INFINITE $B_2[g]$ SEQUENCES

Javier Cilleruelo
Universidad Autónoma de Madrid
28049 Madrid, Spain
franciscojavier.cilleruelo@uam.es

Carlos Trujillo[1]
Universidad del Cauca
Popayan, Colombia

Abstract. We exhibit, for any integer $g \geq 2$, an infinite sequence $A \in B_2[g]$ such that $\limsup_{x\to\infty} \frac{A(x)}{\sqrt{x}} = \frac{3}{2\sqrt{2}}\sqrt{g-1}$. In adition, we obtain better estimates for small values of g. For example, we exhibit an infinite sequence $A \in B_2[2]$ such that $\limsup_{x\to\infty} \frac{A(x)}{\sqrt{x}} = \sqrt{3/2}$

Introduction

For $g \in \mathbb{N}$, $B_2[g]$ denotes the class of all sets $A \subset \mathbb{N}$ such that for all $n \in \mathbb{N}$ the equation $a + a' = n, \quad a, a' \in A \quad a \leq a'$ has at most g solutions. The sets $B_2[1]$ are called *Sidon* sets.

In [6] Erdős proved that if A is an infinite Sidon sequence then $\liminf_{x\to\infty} \frac{A(x)}{x^{1/2}} = 0$ where $A(x) = \#\{a \leq x; \quad a \in A\}$ is the counting function. On the other hand he showed that there exists an infinite Sidon sequence such that $\limsup_{x\to\infty} \frac{A(x)}{x^{1/2}} = 1/2$. This limit was improved to $1/\sqrt{2}$ by Kruckeberg [5]. Much less is known on infinite $B_2[g]$ sequences for $g > 1$. It is conjectured that $\liminf_{x\to\infty} \frac{A(x)}{x^{1/2}} = 0$ for any infinite $B_2[g]$ sequence, but it is unknown even for $g = 2$.

Respect the limit superior, Kolountzakis [4] proved that there is an infinite $B_2[2]$ sequence A such that $\limsup_{x\to\infty} \frac{A(x)}{x^{1/2}} = 1$. Xingde Jia [3] worked on this topic and, although his method doesn't work for usual $B_2[g]$ sequences, he gets interesting upper bounds for sequences such that, fixed m, the number of solutions of $n \equiv a + a' \pmod{m}$, $a \leq a'$, $a, a' \in A$, is less or equal than g, for any integer n.

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We have proceeded in a different way and we improve the previous lower bounds on infinite $B_2[g]$ sequences for $g \geq 2$.

Theorem 1. *For all $g \geq 2$ there exists an infinite $B_2[g]$ sequence A such that* $\limsup_{x\to\infty} \frac{A(x)}{\sqrt{x}} = L_g$ *where*

$$L_g = \begin{cases} \sqrt{3/2}, & g = 2 \\ 3/2, & g = 3 \\ \sqrt{\frac{36}{11}}, & g = 4 \\ \sqrt{\frac{9}{2}}, & g = 5 \\ \sqrt{\frac{100}{17}}, & g = 6 \\ \sqrt{\frac{27}{4}}, & g = 7 \\ \sqrt{8}, & g = 8 \\ \frac{3}{2\sqrt{2}}\sqrt{g-1}, & g \geq 9. \end{cases}$$

Proof of theorem 1.

To prove the theorem we only need to show that any $B_2[g]$ sequence $A_0 = \{n_1 < n_2 < \cdots < n_k\}$ can be extended to a $B_2[g]$ sequence $A = \{n_1 < \cdots < n_k < n_{k+1} < \cdots < n_l\}$ where $\frac{l}{\sqrt{n_l}} = \frac{A(n_l)}{\sqrt{n_l}} \geq L_g + o(1)$.

For the construction of A we need two special sets C_g and B_p, whose properties are stated in the following two Propositions. The proof of the Propositions is postponed to the end of the section.

Proposition 1. *For any prime p, there exists a set $B_p \subset (p^{1/2}, p^2 - p^{1/2})$ such that*

i) If $b + b' \equiv b'' + b'''$ $\pmod{p^2-1}$, $b, b', b'', b''' \in B_p$ then $\{b, b'\} = \{b'', b'''\}$;

ii) $|b - b'| > p^{1/2}$ for all differents $b, b' \in B_p$;

iii) $|B_p| > p - 4p^{1/2}$.

Proposition 2. *For all $g \geq 2$ there exists an integer u_g and a set $C_g \subset [0, u_g]$ such that, if $r(n) = \#\{n = c + c';\quad c, c' \in C_g\}$, then*

i) $r(n) \leq g$ for all integer n;

ii) $r(c) \leq g - 1$ and $r(c-1) \leq g - 1$ for all $c \in C_g$;

iii) $\frac{|C_g|}{\sqrt{u_g+1}} = L_g$, where L_g is defined as in Theorem 1.

The construction of A. Taking $x = n_k$, p a prime, $x^2 < p < 2x^2$ and $m = p^2 - 1$ we define

$$A = A_0 \cup D \text{ where } D = \bigcup_{c \in C_g} (B_p + cm + 2x)$$

and the sets B_p and C_g are defined as in the Propositions 1 and 2.

Obviously A is an extension of A_0. Then we need to prove that A is a $B_2[g]$ sequence satisfying $\frac{|A|}{\sqrt{n_l}} = L_g + o(1)$ where n_l is the last element of A.

Proposition 3. *A is a $B_2[g]$ sequence.*

Proof. If $n \leq 2x$, then all the representations of n as a sum of two elements of A are of the form $a + a'$ $a, a' \in A_0$. Because A_0 is a $B_2[g]$ sequence, there are at most g representations.

If $n > 2x$, there is at most one representation of the form $a + d$, $a \in A_0, d \in D$. Otherwise, if $n = a + d = a' + d'$ then $x > |a - a'| = |d - d'| = |b - b' + (c_i - c_{i'})m| > p^{1/2} > x$. In this inequality we have used the property ii) of Proposition 1 when $c_i = c_{i'}$ and the condition $B_p \subset (p^{1/2}, p^2 - p^{1/2})$ when $c_i \neq c_{i'}$. We consider two cases:

1) $n \in A_0 + D$. Suppose that there are more than g representations,

$$n = a + d = d_1 + d_1' = \cdots = d_g + d_g'$$

$$n = a + (b + cm + 2x) =$$

$$= (b_1 + c_{j_1} m + 2x) + (b_1' + c_{j_1'} m + 2x) = \cdots = (b_g + c_{j_g} m + 2x) + (b_g' + c_{j_g'} m + 2x).$$

We can suppose that d_i and d_i' are such that $b_i \leq b_i'$ and for the property i) of Proposition 1 we have that $b_i = b_j$ and $b_i' = b_j'$ for all i, j.

Then we can write

$$a + b - b_1 - b_1' - 2x + cm = (c_1 + c_1')m = \cdots = (c_g + c_g')m.$$

We observe that $a + b - b_1 - b_1' - 2x < x + (p^2 - p^{1/2}) - p^{1/2} - p^{1/2} - 2x < p^2 - 3p^{1/2} < m$ and $a + b - b_1 - b_1' - 2x > 1 + p^{1/2} - (p^2 - p^{1/2}) - (p^2 - p^{1/2}) - 2x = -2m + 3p^{1/2} - 2x - 1 > -2m$ where we have used $x < p^{1/2}$ in the last inequality. Then, $a + b - b_1 - b_1' - 2x$ is 0 or $-m$. If we divide by m we have g representations of c or $c - 1$ as sums of elements of C_g. But this is impossible because the property ii) of Proposition 2.

2) $n \notin A_0 + D$. Then all the representations are in the form $d + d'$. If we have ordered d and d' as before we have also that $b_i = b_j$ and $b_i' = b_j'$ for all i, j. If there are more than g representations we will have more than g different representations of an integer in the form $c_i + c_i'$, which is impossible because of the propery i) of Proposition 2.

Proposition 4. $\frac{|A|}{\sqrt{n_l}} = L_g + o(1)$, *where n_l is the last element of A.*

Proof. $|A| = |A_0| + |C_g||B_p| \geq |C_g|m^{1/2}(1 + o(1))$ and $|A| \subset [1, (u_g + 1)m + o(m)]$

It is clear that

$$\frac{|A|}{\sqrt{n_l}} = \frac{|C_g|m^{1/2}(1 + o(1))}{\sqrt{m(u_g + 1) + o(m)}} = L_g + o(1),$$

in view of property iii) of Proposition 2.

Proof of Proposition 1. Chowla and Erdős [2] proved that for every prime p there exists a Sidon sequence $B \subset [1, p^2 - 1]$ with p terms such that if $b + b' \equiv b'' + b''' \pmod{p^2 - 1}$ then $\{b, b'\} = \{b'', b'''\}$.

The set B_p we are looking for will be the set B except for the elements lying in the intervals $[1, p^{1/2}] \cup [p^2 - p^{1/2}, p^2 - 1]$ and those b, b' such that $|b - b'| < p^{1/2}$.

Because all the differences $b - b'$ are different we need to pick up at most $4p^{1/2}$ elements from B.

Proof of Proposition 2. We take $C_2 = \{1, 2, 5\}$, $C_3 = \{0, 1, 3\}$ $C_4 = \{0, 1, 2, 4, 7, 10\}$, $C_5 = \{0, 1, 2, 3, 5, 7\}$, $C_6 = \{0, 1, 2, 3, 4, 6, 8, 11, 13, 16\}$, $C_7 = \{0, 1, 2, 3, 4, 5, 7, 9, 11\}$, $C_8 = \{0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 17\}$,

It is easy to prove that these sets satisfy the conditions of Proposition 2.

In [1] it was proved that the set $A^g = \{k; \quad 0 \leq k \leq g - 1\} \cup \{g - 1 + 2k; \quad 1 \leq k \leq [g/2]\}$ satisfies that $r(n) \leq g$ for any integer n.

Then, for $g \geq 9$, we take $C_g = A^{g-1}$ and we have $r(n) \leq g - 1$ for any integer n. In particular, C_g satisfies the conditions i) and ii) of Proposition 2.

Also it is easy to see that iii) of Proposition 2 is satisfied.

References

[1] J.Cilleruelo, I.Ruzsa and C. Trujillo, “Upper and lower bound for finite $B_h[g]$ sequences, $g > 1$ ”, *preprint* .

[2] H.Halberstam and K.F.Roth, “Sequences”, *Springer-Verlag, New York. 1983.*

[3] X.-D. Jia, “$B_h[g]$ sequences with large upper density”, *Journal of Number Theory,* **56** **,**298-308 (1996).

[4] M.N.Kolountzakis, “The density of $B_h[g]$ sequences and the minimun of dense cosine sums”. *Journal of Number Theory* **56** , 4-11 (1996)

[5] F. Kruckeberg, “B_2 Folgen und verwandte Zahlenfolgen”, *J. Reine Angew. Math.* **106** , 53-60 (1961).

[6] A. Stöhr, “Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe”, II, *J. Reine Angew. Math.* **194** 111-140 (1955).