Supplemental Material to
Volcano transition in populations of phase oscillators with random nonreciprocal interactions

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We have redone the simulation by Kimoto and Uezu (K&U) [1], using the same population size $N = 500$, and a coupling strength $\tilde{J} = 1$. Following [1] the frequency dispersion σ was decreased quasiadiabatically, with a step size $\Delta\sigma = 1.5 \times 10^{-4}$, slightly smaller than $\Delta\sigma = \sqrt{\pi/2} \times 1.25 \times 10^{-4} \simeq 1.567 \times 10^{-4}$ used in [1]. The initial σ value was 1.88, almost identical to $\sqrt{\pi/2} \times 1.5 \simeq 1.87997$ used in [1]. Computation time for each σ value was 800 t.u. long, while 10000 t.u. (recording the local fields every time unit) were run at specific σ values, as in [1].

The results for two completely independent numerical simulations are shown in Fig. S1. One data set (crosses) are the locations of the maximum of the histogram of local fields amplitudes, while the red circles are the values of r_* after fitting the histogram to Eq. (27) in the main text. The prediction by K&U (coincident with our extrapolation from a low-rank coupling matrix) is

$$1 = \frac{2}{\pi g(0)} = \sqrt{\frac{8}{\pi}}\sigma_v \qquad \Rightarrow \qquad \sigma_v = \sqrt{\frac{\pi}{8}} \simeq 0.627, \tag{S1}$$

see dashed line in Fig. S1. In our view there is a nonnegligible discrepancy between theory and numerics. Alternatively, we can infer σ_v from our numerical result in Fig. 5 with $N = 400$. There we fixed $\sigma = \sqrt{\pi/2}$ and varied $\tilde{J}$. Now, we can move to σ space obtaining

$$\sigma_v^{emp} = \frac{\sqrt{\pi/2}}{\tilde{J}_v^{emp}}, \tag{S2}$$

where $\tilde{J}_v^{emp}$ denotes the empirical critical coupling for the volcano transition in Fig. 5. According to Fig. 5 in the main text criticality is in the range $1.6 < \tilde{J}_v^{emp} < 1.7$, so we expect

$$0.737 < \sigma_v^{emp} < 0.783 \tag{S3}$$

This estimation is compatible with the results in Fig. S1. Nonetheless, an extensive study with more simulations, and ideally with larger systems sizes is probably in order.
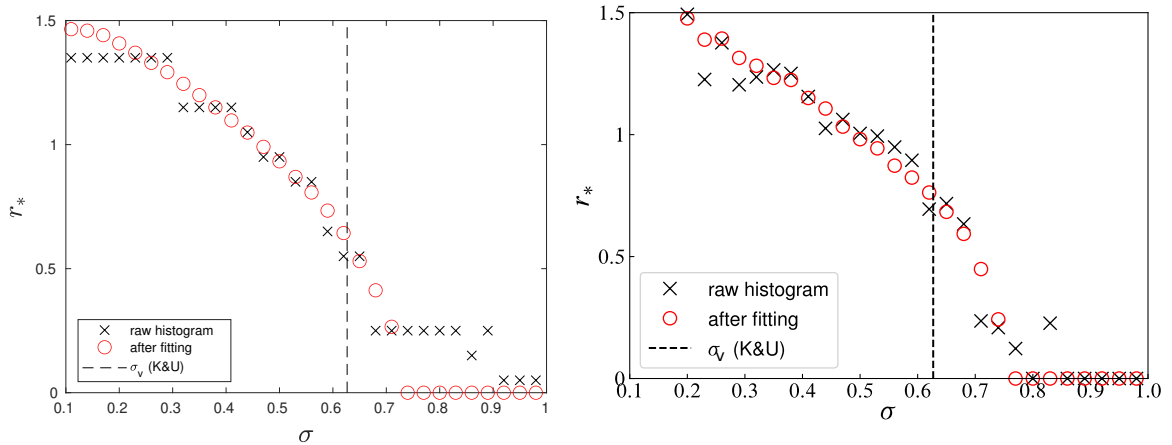

FIG. S1. Maximum of the distribution of local fields r_* as a function of the frequency dispersion. Each panel corresponds to a completely independent numerical implementation (one by each author) using a 4th order Runge-Kutta algorithm of step size 0.1. Confront with Fig. 6(b) of [1].

[1] T. Kimoto and T. Uezu, Phys. Rev. E **100**, 022213 (2019).