Automatic log spectrum restoration of atmospheric seeing

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Summary. This paper presents an automatic method for (i) digital estimation of the width of the atmospheric seeing in astronomical images of extended objects and (ii) image restoration by using the constrained Jansson-Van Cittert deconvolution algorithm. The estimation of the seeing is achieved by computing the radial profile of the averaged log spectrum of the image. The result of this estimation is then applied to compute the Point Spread Function (PSF) used in the deconvolution process. The method is applied to a photographic image of a sunspot. The quality of the restoration assesses the power and usefulness of the method.

Key words: image processing – seeing

1. Introduction

Image restoration requires basically two things in order to allow the obtention of satisfactory results:

1. A powerful restoring algorithm, robust to noise (a general and useful review on this topic may be found in Frieden (1979)). For applications to astronomical images see for instance Heasley (1984).

2. Prior knowledge about the degradation suffered by the image, and an accurate mathematical model of that degradation. This point has the foremost importance for the success of the restoration (Pratt, 1978).

The resolution in astronomical images obtained with ground-based telescopes, is limited by the influence of the earth’s atmosphere rather than by the diffraction limit of telescopes. The PSF (Point Spread Function) affecting these images consists of a “seeing disk” instead of the usual diffraction “Airy disk” (Roddier, 1981). In addition, noise constitutes another degradation always present in detection processes. Assuming this noise \( n(x, y) \) to be additive and if isoplanatism (spatially invariant blur) holds, then an astronomical image \( i(x, y) \) of an object \( o(x, y) \) may be expressed as

\[
i(x, y) = o(x, y) \otimes s(x, y) + n(x, y),
\]

where \( s(x, y) \) is the PSF that will usually be the seeing disk (of approximately gaussian shape (Roddier, 1981)), and \( \otimes \) means convolution. It is possible to know the seeing disk \( s(x, y) \) if it has been measured at a time close enough to that at which the image was recorded; or when there are point sources (stars) inside the image, which enables one to estimate the size and shape of the PSF. However this prior knowledge is not always available, as for example, in the case of images of the sun surface and other extended objects. In such situations, the problem is to extract information about the degradation from the image itself. Estimations about noise can be made by several methods, for instance by means of statistical computations on the background.

The estimation of the PSF is, however a more serious problem.

Several authors have reported their attempts to extract the information about blurs from images. A particularly successful method is based on the computation of the CEPSTRUM (Cannon, 1976; Childers et al., 1977). The bidimensional power Cepstrum \( C_f(p, q) \) of a function \( f(x, y) \) is defined by the expression (Cannon, 1976)

\[
C_f(p, q) = FT^{-1} \{ \log [ FT( f(x, y) ) ] \},
\]

where \( FT \) means Fourier transform. The methods based on the Cepstrum have proved powerful to identify simple blurs such as linear camera motion, (the PSF becomes a straight line), and out-of-focus images, (where the PSF is a circle) (Cannon, 1976; Lehar and Stevens, 1984). The main advantage is that this kind of computations transform the convolution into an addition: Firstly the Fourier transform converts the convolution into a product, and then the logarithm transform the product into an addition. In this way, and neglecting the effects of noise, Eq. (1) can be rewritten in terms of cepstra as

\[
C_i(p, q) = C_o(p, q) + C_s(p, q)
\]

where \( C_i, C_o, \) and \( C_s \) are the cepstra of \( i, o, \) and \( s \) respectively.

As mentioned above, Eq. (3) is useful for such blurs with step edges (circles, lines etc.), which give sets of, more or less, periodic zeros in the spectrum, transformed in cepstral spikes (Cannon, 1976). Unfortunately, common cases present more complicated and smoother PSF. Moreover, the seeing disk presents a bell-shaped profile (approximately Gaussian). This constitutes an additional problem because the Fourier transform of a Gaussian is another Gaussian whose logarithm is a parabola, which has no Fourier transform! Hence Gaussians to not have defined Cepstrum [except as a second derivative of a delta function (Champeney, 1973)].

In this paper we propose a modification of these techniques to be applied to the automatic restoration of atmospheric seeing and other Gaussian blurs. This is based on computing the radial profile of the logarithm of the spectrum (log spectrum or Logtro). In what follows, the basis of the method is established in Sect. 2,
then Sect. 3 will contain the implementation and tests; and, finally, results obtained by applying the logtro to the automatic restoration of a photographic image of a sunspot will be presented in Sect. 4. This is done by evaluating, firstly, the PSF from the image itself and then using the constrained Jansson-Van Cittert deconvolution algorithm.

2. Basis of the method

As mentioned earlier, the Cepstrum of a Gaussian function is equal to the second derivative of a delta function, which is not easy to handle. It is then preferable to stop the computation of the Cepstrum prior to the last inverse Fourier transform and then try to extract the information of the log spectrum, which we shall call Logtro. The Logtro $L_{ij}$ of $f(x,y)$ is defined as

$$L_{ij}(u,v) = \log \{ FT[f(x,y)]^2 \} = TF[C_{ij}(p,q)].$$

Hence $L_{ij}$ is the logarithm of the square (or square of the magnitude) of the Fourier transform of $f(x,y)$, and it is also the Fourier transform of the Cepstrum.

Working with the Logtro instead of the Cepstrum enables us to avoid problems with the Fourier transform of parabolas, while maintaining the advantage of the Cepstrum, namely, that of transforming a convolution product into an addition. In this way, and neglecting the effects of noise, Eq. (1) becomes, in terms of the Logtro, as follows:

$$L_{ij}(u,v) = L_{ij0}(u,v) + L_{ij}(u,v).$$

In order to obtain a deconvolution and, thus, determine $L_{ij0}$ and $L_{ij}$, it should be sufficient to solve Eq. (5). This does not, however, yield a restoration because of the loss of the phase in computing the magnitude of the spectrum. Nevertheless, Eq. (5) has two unknown $L_{ij}$ and $L_{ij0}$, and thus there is no unique solution. However, as will be shown immediately, it suffices to assign adequate functional forms to $L_{ij}$ and $L_{ij0}$ to obtain the solution.

When the PSF is dominated by atmospheric turbulence, the assumption of a Gaussian shape is very good (Roddie, 1981). Since, on the other hand, there is rotation symmetry, the PSF can be expressed as

$$s(r) = k \exp \left( -r^2 / 2\sigma^2 \right).$$

$k$ being a constant; $r$ the radial coordinate and $\sigma$ represents the width of the PSF which is the parameter to be determined.

In order to solve Eq. (5) it is also necessary to make some assumptions about the shape of the spectra of natural objects. Several authors (Vos and Clarke, 1975; Kawaguchi, 1982; Pentland, 1984) have found that the power spectrum of natural objects, (including 1-D spectra such as human speech, thermic noise in electronics, etc), tends to show a functional dependence with the frequency $u$ in the form $u^{-\gamma}$, where $\gamma$ is a real constant. Some authors, (Mandelbrot, 1982; Pentland, 1984), relate this exponent $\gamma$ with the texture and the fractal dimension. These authors also suggest that in most cases $\gamma$ is invariant versus changes of scale; this appears to be true only inside certain ranges of the scale, which varies depending on the case (Pentland, 1984). Invariance versus changes of scale allows the incorporation of another parameter $u_0$ in the functional shape of the spectra in the form $(u/u_0)^{-\gamma}$, where $u_0$ is a scale factor inversely proportional to that of the object (by the scaling property of the Fourier transform). Note that this functional dependence is only a tendency; and, besides, it is not easy to handle due to the divergence at the origin. This problem can be avoided by assuming a Lorentzian dependence, which is well behaved at the origin and tends to $(u/u_0)^{-\gamma}$ as the frequency $u$ increases. In this way, we assume, as will be shown below it is a good approximation for general natural spectra, that the radial profile of spectra $\tilde{O}$ of natural objects tends to the radial dependence:

$$\tilde{O}(\theta) = k'[(1 + (\theta/\theta_0)^2)^{\gamma/2}].$$

$\theta$ being the radial frequency, $\theta_0$ the scale, $k'$ is a constant and $a$ is the exponent which when $(\theta/\theta_0)^2 \gg 1$ (high frequencies) makes $\tilde{O}$ tend to $(\theta/\theta_0)^{-\gamma}$ with $\gamma = 2a$.

As pointed out earlier, Eq. (7) is only a tendency of natural spectra. In order to avoid fluctuations and local discrepancies between real spectra and Eq. (7), and also azimuthal asymmetries, it is necessary to remove, as much as possible, those discrepancies caused by some local features of the object. With this aim it is very useful to apply the method proposed by Welch (1967), in which the image is broken up into many smaller images, then each log spectrum is computed to obtain an average log spectrum. In addition, since the PSF has radial symmetry, the radial profile is computed by averaging in azimuth the averaged log spectrum. In this way, the final radial profile (logtro) has lost most of the local features and fluctuations, so that a good fit to Eq. (7) can be achieved. Note that these partitions of the image and the subsequent radial profile computation do not spoil but enhance the log spectrum of the Gaussian PSF. To prove this point, let $P_{ij}$ be the $(i,j)$ sub-image. If the extent of $P_{ij}$ is much greater than that of the PSF $s(x,y)$, then the logtro of $P_{ij}$ is

$$L_{P_{ij}} = L_{O_{ij}} + L_{s},$$

where $L_{O_{ij}}$ is the logtro of the corresponding sub-object. Taking averages in Eq. (8) we obtain the following expression:

$$\langle L_{P_{ij}} \rangle = \left(1/N \right) \sum_{ij} L_{O_{ij}} + L_{s},$$

where $N$ is the total number of sub-images. Equation (9) shows that the characteristics of the blur (PSF) remain, and in some way are enhanced, whereas the local features of the Logtro of the object tend to disappear by average. The final azimuthal average strengthens this effect.

In order to confirm our assumptions about the functional form of the spectra of natural objects, we have computed the Logtro of several non blurred images. Figure 1 shows four different images whose Logtros have been computed: (a) corresponds to a partial view of the Roque de los Muchachos Observatory on La Palma; (b) is a natural landscape; (c) is a fluorescent image of a photocoagulated retina and (d) is an artificial, high-contrast emblem. Their respective Logtros and their fits to Eq. (7) are shown in Figs. 2a–2d. In all cases the fit is very good, which supports our previous assumptions. Small discrepancies at high frequencies are mainly due to “computing noise” in which the very small values of the logtro are embedded. This point will be discussed in more detail at a later stage of the paper.

Equation (5) can be rewritten in terms of the radial profile of the averaged Logtro. This gives, by using Eqs. (6) and (7) in (9) and after taking logarithms, the following expression:

$$\langle L_{P_{ij}}(\theta) \rangle = C - a \log \left[ 1 + (\theta/\theta_0)^2 \right] - \theta^2 / 2\sigma^2,$$

where $C = \log (k + k')$ is a constant, $a$ is the exponent related with the texture and fractal dimension of the object (Pentland, 1984), $\theta$ is the radial frequency, $\theta_0$ represents the scale factor and $\sigma$ is the width of the spectrum of the PSF ($\sigma = 1 / 2\pi r_{\sigma}$). Once the radial
profile of the Logtro has been computed, in order to evaluate the parameters \( C, a, q_0, \) and \( \sigma_r \), it suffices to fit the data to Eq. (10). The constant \( C \) could be eliminated by normalizing the zero frequency component of the spectrum to 1 before computing each logarithm. However, it is useful to keep \( C \) as a free parameter to obtain better fits.

This method enables the determination, not only of the width of the seeing disk (or gaussian blur), but also of some interesting additional information about averaged texture and fractal dimension of the object. Another parameter, namely the scale factor of the object (inversely proportional to \( q_0 \)), could be interesting in comparing astronomical objects of the same kind but of different size or distance to the earth.

3. Implementation and tests

As briefly mentioned above, the computation of the radial profile of the Logtro is achieved with the following steps:

Firstly, the original image of \( N_x \times N_y \) pixels is broken up into many square sub-images \( P_{ij} \) of \( n \times n \) pixels each one (e.g. 32 \times 32; 64 \times 64 \) or \( 128 \times 128 \)). The choice of \( 64 \times 64 \) pixels has proved to be the best solution considering computer time and quality of the results, except for very spread blurs. The number of sub-images is variable, but usually \( 10 \times 10 \). The sampling is made by taking pieces whose centres are equispaced in both \( x \) and \( y \)-axes. It is very convenient that adjacent sub-images do overlap in order to avoid losses of information at the edges of the sub-images, due to the posterior windowing, and even to obtain some useful redundancy. The windowing reduces edge effects (Cannon, 1976; Pratt 1978). The averaged Logtro is then computed by addition of all the logarithms of the magnitude of the Fourier transform of each sub-image. Finally, the radial profile is computed by azimuthal average of the pixels of the averaged Logtro. The fit of the resulting data to Eq. (10) is made by a \( \chi^2 \) minimization algorithm (Gill and Murray, 1978). These fits are made without considering the first point (zero frequency) of the Logtro for two main reasons: Firstly, we work with normalized spectra, so \( L_r(q = 0) = 0 \) does not contain information. Secondly, images could be added to a constant background which, by Fourier transform, becomes a delta function added to zero frequency component of the spectrum. We have obtained better results without considering this central
component and using the constant $C$ of Eq. (10) as a free parameter.

In order to test the method, several images have been degraded with Gaussian blurs of different $\sigma_i$. Figure 3 shows two blurred versions (with $\sigma_i = 1$ and $\sigma_i = 4$ respectively) of the anagram of the IAC; these can be compared with the original image of Fig. 1d. It should be noted that the power of the method will strongly depend on the accuracy obtained in the fits. On the other hand, natural objects tend to show very high dynamic ranges in their spectra. In other words, natural spectra usually show differences of several orders of magnitude, between their central component (zero frequency) and the highest frequencies. Also, any numerical computing process has round-off errors, or "computing noise", associated to it. In the case of discrete Fourier transforms some aliasing error may be also present, added to that computing noise; specially for high frequencies. These effects make the computing signal-to-noise ratio decrease very rapidly as the spatial frequency increases. Therefore, computing errors could be greater than the value of the spectrum for high frequencies. The problem is then to compute a cut-off frequency in such a way that any frequency beyond this limit will not be considered in further computations. Note also that we are concerned with the log spectrum, that has been computed by using decimal logarithms. This means that a difference, say of 7, in the Logtro, implies 7 orders of magnitude in the spectrum! As the results of the tests will show, the computing errors together with the high dynamic ranges involved, impose the main limitation to the method. The most visible influence upon the data is that the expected decay in the Logtro as the frequency increases is broken by computing noise (see Fig. 4a–d). Our experience with tests indicates that it is always advisable to neglect values, smaller than $-7$, in the normalized Logtro. In other words, the frequency reaching that value should be considered as the cut-off. Moreover, in all fits we have found that the best results were achieved when values of the Logtro less than $-5$ were not considered. We interpret this result as due to the fact that the computing signal-to-noise ratio decreases as the order of magnitude of the points of the spectra decreases. This implies that only the values in the range between 0 and $10^{-5}$ are acceptable. As
Fig. 3a and b. Test images (anagram of the IAC) blurred with two different Gaussian PSF. 
a corresponds to $\sigma_r = 1$ pixels and b to $\sigma_r = 4$ pixels.

Fig. 4a-d. Four different computations (squares) and fits (continuous line) of the log spectrum, corresponding to four progressive blurs of the test image. a $\sigma_r = 1$; b $\sigma_r = \sqrt{2}$; c $\sigma_r = 2$ and d $\sigma_r = 4$ pixels. Note that as $\sigma_r$ increases the fitted lines (continuous) progressively diverge for high frequencies and the number of useful points for the fit decreases.
Figure 5 shows, for the same test image, the computed $\sigma$ versus the real $\sigma$, in the range 1 to 4 pixels. The curve has a straight portion between 1 to nearly 2 pixels in which the computed $\sigma$ coincides, within a good approximation, with its real value. Beyond this point, however, the line becomes curved showing a "saturation" tendency. The computed value differs from the real one more and more as $\sigma$ increases. As it is shown in Table 1, the decrease in the computed value with respect to the real one is made at the cost of a spurious increase in the exponent $a$. Surely this is because too few points remain useful for the fit, and a distinction between Lorentzian and Gaussian becomes harder. This effect may be corrected by appropriate calibrations, but in any case the uncertainty will increase as $\sigma$ increases. A much better solution is to increase the number of useful points of the Logtro by taking larger sub-images, for instance of $128 \times 128$ pixels instead of $64 \times 64$. This is advisable when one suspects that $\sigma$ is greater than 2 pixels.

4. Restoration

To prove the efficiency of the method we have chosen a real example which, on the other hand, could constitute one of the most interesting astrophysical applications. The image to restore corresponds to a sunspot. Sunspots are frequently observed and photographed by the Solar Physics group of the IAC. The image (see Fig. 6) has been recorded in a photographic film on a 40 cm solar telescope at the Izana Observatory (Tenerife) and digitized in $300 \times 300$ pixels by a Perkin-Elmer microdensitometer. The radial profile of the averaged Logtro was computed in the same way as the test images; namely, it was broken up into $10 \times 10$ sub-images, each of them with $64 \times 64$ pixels. The fit, shown in Fig. 7, was $\sigma = 1.56$ pixels and $a = 1.2$. This value is inside the straight portion of Fig. 5, so it was used directly to generate the Gaussian PSF to start the deconvolution.

The restoration algorithm used was the constrained Jansson-Van Cittert procedure (Jansson, 1984; Heasley, 1984). We had great interest in testing this algorithm because so far we have only found images restored by this algorithm in Heasley (1984), notwithstanding Fieden (1979) presents it as a very promising one. However, this method has been more widely applied to spectroscopy (Jansson, 1984).

Basically, the algorithm consists of two steps at each iteration. First the $K$-th result of the iteration $i^{(k)}(x, y)$ is convolved with the PSF $s(x, y)$ to compute a theoretical image $i^{(k)}(x, y)$:

$$i^{(k)}(x, y) = i^{(k)}(x, y) \otimes s(x, y),$$

(11a)

The values of the spectra decrease, with an increase of the frequency, the signal is progressively plunged into noise (the natural decrease is progressively broken). For values less than $10^{-7}$ the resulting signal is saturated by noise.

These effects can be observed in Figs. 4a–4d, where the Logtro (squares) and the fitted curve (continuous line) are shown for four blurred versions, with $\sigma_r = 1, \sqrt{2}, 2,$ and 4 pixels respectively, of the image of Fig. 1. The curves show both a very good fit for low and intermediate frequencies (Logtro greater than $-5$) and the negative influence of computing noise for high frequencies. This negative effect increases as the width of the PSF increases. The effect of the blur is to drastically decrease the values of the Logtro for high frequencies in such a way that, as $\sigma$ increases, the value $-5$ is more promptly reached. Then the problem is that for high $\sigma$, the number of useful points of the Logtro diminishes and so does the accuracy of the fits. This effect can be observed in Figs. 4a–4d which also show how the original and fitted curves tend to diverge more and more of each other for high frequencies as $\sigma$ increases.

<table>
<thead>
<tr>
<th>Actual ($\sigma$)</th>
<th>Computed ($\sigma_r$)</th>
<th>Exponent (a)</th>
<th>Scale factor ($\delta_0$)</th>
<th>Useful points</th>
<th>Accuracy ($\chi^2$)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.13</td>
<td>0.82</td>
<td>0.26</td>
<td>16</td>
<td>$1.07 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>1.398</td>
<td>0.824</td>
<td>0.262</td>
<td>14</td>
<td>$1.08 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>1.817</td>
<td>0.826</td>
<td>0.262</td>
<td>12</td>
<td>$1.75 \times 10^{-3}$</td>
</tr>
<tr>
<td>$2\sqrt{2}$</td>
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<td>0.89</td>
<td>0.292</td>
<td>9</td>
<td>$1.41 \times 10^{-3}$</td>
</tr>
<tr>
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<tr>
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<td>1.03</td>
<td>0.357</td>
<td>7</td>
<td>$2.2 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
then the $K + 1$ result is obtained by means of the operation:

$$o^{k+1}(x, y) = o^{k}(x, y) + r(x, y) \left[ i(x, y) - o^{k}(x, y) \right]$$

where $r(x, y)$ is a constraint function of the form

$$r(x, y) = C \left[ 1 - 2 |B(x, y) - A(x, y)| o^{k}(x, y) - \left| A(x, y) + B(x, y) \right|/2 \right],$$

where $C$ is a constant. It is easy to show that $A(x, y)$ and $B(x, y)$ are respectively lower and upper limits for $o^{k}(x, y)$. In other words, the increment in $o^{k}(x, y)$ in (11b) is weighted. The maximum increment is achieved when $o^{k}(x, y) = \left| A(x, y) + B(x, y) \right|/2$, and the increment is zero or negative when $o^{k}(x, y)$ is either greater than $B(x, y)$ or lower than $A(x, y)$. Our initial guess is the original image $i(x, y)$, and $A(x, y)$ and $B(x, y)$ are computed by subtracting or adding a constraint constant $d$ to $i(x, y)$. In this way $A(x, y) = i(x, y) - d$ and $B(x, y) = i(x, y) + d$. The positivity condition is then imposed to $A(x, y)$. Thus the solution is restricted to be positive. The constraint constant plays a very important role, since the restored image is forced to be within the interval $[i(x, y) - d, i(x, y) + d]$. The choice of $d$ depends on the signal-to-noise ratio of the image. As the noise increases $d$ should be decreased in order to avoid spurious enhancement of noise. Since the signal-to-noise ratio in this image is good, we have chosen a constraint constant equal to 0.2 (20%, since the value of the pixels of the image is in the interval $[0, 1]$).

The convergence criterion is based on the square of the differences between $i^{k}(x, y)$ and $i(x, y)$. Convergence was reached after six iterations. The resulting restored image is shown in Fig. 8. A substantial increase in contrast and a visible increase in resolution is seen, being easily appreciable in the filaments inside the penumbra.

It should be remarked that this restoration was made without any prior knowledge about the PSF. On the other hand, this case is critical for the Gaussian approximation since the image has been recorded on a 40 cm telescope, while the characteristic size of the atmospheric turbulence may be of the order of 20 cm. This means that although the contribution of the atmosphere to the PSF is dominant, the contribution of the telescope is not negligible, as could be on another telescope with larger diameter. These results show that even in this critical case the application of this method is satisfactory.

5. Conclusions

In this paper we have presented a method to estimate the PSF causing the blurring of an image. This method is applicable in cases in which the only knowledge about the blur is that it may be approximated by a Gaussian.
Different test images have confirmed that natural objects (and even many artificial ones) show Lorentzian decays in the radial profile of their averaged log spectra. The fits provide not only the width of the Gaussian blur, but also additional information about texture or fractal dimension and the scale factor of the spectra. This could be useful in astronomical applications.

The method shows “saturation effects”, which we attribute to the computing “noise”, that seems to limit its usefulness and accuracy for large blur images. However this problem could be avoided by using a larger number of pixels in the computations. The case presented above, in which the image has been broken up into 64 x 64 pixel sub-images, has shown to be acceptably accurate for blur sizes \( \sigma_r \) up to 2 pixels. One should expect that when the size of the sub-images is 128 x 128 pixels, the useful range will reach \( \sigma_r = 4 \) pixels.

We have applied the method to the estimation of the PSF in the image of a sunspot. The obtained \( \sigma_r = 1.56 \) has been used to generate a Gaussian PSF, used to deconvolve the image. The algorithm used in the restoration was the constrained Jansson-Van Cittert's. The results show, not only the usefulness of our method, but also that although the J-VC algorithm has not been used so far for restoration of astronomical images, its power and strength versus noise is great and advises its use. On the other hand, this algorithm is easy to implement.

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