldots, \mathbf{8})$:

$$\boxed{0.541667,\ 0.540808,\ 0.542183,\ 0.541950,\ 0.\mathbf{5420}64,\ 0.\mathbf{5420}30,\ 0.\mathbf{5420}43\,.}$$

2. The functions v_n $(n = 2, \ldots, 8)$ are of the form:

$$v_n = \sum_{k=1}^{m} a_{2k}^{(n)} \frac{P_{2k}(\cos\theta)}{r^{2k+1}} \quad (m = 2^{\,n-1})\,.$$

For example, the coefficients of v_2 and v_3 are

$$\begin{bmatrix} 0.1646825397 \\ \\ -0.007142857143 \end{bmatrix}$$

and

$$\begin{bmatrix} \\ 0.1675212043 \\ \\ -0.005902780609 \\ \\ 0.0005170141650 \\ \\ -0.000009712509712 \end{bmatrix},$$

respectively.

In Figure 4 we show the successive differences $u_6 - u_5$, $u_7 - u_6$ and $u_8 - u_7$ evaluated on S. It is seen that the maximum norm of $u_8 - u_7$ is equal to 5×10^{-5} approximately.

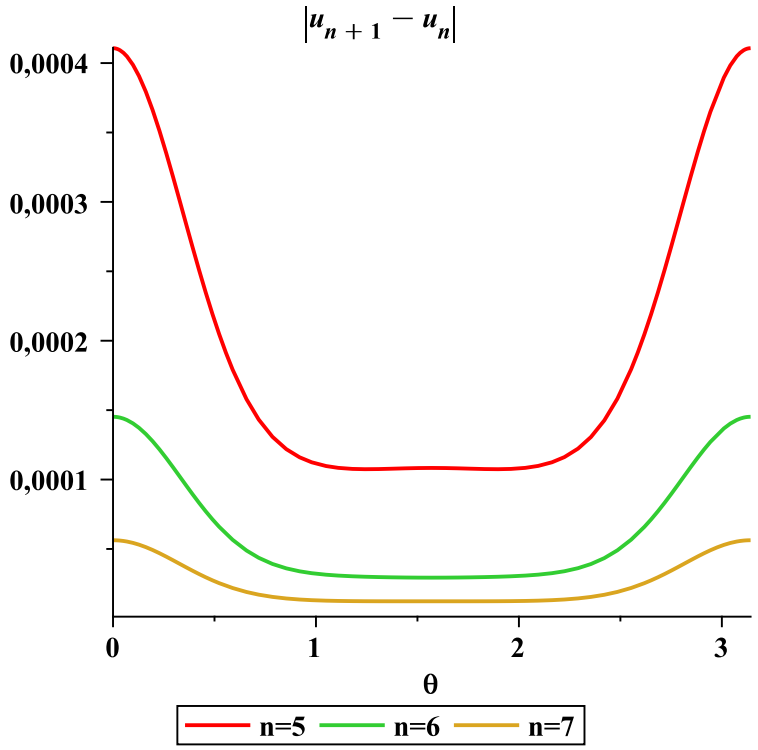

Figure 4. Numerical convergence.

Let $f_n = |\nabla u_n|^2$. Figure 5 displays the functions $|f - f_6|$, $|f - f_7|$ and $|f - f_8|$ where $f = 1 + 3\cos^2\theta$. We have $\|f - f_8\|_\infty \approx 2.5 \times 10^{-4}$. In Figure 6 are plotted the functions $u^* = z/r^3$, the minimal $(-u_8)$ and the maximal (u_8) solutions of the problem (3.1) on the unit sphere. Finally, we draw in Figure 7 the radial components of the gradients of these three functions on S. It is interesting to observe that $(u^*)' = u_8' < 0$ at $\theta = 0$ and $(u^*)' = -u_8' > 0$ at $\theta = \pi$. This is to illustrate a general property of the maximal solution of the Backus problem, namely (see [5, Theorem 2.4]): let $g \in C_+(S)$ and let $u \in K$ satisfy $G(u) = g$. If $u^* \neq -u$ is other solution of the Backus problem then the radial component of ∇u^* changes its sign on S. In particular, there are points $x \in S$ and $y \in S$ such that

$$\langle \nabla u^*(x)\,, x\rangle = \langle \nabla u(x)\,, x\rangle \leq 0 \quad \text{and} \quad \langle \nabla u^*(y)\,, y\rangle = -\langle \nabla u(y)\,, y\rangle \geq 0\,.$$

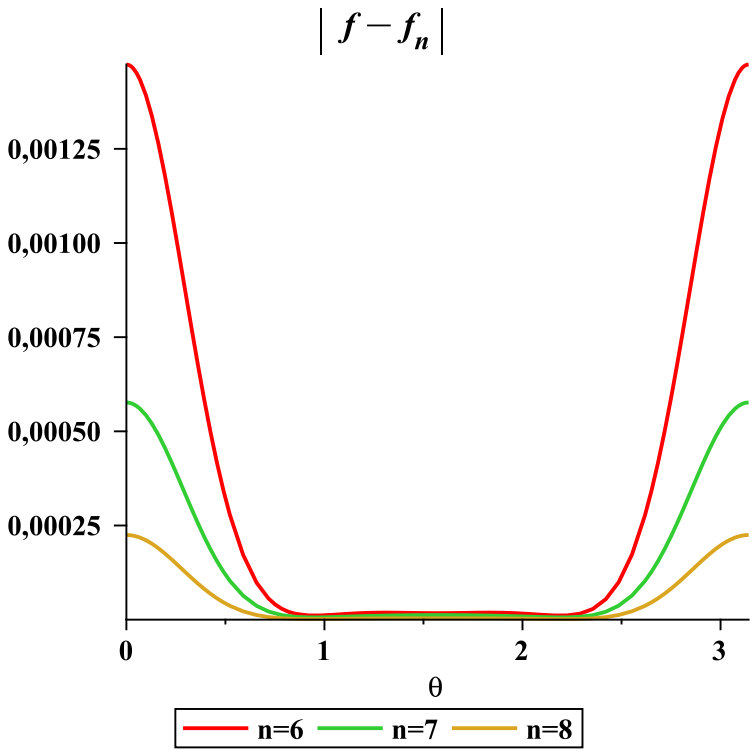

Figure 5. Absolute error.

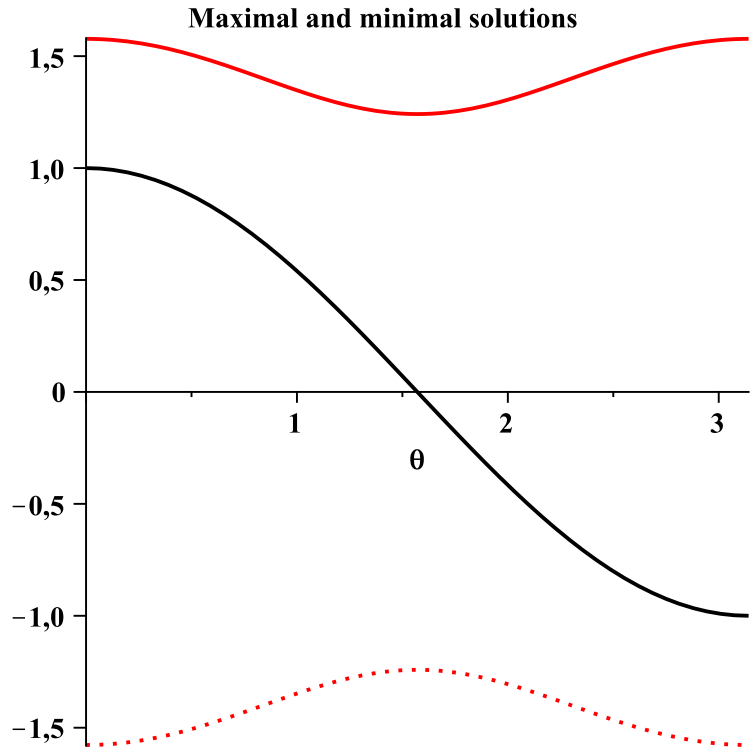

Figure 6. Plot of the functions u_8 (-), $u^* = \cos\theta$ and $-u_8$ (..).

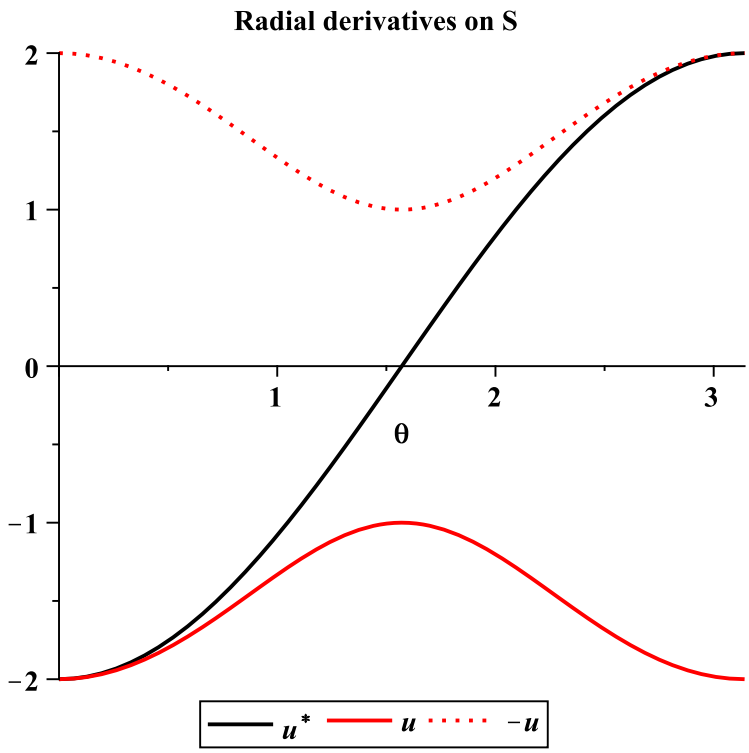

Figure 7. Radial derivatives on S of the functions $-u_8$, u^* and u_8.

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