14th European Workshop on Computational Geometry

CG98

Ferran Hurtado, Joan Trias
(Editors)

Barcelona, 25-26-27 March 1998
Departament de Matemàtica Aplicada II
UNIVERSITAT POLITÈCNICA DE CATALUNYA

Applying Delta/Star Reductions for Checking the Spatial Realizability of Line Drawings

Lluís Ros and Federico Thomas

Institut de Robòtica i Informàtica Industrial
Gran Capità 2-4, 2ª planta, 08034 Barcelona, SPAIN

1 Introduction

A *line drawing* is a drawing containing only *line segments* and *junctions*, points where two or more of these segments meet. A line drawing is said to be *realizable* or *consistent* if it is the orthografic or perspective projection of some three-dimensional scene of polyhedral objects, and *incorrect* or *inconsistent* otherwise. Such a scene is known as a *realization* of the line drawing.

We present a novel approach to solve the problem of deciding whether a line drawing corresponds to the projection of a polyhedron. This work proves that the realizability of a line drawing, without occluding segments, can be verified by checking the concurrence in a point of the lines supporting groups of three segments taken from the drawing or added during the test itself. When compared to former approaches, it exhibits two main advantages. First, the usual problem of superstrictness arising in algebraic approaches can be overcome by simply allowing some error tolerance in all concurrence tests. Additionally, when one of these tests fails, involved vertices can be easily identified and the source of inconsistency, located. Second, consistent edge-labellings are synthesized as a result of the reconstruction, instead of being a required input.

We assume that every junction in a line drawing is common to at least two line segments and hence the segments partition the plane of the drawing into several polygonal *regions*.

The *incidence structure* of a line drawing L is a planar and connected graph $G(L) = (J, S)$ where J is the set of junctions of L and S is the set of line segments. There is a one-to-one correspondence between the elements in a consistent line drawing and the elements of the polyhedral scenes it represents: junctions correspond to *vertices*, line segments to *edges*, and polygonal regions to *faces*.

Although it induces an abuse of language, we will refer to the *line of support* of a given edge l or the *plane of support* of a given face φ by using the terms *line* l and *plane* φ, respectively.

The line drawings are supposed to contain no *occlusive* line segments. That is, every segment represents the intersection of two *adjacent* faces in 3-D. This even holds for the segments of the outer contour of the drawing which, hence, represent coplanar edges in 3-D.

Under these assumptions, the realization of a correct drawing is a spherical polyhedron and, therefore, the class of drawings considered is restricted to those whose incidence structure is planar, edge 3-connected and vertex 2-connected .

2 Edge-concurrence subsumes all projective conditions

There are several well-known projective conditions that a line drawing should accomplish in order to correctly represent the projection of a polyhedral scene.

The *edge alignment condition* (fig. 1a, left) states that if two different edges of a polyhedron share the same two faces, then the two edges must be aligned.

The *edge concurrence condition* (fig. 1b, left) says that given three faces of a polyhedron, if any two of them share a common edge, then these edges must all be concurrent to the same point. This is true even in the case that the edges are parallel, since we can view them as embedded in the projective 3-space, allowing the existence of "points at infinity" [18].

The *n-calotte condition* which imposes constraints on the concurrence of all edges incident to a given n-gonal face of a polyhedron. We describe here the case $n = 4$. Given a quadrilateral face of a polytope such as face ε in fig. 1c, left, consider the point M of intersection of line $\delta - \alpha$ with line $\delta - \gamma$, and point N, where $\alpha - \beta$ intersects with $\gamma - \beta$. Then, M and N must be aligned with O, where $\alpha - \varepsilon$ intersects with $\varepsilon - \gamma$, because M, N and O must lie on the intersection of planes α and γ.

The above three conditions involve two types of geometric elements of the line drawing: regions and line segments. Then, it seems natural to express them in terms of the topology of what we call the *constraint graph*, a multigraph $G_r(L) = (V, E)$ which contains a vertex in V for each region in the line drawing, and an edge in E for each line segment separating two adjacent regions.

14th European Workshop on Computational Geometry
Ferran Hurtado and Joan Trias (Editors)
Barcelona 25-27 March 1998 - Pages 127-129

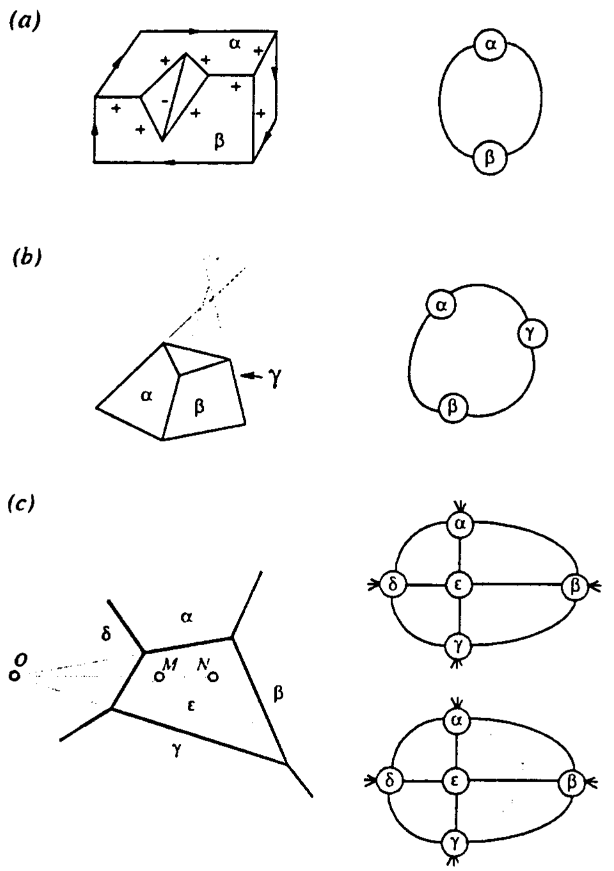

Fig. 1. *Edge-alignment (a), edge-concurrence (b) and n-calotte (c) conditions.*

Now, to test the edge-alignment condition, we just gather all pairs of vertices $v \in V$ with parallel edges between them and judge whether the coefficients of their corresponding segments are equal or not (fig. 1a, right).

In order to test the edge-concurrence condition, we detect all cycles of length three in $G_r(L)$ and judge whether the corresponding segments meet at a single point.

The 4-calotte condition in fig. 1c clearly reduces to the three edge-concurrence tests for the sets of lines ($\delta-\alpha$, $\delta-\gamma$, *MN*), ($\alpha-\beta$, $\gamma-\beta$, *MN*) and ($\alpha-\varepsilon$, $\gamma-\varepsilon$, *MN*). This cannnot be directly expressed in terms of the topology of $G_r(L)$ (fig. 1c, top-right) because the drawing does not provide the projection of line *MN*. However, by simply adding a *fictitious edge* to $G_r(L)$ corresponding to the unknown projection of line $\alpha-\gamma$ (fig. 1c, bottom-left), all relevant 3-cycles emerge. Edge-concurrence of the 3-cycles $\alpha, \delta, \gamma, \alpha$, and $\alpha, \beta, \gamma, \alpha$ constrains fictitious line $\alpha-\gamma$ to meet *M* and *N*, respectively. Once the location of $\alpha-\gamma$ is known, the 4-calotte condition is finally verified when checking the 3-cycle $\alpha, \varepsilon, \gamma, \alpha$.

Note that, in general, once a fictitious edge is fixed, this information can be *propagated* and used to fix other fictitious edges.

We say that a geometric constraints graph is *globally consistent* if we can find values for its fictitious edges in such a way that all edge-concurrence conditions corresponding to all 3-cycles in the graph are satisfied.

Let $G_r(L)$ be the constraint graph of a line drawing. Then, arbitrarily choose one face φ of the polyhedron represented by L and construct a new constraint graph, $G_r^e(L)$, by extending $G_r(L)$ with all fictitious edges of the form (φ, x) that represent the intersection of φ with any other face x of the polyhedron. Although explained in a different language, W. Whiteley proved that L is realizable if and only if $G_r^e(L)$ is globally consistent.

3 Delta/star reductions

It is always possible to reduce any spherical polyhedron to a simplex by applying a finite sequence of the two following operations:

- The simplicial completion (fig. 2a), which eliminates a triangular face by extending its three neighboring faces as far as their common point of intersection.
- The simplicial elimination (fig. 2b), which simply removes a trihedral vertex by cutting it through the plane defined by its three neigboring vertices, obtaining a new triangular face.

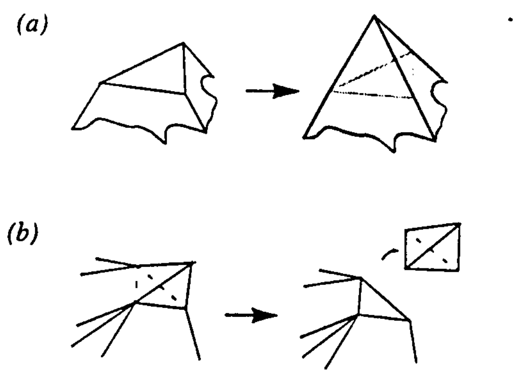

Fig. 2. *Simplicial completion (a) and elimination (b).*

If a line drawing is consistent, then it has to be possible to transform it to the projection of a simplex by means of some reduction steps analogous to the simplicial operations in the 3D case. This leads us to the definition of the delta/star reductions (Δ / Y reductions, for short).

The projection of the simplicial completion operation onto a plane induces the four different types of *delta-to-star*

($\Delta \rightarrow Y$, for short) reductions shown in fig. 3a. Each of these operations adds a new junction which corresponds to the new trihedral vertex appearing in the polyhedron. Those segments that meet at a junction of degree 3 in the original triangle will be called *simple segments*.

The projection of the simplicial elimination operation leads to four types of *star-to-delta* ($Y \rightarrow \Delta$, for short) reductions (fig. 3b). A new triangular region appears within

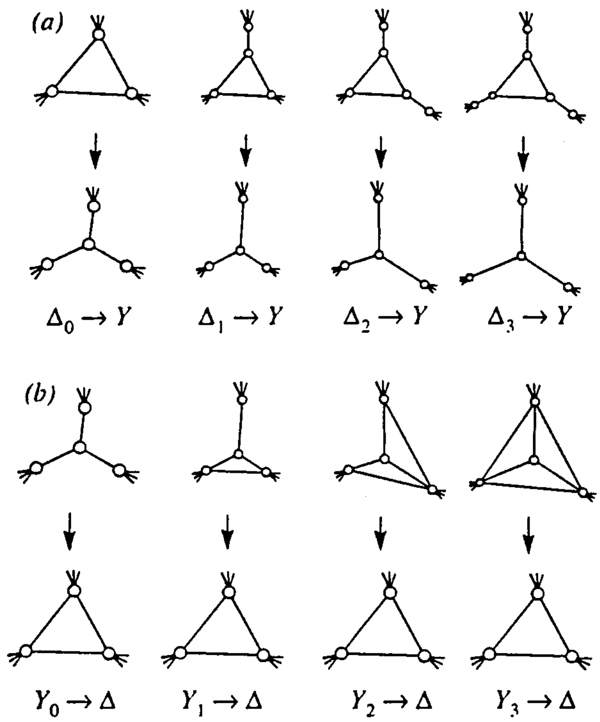

Fig. 3. $\Delta \rightarrow Y$ *(a) and* $Y \rightarrow \Delta$ *(b) opera-*

the three original junctions of the star.

It is always possible to reduce a correct line drawing of a polyhedron to the projection of a simplex by applying these operations.

Δ / Y reductions may add new fictitious edges and vertices to the constraint graph, thus creating new 3-cycles and inducing new edge-concurrence tests. We show that the overall consistency of the drawing can be checked by simply verifying all concurrence tests implicit in the geometric constraint graph, once this has been extended with all fictitious edges and faces corresponding to Δ / Y reductions. Let us see an example.

The drawing in fig. 5a can be easily reduced to the projection of a simplex by means of four $\Delta \rightarrow Y$ reductions, one for each of the four triangles in it. Let us suppose that the first reduction is applied over the central triangle, which has no simple edges. This is a $\Delta_0 \rightarrow Y$ reduction and, for the moment, it is impossible to tell where the new central

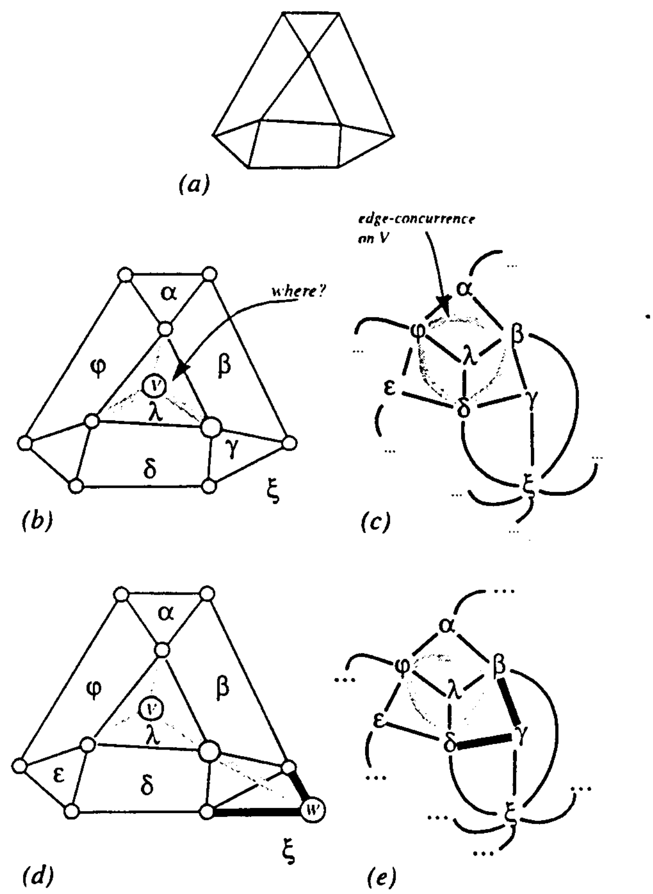

Fig. 4. *Checking the consistency of a truncated pyramid.*

junction V should lie to keep the consistence of the drawing (fig. 5b). In the corresponding constraint graph this is translated into the addition of a new 3-cycle with three fictitious edges (the thick grey ones in fig. 5c).

We go on by applying a $\Delta_2 \rightarrow Y$ reduction to the lower-right triangle (fig. 5d). This time, the position of the new junction is completely determined. Actually, we have three edges that must be concurrent, namely (β, ξ), (ξ, δ) and (δ, β). Since from the line drawing we know the location of (β, ξ) and (ξ, δ), we are able to fix edge (δ, β). This corresponds to the verification of the edge-concurrence condition on the 3-cycle $\delta, \xi, \beta, \delta$. This allows us to "propagate" this new information to fix edge (δ, β) in the first 3-cycle $\varphi, \delta, \beta, \varphi$.

We now proceed analogously with another $\Delta \rightarrow Y$ operation over the lower-left triangle. Again, the edge-concurrence condition on the 3-cycle $\xi, \varphi, \delta, \xi$ fixes the location of edge (φ, δ), which permits to fix the position of vertex V. Finally, a $\Delta \rightarrow Y$ operation on the upper triangle leads to the unique consistency test of the whole process: if the drawing is consistent, edge (φ, β) must be concurrent with the two previously determined edges (φ, δ) and (δ, β)