QCD Description of Hadronic Tau Decays

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The QCD analysis of hadronic τ decays is reviewed and a summary of the present phenomenological status is presented. The following topics are discussed: the determination of $\alpha_s(m_\tau^2) = 0.338 \pm 0.012$ from the inclusive τ hadronic width, the measurement of $|V_{us}|$ through the Cabibbo-suppressed decays of the τ, and the extraction of chiral-perturbation-theory couplings from the spectral tau data.

1. THEORETICAL FRAMEWORK

The hadronic τ decays turn out to be a beautiful laboratory for studying strong interaction effects at low energies [1–3]. The τ is the only known lepton massive enough to decay into hadrons. Its semileptonic decays are then ideally suited to investigate the hadronic weak currents.

The inclusive character of the total τ hadronic width renders possible an accurate calculation of the ratio [4–8]

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \nu_\tau \,\text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e]} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \,.$$

The theoretical analysis involves the two-point correlation functions for the vector $V_{ij}^\mu = \bar{\psi}_j \gamma^\mu \psi_i$ and axial-vector $A_{ij}^\mu = \bar{\psi}_j \gamma^\mu \gamma_5 \psi_i$ colour-singlet quark currents ($i, j = u, d, s$):

$$\Pi^{\mu\nu}_{ij,\mathcal{J}}(q) \equiv i \int d^4x \, e^{iqx} \langle 0 | T(\mathcal{J}^\mu_{ij}(x) \mathcal{J}^\nu_{ij}(0)^\dagger) | 0 \rangle, \quad (1)$$

which have the Lorentz decompositions

$$\Pi^{\mu\nu}_{ij,\mathcal{J}}(q) = \left(-g^{\mu\nu} q^2 + q^\mu q^\nu\right) \Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu \, \Pi^{(0)}_{ij,\mathcal{J}}(q^2) \,, \quad (2)$$

where the superscript ($J = 0, 1$) denotes the angular momentum in the hadronic rest frame.

The imaginary parts of $\Pi^{(J)}_{ij,\mathcal{J}}(q^2)$ are proportional to the spectral functions for hadrons with the corresponding quantum numbers. The hadronic decay rate of the τ can be written as an integral of these spectral functions over the invariant mass s of the final-state hadrons:

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\Pi^{(1)}(s) + \mathrm{Im}\Pi^{(0)}(s)\right]. \quad (3)$$

The appropriate combinations of correlators are

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) + |V_{us}|^2 \left(\Pi^{(J)}_{us,V}(s) + \Pi^{(J)}_{us,A}(s)\right). \quad (4)$$

The contributions coming from the first two terms correspond to $R_{\tau,V}$ and $R_{\tau,A}$ respectively, while $R_{\tau,S}$ contains the remaining Cabibbo-suppressed contributions.

The integrand in Eq. (3) cannot be calculated at present from QCD. Nevertheless the integral itself can be calculated systematically by exploiting the analytic properties of the correlators $\Pi^{(J)}(s)$. They are analytic functions of s except along the positive real s-axis, where their imaginary parts have discontinuities. R_τ can then be written as a contour integral in the complex s-plane running counter-clockwise around the circle $|s| = m_\tau^2$ [6]:

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2} \Pi^{(0)}(s)\right]. \quad (5)$$

This expression requires the correlators only for complex s of order m_τ^2, which is significantly larger than the scale associated with non-perturbative effects. Using the Operator Product

Expansion (OPE), $\Pi^{(J)}(s) = \sum_D C_D^{(J)}/(-s)^{D/2}$, to evaluate the contour integral, R_τ can be expressed as an expansion in powers of $1/m_\tau^2$. The uncertainties associated with the use of the OPE near the time-like axis are heavily suppressed by the presence in (5) of a double zero at $s = m_\tau^2$.

In the chiral limit ($m_{u,d,s} = 0$), the vector and axial-vector currents are conserved. This implies $s\,\Pi^{(0)}(s) = 0$. Therefore, only the correlator $\Pi^{(0+1)}(s)$ contributes to Eq. (5). Since $(1-x)^2(1+2x) = 1 - 3x^2 + 2x^3$ $[x \equiv s/m_\tau^2]$, Cauchy's theorem guarantees that, up to tiny logarithmic running corrections, the only non-perturbative contributions to the circle integration in (5) originate from operators of dimensions $D = 6$ and 8. The usually leading $D = 4$ operators can only contribute to R_τ with an additional suppression factor of $\mathcal{O}(\alpha_s^2)$, which makes their effect negligible [6].

2. DETERMINATION OF α_s

The Cabibbo-allowed combination $R_{\tau,V+A}$ can be written as [6]

$$R_{\tau,V+A} \;=\; N_C\, |V_{ud}|^2\, S_{\rm EW} \left\{1 + \delta_{\rm P} + \delta_{\rm NP}\right\}, \qquad (6)$$

where $N_C = 3$ is the number of quark colours and $S_{\rm EW} = 1.0201 \pm 0.0003$ contains the electroweak radiative corrections [9–11]. The dominant correction ($\sim 20\%$) is the perturbative QCD contribution $\delta_{\rm P}$, which is already known to $O(\alpha_s^4)$ [6, 12]. Quark mass effects [6, 13, 14] are tiny for the Cabibbo-allowed current and amount to a negligible correction smaller than 10^{-4} [6, 15].

Non-perturbative contributions are suppressed by six powers of the τ mass [6] and, therefore, are very small. Their numerical size has been determined from the invariant-mass distribution of the final hadrons in τ decay, through the study of weighted integrals [16],

$$R_\tau^{kl} \;\equiv\; \int_0^{m_\tau^2} ds\, \left(1 - \frac{s}{m_\tau^2}\right)^k \left(\frac{s}{m_\tau^2}\right)^l \frac{dR_\tau}{ds}\,, \qquad (7)$$

which can be calculated theoretically in the same way as R_τ, but are more sensitive to OPE corrections. The predicted suppression [6] of the non-perturbative corrections to R_τ has been confirmed by ALEPH [17], CLEO [18] and OPAL [19]. The most recent analysis gives [20]

$$\delta_{\rm NP} \;=\; -0.0059 \pm 0.0014\,. \qquad (8)$$

The QCD prediction for $R_{\tau,V+A}$ is then completely dominated by $\delta_{\rm P}$; non-perturbative effects being smaller than the perturbative uncertainties from uncalculated higher-order corrections. Assuming lepton universality, the measured values of the τ lifetime and leptonic branching ratios imply $R_\tau = 3.6291 \pm 0.0086$ [21]. Subtracting the Cabibbo-suppressed contribution $R_{\tau,S} = 0.1613 \pm 0.0028$ [21], one obtains $R_{\tau,V+A} = 3.4678 \pm 0.0090$. Using $|V_{ud}| = 0.97425 \pm 0.00022$ [22] and (8), the pure perturbative contribution to R_τ is determined to be:

$$\delta_{\rm P} = 0.1997 \pm 0.0035\,. \qquad (9)$$

The predicted value of $\delta_{\rm P}$ turns out to be very sensitive to $\alpha_s(m_\tau^2)$, allowing for an accurate determination of the fundamental QCD coupling [5, 6]. The calculation of the $\mathcal{O}(\alpha_s^4)$ contribution [12] has triggered a renewed theoretical interest on the $\alpha_s(m_\tau^2)$ determination, since it allows to push the accuracy to the four-loop level. However, as shown in Table 1, the recent theoretical analyses slightly disagree on the final result. The differences are larger than the claimed $\mathcal{O}(\alpha_s^4)$ accuracy and originate in the different inputs or theoretical procedures which have been adopted.

2.1. Perturbative contribution to R_τ

In the chiral limit, the result is more conveniently expressed in terms of the logarithmic derivative of the two-point correlation function of the vector (axial) current, $\Pi(s) = \frac{1}{2}\,\Pi^{(0+1)}(s)$, which satisfies an homogeneous renormalization–group equation:

$$D(s) \equiv -s\frac{d}{ds}\Pi(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n . \qquad (10)$$

With the choice of renormalization scale $\mu^2 = -s$ all logarithmic corrections, proportional to powers of $\log(-s/\mu^2)$, have been summed into the running coupling. For three flavours, the known coefficients take the values: $K_0 = K_1 = 1$; $K_2 = 1.63982$; $K_3(\overline{MS}) = 6.37101$ and $K_4(\overline{MS}) = 49.07570$ [12].

Table 1
$\mathcal{O}(\alpha_s^4)$ determinations of $\alpha_s(m_\tau^2)$. The assumed value of δ_P is also given (if quoted by the authors).

Reference	Method	δ_P	$\alpha_s(m_\tau^2)$	$\alpha_s(M_Z^2)$
Baikov et al. [12]	CIPT, FOPT	0.1998 ± 0.0043	0.332 ± 0.016	0.1202 ± 0.0019
Davier et al. [23]	CIPT	0.2066 ± 0.0070	0.344 ± 0.009	0.1212 ± 0.0011
Beneke-Jamin [15]	BSR + FOPT	0.2042 ± 0.0050	0.316 ± 0.006	0.1180 ± 0.0008
Maltman-Yavin [24]	PWM + CIPT	—	0.321 ± 0.013	0.1187 ± 0.0016
Menke [25]	CIPT, FOPT	0.2042 ± 0.0050	$0.342\,^{+\,0.011}_{-\,0.010}$	0.1213 ± 0.0012
Caprini-Fischer [26]	BSR + CIPT	0.2042 ± 0.0050	$0.320\,^{+\,0.011}_{-\,0.009}$	—
Cvetič et al. [27]	β_exp + CIPT	0.2040 ± 0.0040	0.341 ± 0.008	0.1211 ± 0.0010
Pich [1]	CIPT	0.2038 ± 0.0040	0.342 ± 0.012	0.1213 ± 0.0014

Table 2
Exact results for $A^{(n)}(\alpha_s)$ ($n \le 4$) at different β-function approximations, and corresponding values of $\delta_\mathrm{P} = \sum_{n=1}^4 K_n\, A^{(n)}(\alpha_s)$, for $a_\tau \equiv \alpha_s(m_\tau^2)/\pi = 0.11$. The last row shows the FOPT estimates at $\mathcal{O}(a_\tau^4)$.

	$A^{(1)}(\alpha_s)$	$A^{(2)}(\alpha_s)$	$A^{(3)}(\alpha_s)$	$A^{(4)}(\alpha_s)$	δ_P
$\beta_{n>1} = 0$	0.14828	0.01925	0.00225	0.00024	0.20578
$\beta_{n>2} = 0$	0.15103	0.01905	0.00209	0.00020	0.20537
$\beta_{n>3} = 0$	0.15093	0.01882	0.00202	0.00019	0.20389
$\beta_{n>4} = 0$	0.15058	0.01865	0.00198	0.00018	0.20273
$\mathcal{O}(a_\tau^4)$	0.16115	0.02431	0.00290	0.00015	0.22665

The perturbative component of R_τ is given by

$$\delta_\mathrm{P} \; = \; \sum_{n=1} K_n\, A^{(n)}(\alpha_s)\, , \qquad (11)$$

where the functions [7]

$$A^{(n)}(\alpha_s) \; = \; \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(\frac{\alpha_s(-s)}{\pi}\right)^n \times \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8}\right) \qquad (12)$$

are contour integrals in the complex plane, which only depend on $a_\tau \equiv \alpha_s(m_\tau^2)/\pi$. Using the exact solution (up to unknown $\beta_{n>4}$ contributions) for $\alpha_s(-s)$ given by the renormalization-group β-function equation, they can be numerically computed with a very high accuracy [7]. Table 2 gives the numerical values for $A^{(n)}(\alpha_s)$ ($n \le 4$) obtained at the one-, two-, three- and four-loop approximations (i.e. $\beta_{n>1} = 0$, $\beta_{n>2} = 0$, $\beta_{n>3} = 0$ and $\beta_{n>4} = 0$, respectively), together with the corresponding results for $\delta_\mathrm{P} = \sum_{n=1}^4 K_n\, A^{(n)}(\alpha_s)$, taking $a_\tau = 0.11$. The perturbative convergence is very good and the results are stable under changes of the renormalization scale. The error induced by the truncation of the β function at fourth order can be conservatively estimated through the variation of the results at five loops, assuming $\beta_5 = \pm\beta_4^2/\beta_3 = \mp 443$, i.e. a geometric growth of the β function.

Higher-order contributions to the Adler function $D(s)$ will be taken into account adding the fifth-order term $K_5\, A^{(5)}(\alpha_s)$ with $K_5 = 275 \pm 400$. Moreover, we will include the 5-loop variation with changes of the renormalization scale in the range $\mu^2/(-s) \in [0.5, 1.5]$. Adopting this very conservative procedure, the experimental value of δ_P given in Eq. (9) implies

$$\alpha_s(m_\tau^2) = 0.338 \pm 0.012\, . \qquad (13)$$

The result is slightly lower than the one given in Ref. [1], due to the smaller value of δ_P.

The strong coupling measured at the τ mass scale is significantly larger than the values obtained at higher energies. From the hadronic decays of the Z, one gets $\alpha_s(M_Z^2) = 0.1190 \pm 0.0027$ [28], which differs from $\alpha_s(m_\tau^2)$ by $18\,\sigma$. After evolution up to the scale M_Z [29], the strong coupling constant in (13) decreases to

$$\alpha_s(M_Z^2) \;=\; 0.1209 \pm 0.0014\,, \qquad (14)$$

in excellent agreement with the direct measurements at the Z peak and with a better accuracy. The comparison of these two determinations of α_s in two very different energy regimes, m_τ and M_Z, provides a beautiful test of the predicted running of the QCD coupling; i.e., a very significant experimental verification of *asymptotic freedom*.

2.2. Fixed-order perturbation theory

The integrals $A^{(n)}(\alpha_s)$ can be expanded in powers of a_τ, $A^{(n)}(\alpha_s) = a_\tau^n + \mathcal{O}(a_\tau^{n+1})$. One recovers in this way the naive perturbative expansion [7]

$$\delta_{\rm P} \;=\; \sum_{n=1} (K_n + g_n)\, a_\tau^n \;\equiv\; \sum_{n=1} r_n\, a_\tau^n\,. \qquad (15)$$

This approximation is known as *fixed-order perturbation theory* (FOPT), while the improved expression (11), keeping the non-expanded values of $A^{(n)}(\alpha_s)$, is usually called *contour-improved perturbation theory* (CIPT) [7,30].

As shown in the last row of Table 2, even at $\mathcal{O}(a_\tau^4)$, FOPT gives a rather bad approximation to the integrals $A^{(n)}(\alpha_s)$, overestimating $\delta_{\rm P}$ by 12% at $a_\tau = 0.11$. The long running of $\alpha_s(-s)$ along the circle $|s| = m_\tau^2$ generates very large g_n coefficients, which depend on $K_{m<n}$ and $\beta_{m<n}$ [7]: $g_1 = 0$, $g_2 = 3.56$, $g_3 = 19.99$, $g_4 = 78.00$, $g_5 = 307.78$. These corrections are much larger than the original K_n contributions and lead to values of $\alpha_s(m_\tau^2)$ smaller than (13). FOPT suffers from a large renormalization-scale dependence [7] and its actual uncertainties are much larger than usually estimated [25].

The origin of this bad behaviour can be understood analytically at one loop [7]. In FOPT one makes within the contour integral the series expansion ($\log(-s/m_\tau^2) = i\phi$, $\phi \in [-\pi, \pi]$)

$$\frac{\alpha_s(-s)}{\pi} \approx \frac{a_\tau}{1 - i\beta_1 a_\tau \phi/2} \approx a_\tau \sum_n \left(\frac{i}{2}\beta_1 a_\tau \phi\right)^n, \qquad (16)$$

which is only convergent for $a_\tau < 0.14$. At the four-loop level the radius of convergence is slightly smaller than the physical value of a_τ. Thus, FOPT gives rise to a pathological non-convergent series. The long running along the circle makes compulsory to resum the large logarithms, $\log^n(-s/m_\tau^2)$, using the renormalization group. This is precisely what CIPT does.

2.3. Renormalon hypothesis

The perturbative expansion of the Adler function is expected to be an asymptotic series. If its Borel transform, $B(t) \equiv \sum_{n=0} K_{n+1} t^n/n!$, were well-behaved, one could define $D(s)$ through the Borel integral

$$D(s) = \frac{1}{4\pi^2}\left\{1 + \int_0^\infty dt\, {\rm e}^{-\pi t/\alpha_s(s)} B(t)\right\}. \qquad (17)$$

However, $B(t)$ has pole singularities at positive (infrared renormalons) and negative (ultraviolet renormalons) integer values of the variable $u \equiv -\beta_1 t/2$, with the exception of $u = 1$ [31]. The infrared renormalons at $u = +n$ are related to OPE corrections of dimension $D = 2n$. The renormalon poles closer to the origin dominate the large-order behaviour of $D(s)$.

It has been argued that, once in the asymptotic regime (large n), the renormalonic behaviour of the K_n coefficients could induce cancelations with the running g_n corrections, which would be missed by CIPT. In that case, FOPT could approach faster the 'true' result provided by the Borel summation of the full renormalon series (BSR) [15]. This happens actually in the large–β_1 limit [32,33], which however does not approximate well the known K_n coefficients. A model of higher-order corrections with this behaviour has been recently advocated [15]. The model mixes three different types of renormalons ($n = -1$, 2 and 3) plus a linear polynomial. It contains 5 free parameters which are determined by the known values of $K_{1,2,3,4}$ and the assumption $K_5 = 283$. One gets in this way a larger $\delta_{\rm P}$, implying a smaller value for $\alpha_s(m_\tau^2)$. The result looks however model dependent [34].

The implications of a renormalonic behaviour have been put on more solid grounds, using an optimal conformal mapping in the Borel plane, which achieves the best asymptotic rate of convergence, and properly implementing the CIPT procedure within the Borel transform [26]. Assuming that the known fourth-order series is already governed by the $u = -1$ and $u = 2$ renormalons, the conformal mapping generates a full series expansion ($K_5 = 256$, $K_6 = 2929 \ldots$) which results, after Borel summation, in a larger value of $\delta_{\rm P}$; i.e. the $K_{n>4}$ terms give a positive contribution to $\delta_{\rm P}$ implying a smaller $\alpha_s(m_\tau^2)$ [26].

Renormalons provide an interesting guide to possible higher-order corrections, making apparent that the associated uncertainties have to be carefully estimated. However, one should keep in mind the adopted assumptions. In fact, there are no visible signs of renormalonic behaviour in the presently known series: the $n = -1$ ultraviolet renormalon is expected to dominate the asymptotic regime, implying an alternating series, while all known K_n coefficients have the same sign. One could either assume that renormalons only become relevant at higher orders, for instance at $n = 7$, and apply the conformal mapping with arbitrary input values for K_5 and K_6. Different assumptions about these two unknown coefficients would result in different central values for $\alpha_s(m_\tau^2)$.

A different reshuffling of the perturbative series, not related to renormalons, has been recently proposed [27]. Instead of the usual expansion in powers of the strong coupling, one expands in terms of the β function and its derivatives ($\beta_{\rm exp}$), which effectively results in a different estimate of higher-order corrections. One gets in this way a weaker dependence on the renormalization scale and a value of $\alpha_s(m_\tau^2)$ similar to the standard CIPT result.

2.4. Non-perturbative corrections

At the presently achieved precision, one should worry about the small non-perturbative corrections. In fact, a proper definition of the infrared renormalon contributions is linked to the corresponding OPE corrections with $D = 2n$. A recent re-analysis of the ALEPH data [24], with pinched-weight moments of the hadronic distribution (PWM) and CIPT, obtains $\alpha_s(m_\tau^2) = 0.321 \pm 0.013$. This smaller value originates in a different estimate of the non-perturbative contributions. Unfortunately, Ref. [24] does not quote any explicit values for $\delta_{\rm NP}$ and $\delta_{\rm P}$. From the information given in that reference, I deduce $\delta_{\rm NP} = 0.012 \pm 0.018$. Although compatible with (8), the central value is larger and has the opposite sign. This shift implies a smaller $\delta_{\rm P}$ and, therefore, a slightly smaller strong coupling.

The so-called duality violation effects, i.e. the uncertainties associated with the use of the OPE to approximate the exact correlator, have been also investigated [23, 35]. Owing to the presence in (5) of a double zero at $s = m_\tau^2$, these effects are quite suppressed in R_τ. They are smaller than the errors induced by $\delta_{\rm NP}$, which are in turn subdominant with respect to the leading perturbative uncertainties.

3. $|\mathbf{V_{us}}|$ DETERMINATION

The separate measurement of the $|\Delta S| = 0$ and $|\Delta S| = 1$ tau decay widths provides a very clean determination of V_{us} [36, 37]. To a first approximation the Cabibbo mixing can be directly obtained from experimental measurements, without any theoretical input. Neglecting the small SU(3)-breaking corrections from the $m_s - m_d$ quark-mass difference, one gets:

$$|V_{us}|^{\rm SU(3)} = |V_{ud}| \left(\frac{R_{\tau,S}}{R_{\tau,V+A}} \right)^{1/2} = 0.210 \pm 0.002 \, .$$

The new branching ratios measured by BaBar and Belle are all smaller than the previous world averages, which translates into a smaller value of $R_{\tau,S}$ and $|V_{us}|$. For comparison, the previous value $R_{\tau,S} = 0.1686 \pm 0.0047$ [20] resulted in $|V_{us}|^{\rm SU(3)} = 0.215 \pm 0.003$.

This rather remarkable determination is only slightly shifted by the small SU(3)-breaking contributions induced by the strange quark mass. These effects can be estimated through a QCD analysis of the differences [13, 14, 36–43]

$$\delta R_\tau^{kl} \;\equiv\; \frac{R_{\tau,V+A}^{kl}}{|V_{ud}|^2} - \frac{R_{\tau,S}^{kl}}{|V_{us}|^2} \, . \qquad (18)$$

The only non-zero contributions are proportional to the mass-squared difference $m_s^2 - m_d^2$ or to vacuum expectation values of SU(3)-breaking operators such as $\delta O_4 \equiv \langle 0|m_s\bar{s}s - m_d\bar{d}d|0\rangle \approx (-1.4 \pm 0.4) \cdot 10^{-3}\ \mathrm{GeV}^4$ [13, 36]. The dimensions of these operators are compensated by corresponding powers of m_τ^2, which implies a strong suppression of δR_τ^{kl} [13]:

$$\delta R_\tau^{kl} \;\approx\; 24\, S_{\mathrm{EW}} \left\{ \frac{m_s^2(m_\tau^2)}{m_\tau^2} \left(1-\epsilon_d^2\right) \Delta_{kl}(\alpha_s) - 2\pi^2 \,\frac{\delta O_4}{m_\tau^4}\, Q_{kl}(\alpha_s) \right\} , \qquad (19)$$

where $\epsilon_d \equiv m_d/m_s = 0.053 \pm 0.002$ [44]. The perturbative corrections $\Delta_{kl}(\alpha_s)$ and $Q_{kl}(\alpha_s)$ are known to $O(\alpha_s^3)$ and $O(\alpha_s^2)$, respectively [13, 14].

The $J = 0$ contribution to $\Delta_{00}(\alpha_s)$ shows a rather pathological behaviour, with clear signs of being a non-convergent perturbative series. Fortunately, the corresponding longitudinal contribution to $\delta R_\tau \equiv \delta R_\tau^{00}$ can be estimated phenomenologically with a much better accuracy, $\delta R_\tau|^L = 0.1544 \pm 0.0037$ [36, 45], because it is dominated by far by the well-known $\tau \to \nu_\tau \pi$ and $\tau \to \nu_\tau K$ contributions. To estimate the remaining transverse component, one needs an input value for the strange quark mass. Taking the range $m_s(m_\tau) = (100 \pm 10)\ \mathrm{MeV}$ $[m_s(2\ \mathrm{GeV}) = (96 \pm 10)\ \mathrm{MeV}]$, which includes the most recent determinations of m_s from QCD sum rules and lattice QCD [45], one gets finally $\delta R_{\tau,th} = 0.216 \pm 0.016$ [37], which implies

$$|V_{us}| \;=\; \left(\frac{R_{\tau,S}}{\frac{R_{\tau,V+A}}{|V_{ud}|^2} - \delta R_{\tau,\mathrm{th}}} \right)^{1/2} \;=\; 0.2166 \pm 0.0019_{\,\mathrm{exp}} \pm 0.0005_{\,\mathrm{th}} . \qquad (20)$$

A larger central value, $|V_{us}| = 0.2217 \pm 0.0032$, is obtained with the old world average for $R_{\tau,S}$.

Sizeable changes on the experimental determination of $R_{\tau,S}$ could be expected from the full analysis of the huge BaBar and Belle data samples. In particular, the high-multiplicity decay modes are not well known at present. The recent decrease of several experimental tau branching ratios is also worrisome. As pointed out by the PDG [22], 15 of the 16 branching fractions measured at the B factories are smaller than the previous non-B-factory values. The average normalized difference between the two sets of measurements is $-1.36\,\sigma$. Thus, the result (20) could easily fluctuate in the near future. In fact, combining the measured Cabibbo-suppressed τ distribution with electroproduction data, a slightly larger value of $|V_{us}|$ is obtained [46].

The final error of the V_{us} determination from τ decay is dominated by the experimental uncertainties. If $R_{\tau,S}$ is measured with a 1% precision, the resulting V_{us} uncertainty will get reduced to around 0.6%, i.e. ± 0.0013, making τ decay the best source of information about V_{us}.

An accurate measurement of the invariant-mass distribution of the final hadrons could make possible a simultaneous determination of V_{us} and the strange quark mass, through a correlated analysis of several weighted differences δR_τ^{kl}. However, the extraction of m_s suffers from theoretical uncertainties related to the convergence of the perturbative series $\Delta_{kl}(\alpha_s)$. A better understanding of these corrections is needed.

4. CHIRAL SUM RULES

When $m_{u,d,s} = 0$, the QCD Lagrangian has an independent $SU(3)$ flavour invariance for the left and right quark quiralities. The two quiralities have exactly the same strong interaction, but they are completely decoupled. This chiral invariance guarantees that the two-point correlation function of a left-handed and a right-handed quark currents, $\Pi_{LR}(s) = \Pi_{ud,V}^{(0+1)}(s) - \Pi_{ud,A}^{(0+1)}(s)$, vanishes identically to all orders in perturbation theory (the vector and axial-vector correlators receive identical perturbative contributions). The non-zero value of $\Pi_{LR}(s)$ originates in the spontaneous breaking of chiral symmetry by the QCD vacuum. At large momenta, the corresponding OPE only receives contributions from operators with dimension $d \geq 6$,

$$\Pi_{LR}^{\mathrm{OPE}}(s) = -\frac{\mathcal{O}_6}{s^3} + \frac{\mathcal{O}_8}{s^4} + \cdots \qquad (21)$$

The non-zero up and down quark masses induce tiny corrections with dimensions two and four, which are negligible at high energies.

At very low momenta, Chiral Perturbation Theory (χPT) dictates the low-energy expansion of $\Pi_{LR}(s)$ in terms of the pion decay constant and the χPT couplings L_{10} $[\mathcal{O}(p^4)]$ and C_{87} $[\mathcal{O}(p^6)]$.

Analyticity relates the short- and long-distance regimes through the dispersion relation

$$\frac{1}{2\pi i}\oint_{|s|=s_0} ds\, w(s)\, \Pi_{LR}(s) \;=\; -\int_{s_{th}}^{s_0} ds\, w(s)\, \rho(s) + 2f_\pi^2\, w(m_\pi^2) + \mathrm{Res}[w(s)\Pi_{LR}(s), s=0] \;, \qquad (22)$$

where $\rho(s) \equiv \frac{1}{\pi}\,\mathrm{Im}\Pi_{LR}(s)$ and $w(s)$ is an arbitrary weight function that is analytic in the whole complex plane except in the origin (where it can have poles). The last term in (22) accounts for the possible residue at the origin.

For $s_0 \le m_\tau^2$, the integral along the real axis can be evaluated with the measured tau spectral functions. Taking $w(s) = s^n$ with $n \ge 0$, there is no residue at the origin and, with s_0 large enough so that the OPE can be applied in the entire circle $|s| = s_0$, the OPE coefficients are directly related to the spectral function integration. With $n = 0$ and 1, there is no OPE contribution in the chiral limit and one gets the celebrated first and second Weinberg sum rules [47]. For negative values of the integer n, the OPE does not contribute either while the residues at zero are determined by the χPT low-energy couplings, which can be then experimentally determined [48].

Moreover, the absence of perturbative contributions makes (22) and ideal tool to investigate possible quark-hadron duality effects, formally defined through [49–51]

$$\mathrm{DV}_w \;\equiv\; \frac{1}{2\pi i}\oint_{|s|=s_0} ds\; w(s)\left(\Pi_{LR}(s) - \Pi_{LR}^{\mathrm{OPE}}(s)\right) \;=\; \int_{s_0}^{\infty} \mathrm{d}s\; w(s)\; \rho(s)\,. \qquad (23)$$

This has been thoroughly studied in Ref. [52], using for the spectral function beyond $s_z \sim 2.1$ GeV the parametrization [35, 49–51]

$$\rho(s \ge s_z) \;=\; \kappa\; e^{-\gamma s}\sin(\beta(s - s_z))\,, \qquad (24)$$

and finding the region in the 4-dimensional $(\kappa, \gamma, \beta, s_z)$ parameter space that is compatible with the most recent experimental data [17] and the following theoretical constraints at $s_0 \to \infty$: first and second Weinberg sum sules [47] and the sum rule of Das et al. [53] that determines the pion electromagnetic mass difference.

Ref. [52] performs a statistical analysis, scanning the parameter space $(\kappa, \gamma, \beta, s_z)$ and selecting those 'acceptable' spectral functions which satisfy the experimental and theoretical constraints. From a generated initial sample of $160,000$ tuples, one finds $1,789$ acceptable distributions compatible with QCD and the data. The differences among them determine how much freedom is left for the behaviour of the spectral function beyond the kinematical end of the τ data. For each acceptable spectral function one calculates the parameters L_{10}, C_{87}, $\mathcal{O}_6$ and $\mathcal{O}_8$, obtained through the dispersion relation (22) with the appropriate weight functions. The resulting statistical distributions determine their finally estimated values; the dispersion of the numerical results provides a good quantitative assessment of the actual uncertainties.

The study has been also performed with pinched weight functions of the form $w(s) = s^n(s - s_z)^m$ $(m > 0)$ that vanish at $s = s_z$. As expected, these weights are found to minimize the uncertainties from duality-violation effects, allowing for a more precise determination of the hadronic parameters. One finally obtains [52],

$$\begin{aligned}
L_{10}^r(M_\rho) \;&=\; -(4.06 \pm 0.39)\cdot 10^{-3}\,,\\
C_{87}^r(M_\rho) \;&=\; (4.89 \pm 0.19)\cdot 10^{-3}\ \mathrm{GeV}^{-2}\,,\\
\mathcal{O}_6 \;&=\; (-4.3\,{}^{+\,0.9}_{-\,0.7})\cdot 10^{-3}\ \mathrm{GeV}^6\,,\\
\mathcal{O}_8 \;&=\; (-7.2\,{}^{+\,4.2}_{-\,5.3})\cdot 10^{-3}\ \mathrm{GeV}^8\,.
\end{aligned} \qquad (25)$$

The determination of the two χPT couplings is in good agreement with (but more precise than) recent theoretical calculations, using Resonance Chiral Effective Theory [54] and large–N_C techniques at the next-to-leading order [55], which predict [56]: $L_{10}^r(M_\rho) = -(4.4 \pm 0.9)\cdot 10^{-3}$ and $C_{87}^r(M_\rho) = (3.6 \pm 1.3)\cdot 10^{-3}\ \mathrm{GeV}^{-2}$. It also agrees with the present lattice estimates of $L_{10}^r(M_\rho)$ [57].

Duality-violation effects have very little impact on the determination of L_{10} and C_{87} because the corresponding sum rules are dominated by the low-energy region where the data sits. Thus, one obtains basically the same results with pinched and non-pinched weight functions. This is nolonger true for $\mathcal{O}_6$ and $\mathcal{O}_8$, which are sensitive to

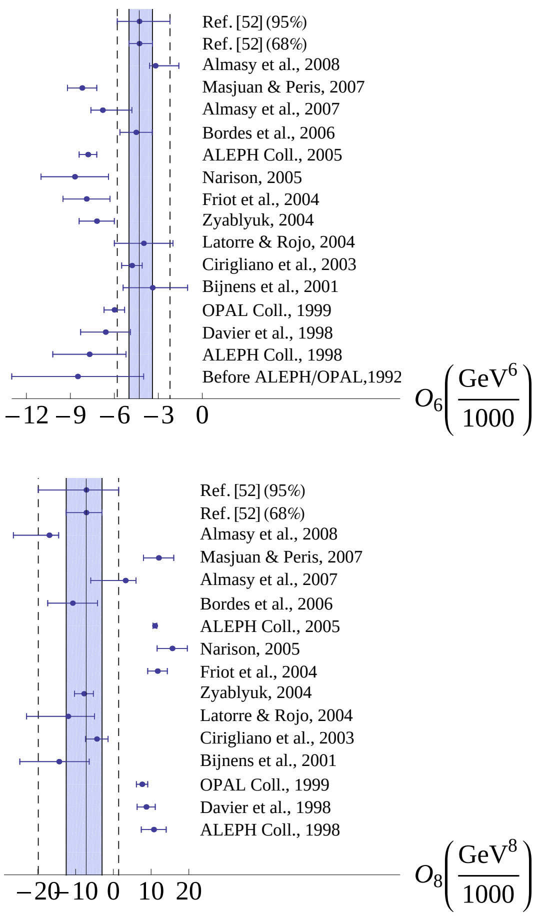

Figure 1. Published results for $\mathcal{O}_6$ and $\mathcal{O}_8$ [52].

the high-energy behaviour of the spectral function; pinched-weights provide then a much better accuracy. This could explain the numerical differences among previous estimates [17, 19, 48, 58], shown in Fig. 1, where duality violation uncertainties were not properly assessed. The results (25) fix with accuracy the value of $\mathcal{O}_6$ and determine the sign of $\mathcal{O}_8$. This information is needed to calculate the electromagnetic penguin contribution to the CP-violating ratio $\varepsilon'_K/\varepsilon_K$ [59].

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