Strictly join irreducible varieties of residuated lattices

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Abstract

We study (strictly) join irreducible varieties in the lattice of subvarieties of residuated lattices. We explore the connections with well-connected algebras and suitable generalizations, focusing in particular on representable varieties. Moreover, we find weakened notions of Halldén completeness that characterize join irreducibility. We characterize strictly join irreducible varieties of basic hoops and use the generalized rotation construction to find strictly join irreducible varieties in subvarieties of MTL-algebras. We also obtain some general results about linear varieties of residuated lattices, with a particular focus on representable varieties, and a characterization for linear varieties of basic hoops.

Keywords: Residuated lattices, substructural logics, lattice of subvarieties, Halldén completeness, basic hoops, linear varieties

1 Introduction

Substructural logics constitute a large class of logical systems algebraizable in the sense of Blok–Pigozzi, where the semantical characterization of provability of the Lindenbaum–Tarski algebraization extends to a characterization of logical deducibility via the algebraic equational consequence (see [26] for a detailed investigation). Substructural logics encompass classical logic, intuitionistic logic, fuzzy logics, relevance logics and many other systems, all seen as logical extensions of the Full Lambek calculus $\mathcal{FL}$. As a consequence of algebraizability, all extensions of $\mathcal{FL}$ are also algebraizable, and the lattice of axiomatic extensions is dually isomorphic to the subvariety lattice of the algebraic semantics, given by the variety of FL-algebras. In this work, we are interested in the positive fragment of $\mathcal{FL}$ (the system obtained by removing the constant 0, and consequently negation, from the language), $\mathcal{FL}^+$, whose corresponding algebraic semantics is given by the variety of residuated lattices RL.

Our investigation will be carried on in the algebraic framework and goes in the direction of gaining a better understanding of the lattice of subvarieties of residuated lattices (thus, equivalently, the lattice of axiomatic extensions of the corresponding logics). In particular, we study properties, and in some relevant cases we find characterizations, of those varieties that in the lattice of subvarieties are join irreducible or strictly join irreducible. Notice that this study is extremely relevant, since the lattice of subvarieties of any variety V, $\Lambda(\mathsf{V})$, is a dually algebraic lattice [34], and thus every variety in $\Lambda(\mathsf{V})$ is the join of a set of strictly join irreducible varieties in $\Lambda(\mathsf{V})$.

Kihara and Ono showed that in the presence of integrality and commutativity, join irreducibility of a variety is characterized by both a logical property, Halldén completeness, and an algebraic property of the generating algebras, well-connectedness ([36], anticipated in [26]).

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We show how both these notions can be generalized for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties and ℓ-groups. Moreover, we answer two questions left open in [36] in our more general setting. In particular, we show that in our more general framework being join irreducible is equivalent to being generated by a subdirectly irreducible algebra and that in general not all subdirectly irreducible algebras generate strictly join irreducible varieties. A key role is played by results implicit in [24] concerning the axiomatization of the join of varieties of residuated lattices.

We then focus on representable varieties, where well-connectedness is shown to correspond to being totally ordered. In the further presence of divisibility, we use the ordinal sum construction to find sufficient conditions for strict join irreducibility. Then we focus on the variety of basic hoops, where a characterization of strictly join irreducible varieties is given. The representation shows to be analogous to the one found in [11] for BL-algebras (of which basic hoops are 0-free subreducts) and is constructed based on the representation of totally ordered hoops with finite index given in [5]. Given the results in [7], using the generalized rotation construction in [16], we are able to lift the results found about the lattice of subvarieties of basic hoops to particular intervals of the lattice of subvarieties of MTL-algebras (representable, commutative, integral FL-algebras). In the end, we focus on particular join irreducible varieties, namely linear varieties, finding some general characterizations in the presence of representability and a characterization for linear varieties of basic hoops, which again we lift to particular subvarieties of MTL.

2 Preliminaries

A **residuated lattice** is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, /, \backslash, 1\rangle$ where

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot, 1\rangle$ is a monoid;
3. $/$ and $\backslash$ are the right and left divisions w.r.t. $\cdot$, i.e. $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$.

Residuated lattices form a variety RL and an axiomatization, together with the many equations holding in these very rich structures, can be found in [15].

A residuated lattice $\mathbf{A}$ is **integral** if it satisfies the equation $x \leq 1$; it is **commutative** if $\cdot$ is commutative; it is **divisible** if the ordering of $\mathbf{A}$ is the *inverse divisibility ordering* i.e. for $a, b \in A$

$$a \leq b \qquad \text{if and only if} \qquad \text{there are } c, d \in A \text{ with } a = cb = bd.$$

The first two properties are clearly equational, while the third is equivalent to the equations [35]

$$(x/y)y \approx x \wedge y \approx y(y\backslash x).$$

Finally, a residuated lattice is **cancellative** if the underlying monoid is cancellative in the usual sense; it turns out that this property is equational as well [13]. A residuated lattice is cancellative if and only if it satisfies

$$xy/y \approx x \qquad x\backslash xy \approx y.$$

So the classes of residuated lattices that satisfy any combination of integrality, commutativity, divisibility or cancellativity are subvarieties of RL. We shall call the variety of integral residuated lattices IRL, commutative residuated lattices CRL and their intersection CIRL.

Residuated lattices with an extra constant 0 in the language are called FL-algebras, since they are the equivalent algebraic semantics of the full Lambek calculus $\mathcal{FL}$ (for a precise definition of this calculus, or more details on algebraizability, see [26]). Residuated lattices are then the equivalent algebraic semantics of the positive (i.e. 0-free) fragment of $\mathcal{FL}$, $\mathcal{FL}^+$.

In particular, for every variety of residuated lattices V over a language $\mathscr{L}$, its corresponding logic is $\mathcal{L}_{\mathsf{V}} = \{\varphi \in Fm_{\mathscr{L}} : \mathsf{V} \models \varphi \geq 1\}$, where $Fm_{\mathscr{L}}$ is the set of formulas over $\mathscr{L}$. Conversely, given any extension $\mathcal{L}$ of $\mathcal{FL}^+$ axiomatized by the set of formulas Φ, its equivalent algebraic semantics is the subvariety $\mathsf{V}_{\mathcal{L}}$ of FL axiomatized by the set of equations $\{\varphi \geq 1 : \varphi \in \Phi\}$. Moreover, the following is shown in [28].

THEOREM 2.1
The maps $\mathbf{L} : \mathsf{V} \mapsto \mathcal{L}_{\mathsf{V}}$ and $\mathbf{V} : \mathcal{L} \mapsto \mathsf{V}_{\mathcal{L}}$ are mutually inverse dual lattice isomorphisms between the lattice of subvarieties of RL and the lattice of logical extensions of $\mathcal{FL}^+$.

It is useful to observe that a conjunction of a finite number of equations is equivalent in RL to a single inequality of the kind $p \geq 1$ or equivalently to an equation of the kind $p \wedge 1 \approx 1$ [25, Lemma 3.1].

With respect to their structure theory, residuated lattices are congruence permutable and congruence point-regular at 1 and also congruence distributive (since they have lattice terms). It follows that FL-algebras and residuated lattices are *ideal determined* [30] and have a good theory of ideals in the sense of [8]. The role of the ideals is played by the **congruence filters** that are particular lattice filters; in order to describe congruence filters, let us first define a **filter** F of $\mathbf{A}$ to be a subset of A that is a lattice filter containing 1 and is closed under multiplication. Both filters and congruence filters form algebraic lattices; in particular, if $A^+ = \{a : a \geq 1\}$, then for any $\theta \in \mathrm{Con}(\mathbf{A})$,

$$A^+/\theta = \bigcup\{a/\theta : a \geq 1\}$$

is a congruence filter. Moreover, the two mappings

$$\theta \longmapsto A^+/\theta \qquad\qquad F \longmapsto \theta_F = \{(a, b) : a/b, b/a \in F\}$$

are mutually inverse order preserving maps from $\mathrm{Con}(\mathbf{A})$ to the congruence filter lattice of $\mathbf{A}$ that we shall write as $\mathrm{Fil}(\mathbf{A})$. Thus, $\mathrm{Con}(\mathbf{A})$ and $\mathrm{Fil}(\mathbf{A})$ are therefore isomorphic as lattices. Notice that A^+/θ defines the same set as $\uparrow[1]_\theta$, used e.g. in [26].

A residuated lattice is **normal** if every filter is a congruence filter and a variety of residuated lattices is **normal** if each of its members is normal. To determine which varieties are normal, a description of the congruence filter generated by a subset is in order. Following [35], we define $l_a(x) = a\backslash xa \wedge 1$ and $r_a(x) = ax/a \wedge 1$ and we call them the **left and right conjugates** of x with respect to a.

LEMMA 2.2
[35] Let $\mathbf{A}$ be a residuated lattice; then a filter F of $\mathbf{A}$ is a congruence filter if and only if for all $a \in F$ and $b \in A$, $l_b(a), r_b(a) \in F$.

This lemma allows a handy description of the congruence filter generated by a subset X of a residuated lattice $\mathbf{A}$; an **iterated conjugate** in $\mathbf{A}$ is a unary term $\gamma_{a_1}(\gamma_{a_2}(\ldots\gamma_{a_n}(x)))$ where $a_1, \ldots, a_n \in A$ and $\gamma_{a_i} \in \{l_{a_i}, r_{a_i}\}$ for $i = 1 \ldots n, n \in \mathbb{N}$. We will denote with $\Gamma^n(\mathbf{A})$ the set of iterated conjugates in $\mathbf{A}$ of *length* n (i.e. a composition of n left and right conjugates).

COROLLARY 2.3
Let $\mathbf{A}$ be a residuated lattice and let $X \subseteq A$; then the congruence filter generated by X in $\mathbf{A}$ is the set

$$\mathrm{Fil}_{\mathbf{A}}(X) = \{b \in A : \gamma_1(a_1)\ldots\gamma_n(a_n) \leq b,\ n \in \mathbb{N},\ a_1,\ldots,a_n \in X \cup \{1\},$$
$$\gamma_i \in \Gamma^k(\mathbf{A}) \text{ for some } k \in \mathbb{N}, i = 1\ldots n\}.$$

Section 4 of [27] contains an implicit characterization of normal residuated semilattices. Let us make it explicit for residuated lattices.

LEMMA 2.4
For a residuated lattice $\mathbf{A}$, the following are equivalent:

1. $\mathbf{A}$ is normal;
2. every principal filter of $\mathbf{A}$ is a congruence filter;
3. for all $a \in A$, given any $b \in A$ there exist $n, m \in \mathbb{N}$ (possibly depending on b) such that $(a \wedge 1)^n b \leq ba$ and $b(a \wedge 1)^m \leq ab$.

PROOF. (1) clearly implies (2). Assume then (2), and let $a \in A$; then the principal filter F generated by a is a congruence filter. If $a \geq 1$ (3) can be easily checked. From Lemma 2.2, it follows that for any $b \in A$ $l_b(a), r_b(a) \in F$. This implies that there are $n, m \in \mathbb{N}$ such that $a^m \leq l_b(a)$ and $a^n \leq r_b(a)$, from which (3) follows by residuation and order preservation. Finally, assume (3) and let F be a filter of $\mathbf{A}$; if $a \in F$ and $b \in A$ then there is an $n \in \mathbb{N}$ with $(a \wedge 1)^n b \leq ba$. It follows that $(a \wedge 1)^n \leq ba/b$ and so $r_b(a) \in F$ and, by similar argument, $l_b(a) \in F$ as well. By Lemma 2.2, F is a congruence filter and (1) holds. □

Of course, there might be no bound on n in the above lemma; but if there is one (for instance, if $\mathbf{A}$ is finite), then $\mathbf{V}(\mathbf{A})$ is a normal variety of residuated lattices. On the other hand, if an entire variety is normal then we can say more. Let $\mathsf{RL}^{\hat{n}}$ be the variety of residuated lattices satisfying

$$(x \wedge 1)^n y \leq yx$$
$$y(x \wedge 1)^n \leq xy.$$

LEMMA 2.5
A variety V of residuated lattices is normal if and only if $\mathsf{V} \subseteq \mathsf{RL}^{\hat{n}}$ for some n.

PROOF. One direction is obvious. Assume then that V is normal and let $\mathbf{F}_{\mathsf{V}}(x,y)$ be the free 2-generated algebra in V. Then $\mathbf{F}_{\mathsf{V}}(x,y)$ is normal and by Lemma 2.4 there are m, k with $(x \wedge 1)^m y \leq yx$ and $y(x \wedge 1)^k \leq xy$. Take $n = \max\{m,k\}$; then in $\mathbf{F}_{\mathsf{V}}(x,y)$

$$(x \wedge 1)^n y \leq yx \qquad y(x \wedge 1)^n \leq xy.$$

But since $\mathbf{F}_{\mathsf{V}}(x,y)$ is free, these inequalities hold in the entire V, i.e. $\mathsf{V} \subseteq \mathsf{FL}^{\hat{n}}$. □

Thus, in particular, every variety of commutative residuated lattices is normal.

3 Join irreducible varieties and iterated conjugates

Given any variety V, we can consider its lattice of subvarieties $\Lambda(\mathsf{V})$; we will say that a subvariety $\mathsf{W} \subseteq \mathsf{V}$ is **join irreducible** (**strongly join irreducible**) if W is a join irreducible (strongly join irreducible) member of $\Lambda(\mathsf{V})$. We will see how in relevant cases (strong) join irreducibility can be characterized by both a property of the corresponding logic and an algebraic description of the generating algebras.

In particular, we start by discussing the concept of well-connected algebra that was introduced by L. Maksimova [37] to characterize the disjunction property in intermediate logics, i.e. those logics $\mathcal{L}$ for which $\mathsf{V}_{\mathcal{L}}$ is a variety of Heyting algebras. A substructural logic $\mathcal{L}$ has the **disjunction property** if whenever $\varphi \vee \psi$ is a theorem of $\mathcal{L}$, in symbols $\mathcal{L} \vdash \varphi \vee \psi$, then either $\mathcal{L} \vdash \varphi$ or $\mathcal{L} \vdash \psi$. Likewise, an integral residuated lattice $\mathbf{A}$ is **well-connected** if 1 is join irreducible, i.e. $a \vee b = 1$ implies either $a = 1$ or $b = 1$. A weaker property is Halldén completeness; a logic $\mathcal{L}$ is **Halldén complete** if it has the disjunction property w.r.t. to any pair of formulas that have no variables in common. Classical logic is Halldén complete but does not have the disjunction property, thus differentiating the two concepts. As shown in [36], these concepts are connected in commutative integral residuated lattices.

THEOREM 3.1
(Theorem 2.5 in [36]) For a variety V of commutative and integral residuated lattices the following are equivalent:

1. $\mathcal{L}_{\mathsf{V}}$ is Halldén complete;
2. V is join irreducible;
3. $\mathsf{V} = \mathsf{V}(\mathbf{A})$ for some well-connected algebra $\mathbf{A}$.

How can we extend the definition of well-connected to the non-integral case? The solution proposed in [36] (and later followed in [17]) is to define a residuated lattice $\mathbf{A}$ to be **well-connected** if 1 is join prime in $\mathbf{A}$, i.e. $a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

We observe straight away that neither integrality nor commutativity is needed to prove that (3) implies (2).

LEMMA 3.2
Let V be a variety of residuated lattices; if $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some well-connected algebra $\mathbf{A} \in \mathsf{V}$ then V is join irreducible.

PROOF. Let $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some well-connected algebra $\mathbf{A} \in \mathsf{V}$, and suppose by way of contradiction that V is not join irreducible, i.e. $\mathsf{V} = \mathsf{W} \vee \mathsf{Z}$ for some proper subvarieties W and Z. Then W and Z must be incomparable; hence, there is an equation $p(x_1, \ldots, x_n) \geq 1$ holding in W but not in Z and an equation $q(y_1, \ldots, y_m) \geq 1$ holding in Z but not in W. Clearly, neither equation can hold in V, so there are $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ with $p(a_1, \ldots, a_n) \not\geq 1$ and $q(b_1, \ldots, b_m) \not\geq 1$; since $\mathbf{A}$ is well-connected we must have $p(a_1, \ldots, a_n) \vee q(b_1, \ldots, b_m) \not\geq 1$ hence $\mathsf{V} = \mathbf{V}(\mathbf{A}) \not\vDash p(x_1, \ldots, x_n) \vee q(y_1, \ldots, y_m) \geq 1$. On the other hand clearly $\mathsf{W}, \mathsf{Z} \vDash p(x_1, \ldots, x_n) \vee q(y_1, \ldots, y_m) \geq 1$ and since $\mathsf{V} = \mathsf{W} \vee \mathsf{Z}$ we must have $\mathsf{V} \vDash p(x_1, \ldots, x_n) \vee q(y_1, \ldots, y_m) \geq 1$. This is a contradiction, derived from the assumption that V was not join irreducible. □

The other implications in the general case however do not hold; an analysis of the Kihara–Ono construction reveals at once that there are two critical points. If V is a variety of commutative residuated lattices then

- every subdirectly irreducible algebra in V is well-connected [36, Lemma 2.2];
- if W, Z are subvarieties of V axiomatized (relative to V) by $p \geq 1$ and $q \geq 1$ (and we make sure that p and q have no variables in common), then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized relative to V by $p \vee q \geq 1$ [36, Lemma 2.1].

Both statements are false if we remove commutativity; for the first it is easy to find a finite and integral residuated lattice that is simple but not well-connected (for instance the example below [26, Lemma 3.60]), while the second fails fore more general reasons discussed at length in [25].

In what follows, we will describe classes of varieties of residuated lattices for which an analogous of Theorem 3.1 can be proved. To do so, we will adapt to our purpose part of the theory developed in [25] about satisfaction of formulas generated by iterated conjugates.

We define a set $B^n(x,y)$ of equations in two variables x, y for all $n \in \mathbb{N}$ in the following way; let Γ^n be the set of iterated conjugates of *length n* (i.e. a composition of n left and right conjugates) over the appropriate language, with $\Gamma^0 = \{l_1\}$ (for a more general definition, here not needed, see [25, p. 229]). For all $n \in \mathbb{N}$

$$B^n(x,y) = \{\gamma_1(x) \vee \gamma_2(y) \approx 1 \ : \ \gamma_1, \gamma_2 \in \Gamma^n\}.$$

Let $\mathbf{A}$ be a residuated lattice and $a, b \in A$; we say that $\mathbf{A}$ **satisfies** $B^n(a,b)$, in symbols $\mathbf{A} \vDash B^n(a,b)$ if $\mathbf{A}, a, b \vDash B^n(x,y)$, i.e. $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$. We say that $\mathbf{A}$ satisfies $(G_{n,k})$ if for all $a, b \in A$, if $\mathbf{A} \vDash B^n(a,b)$, then $\mathbf{A} \vDash B^k(a,b)$.

Lemma 3.3

Let $\mathbf{A}$ be a residuated lattice.

1. for all $n \in \mathbb{N}$, for all $a, b \in A$, if $\mathbf{A} \vDash B^n(a,b)$ then $\mathbf{A} \vDash B^{n-1}(a,b)$;
2. for all $n \in \mathbb{N}$ if $\mathbf{A}$ satisfies $(G_{n,n+1})$ then it satisfies $(G_{n,k})$ for all $k \geq n$.

Proof. The proof of (1) is straightforward, once we have observed that $l_1(a) = a \wedge 1$ and all conjugates are smaller than 1.

For (2), suppose that $\mathbf{A}$ satisfies $(G_{n,n+1})$, we will show that $\mathbf{A}$ satisfies $(G_{n+1,n+2})$. As the relation $\to$ is clearly transitive this implies that $(G_{n,n+2})$ and hence, by induction, $(G_{n,k})$ for $k \geq n$.

Let then $a, b \in A$ such that $\mathbf{A} \vDash B^{n+1}(a,b)$. Given any $\hat{\gamma}_1, \hat{\gamma}_2 \in \Gamma^{n+2}$, there are $\delta_1, \delta_2, \delta_1', \delta_2' \in \Gamma^1$, $\gamma_1, \gamma_2 \in \Gamma^n$ such that

$$\hat{\gamma}_1(x) = \delta_1\gamma_1\delta_2(x) \qquad \hat{\gamma}_2(y) = \delta_1'\gamma_2\delta_2'(y).$$

Since $\mathbf{A} \vDash B^{n+1}(a,b)$, then $\gamma_1\delta_2(a) \vee \gamma_2\delta_2'(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$. If $u = \delta_2(a)$ and $v = \delta_2'(b)$, then $\mathbf{A}$ satisfies $B^n(u,v)$; but since $\mathbf{A}$ satisfies $(G_{n,n+1})$, then it satisfies $B^{n+1}(u,v)$ as well. This implies

$$\hat{\gamma}_1(a) \vee \hat{\gamma}_2(b) = \delta_1\gamma_1(u) \vee \delta_1'\gamma_2(v) = 1,$$

which in turn yields that $\mathbf{A} \vDash B^{n+2}(a,b)$, and then $\mathbf{A}$ satisfies $(G_{n+1,n+2})$. Thus, by transitivity and induction, it satisfies $(G_{n,k})$ for $k \geq n$. □

Let us look closely at the condition $(G_{0,1})$; first we observe the following.

Lemma 3.4

Let V be a variety of residuated lattices; V satisfies $(G_{0,1})$ (i.e. $\mathbf{A}$ satisfies $(G_{0,1})$ for all $\mathbf{A} \in \mathsf{V}$) if and only if it satisfies the quasi equation

$$x \vee y \approx 1 \qquad \Longrightarrow \qquad l_w(x) \vee r_z(y) \approx 1. \tag{G}$$

Proof. One direction is easy to check. Indeed, suppose V satisfies $(G_{0,1})$, and let $\mathbf{A} \in \mathsf{V}, a, b \in A$ such that $a \vee b = 1$. Then $a = a \wedge 1, b = b \wedge 1$, and (G) directly follows from $(G_{0,1})$ that corresponds to the set of quasi equations:

$$(x \wedge 1) \vee (y \wedge 1) \approx 1 \qquad \Longrightarrow \qquad \gamma_1(x) \vee \gamma_2(y) \approx 1, \text{ for } \gamma_1, \gamma_2 \in \Gamma^1.$$

For the other, suppose (G) holds, and let $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$; then $\mathbf{A} \models B^0(a,b)$ if and only if $(a \wedge 1) \vee (b \wedge 1) = 1$. Then by (G) for all $u, v \in A$

$$1 = l_u(a \wedge 1) \vee r_v(b \wedge 1) \leq l_u(a) \vee r_v(b) \leq 1.$$

Thus $l_u(a) \vee r_v(b) = 1$, and in particular for all $u \in A$, $l_u(a) \vee r_1(b) = l_u(a) \vee (b \wedge 1) = 1$ and so, applying again the same reasoning since $l_u(a) \leq 1$, for all $v \in A$

$$1 = l_u(a) \vee l_v(b \wedge 1) \leq l_u(a) \vee l_v(b) \leq 1.$$

The proof that $r_u(a) \vee r_v(b) = 1$ is similar, so we conclude that $\mathbf{A} \models B^1(a,b)$; therefore $\mathbf{A}$ satisfies $(G_{0,1})$ and so does V. □

(G) is trivially satisfied by any commutative variety of residuated lattices, but moreover

LEMMA 3.5
Any normal variety V of residuated lattices satisfies (G).

PROOF. Let $\mathbf{A} \in \mathsf{V}$, and consider $x, y \in A$ with $x \vee y = 1$, thus clearly $x, y \leq 1$. Since both conjugates are below 1, $l_w(x) \vee r_z(y) \leq 1$. Thus, we need to show that $1 \leq l_w(x) \vee r_z(y)$.

By Lemma 2.5, $(y \wedge 1)^n z \leq zy$ and $w(x \wedge 1)^n \leq xw$, thus $(x \wedge 1)^n \leq l_w(x)$ and $(y \wedge 1)^n \leq r_z(y)$. From [25, Lemma 3.20], we get that if $x \vee y = 1$ then $x^n \vee y^n = (x \wedge 1)^n \vee (y \wedge 1)^n = 1$. Thus,

$$1 \leq (x \wedge 1)^n \vee (y \wedge 1)^n \leq l_w(x) \vee r_z(y)$$

and the proof is completed. □

There are also non-normal varieties satisfying (G), for instance, *representable* varieties (see Section 6 below). We observe also that the variety of ℓ-groups satisfies $(G_{1,2})$ (see [25, p. 235]). The last lemma we need is implicit in [25].

LEMMA 3.6
Let V be a variety of residuated lattices and let $p(x_1, \ldots, x_n) \geq 1$, $q(y_1, \ldots, y_m) \geq 1$ be two inequalities not holding in V. If W and Z are the subvarieties axiomatized by $p \wedge 1 \approx 1$ and $q \wedge 1 \approx 1$, respectively, then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized by the set $B(p,q) = \bigcup_{n \in \mathbb{N}} B^n(p,q)$. Moreover, if V satisfies $(G_{l,l+1})$ for some $l \in \mathbb{N}$ then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized by the finite set $B^l(p,q)$.

PROOF. The first part is a consequence of [25, Corollary 4.3]. The second part follows from [25, Theorem 4.4 (1)], where it is shown that $\mathsf{W} \vee \mathsf{Z}$ is axiomatized by $B^l(p,q)$ plus the (finite set of) equations implying $(G_{l,l+1})$, which here can be omitted since $(G_{l,l+1})$ holds in V. □

4 Γ-connectedness and Γ-completeness

The **negative cone** of a residuated lattice $\mathbf{A}$ is an integral residuated lattice $\mathbf{A}^-$ whose universe is $A^- = \{a \in A : a \leq 1\}$ and the operations are defined as follows: the lattice operations and the product are the same as in $\mathbf{A}$ but the residuations are defined as $a/_1 b := (a/b) \wedge 1$ and $a\backslash_1 b := (a\backslash b) \wedge 1$. The connections between $\mathbf{A}^-$ and $\mathbf{A}$ are strict; for instance, it can be shown that $\mathrm{Con}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})^-$ are isomorphic [26]. A residuated lattice $\mathbf{A}$ is **weakly well-connected** if $\mathbf{A}^-$ is well-connected; we have the following obvious lemma.

LEMMA 4.1
For a residuated lattice **A**, the following are equivalent:

1. **A** is weakly well-connected;
2. 1 is join irreducible in **A**;
3. for $a, b \in A$ if $(a \wedge 1) \vee (b \wedge 1) = 1$, then either $a \geq 1$ or $b \geq 1$.

Clearly, if **A** is well-connected, then it is weakly well-connected and the two concepts coincide in varieties that are 1-distributive, i.e. satisfy the equation

$$(x \vee y) \wedge 1 \leq (x \wedge 1) \vee (y \wedge 1).$$

We want to generalize these concepts; we say that a residuated lattice **A** is Γ^n**-connected** if for all $a, b \in A$, if $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma_n(\mathbf{A})$, then either $a \geq 1$ or $b \geq 1$. To prove the next result, we need a lemma.

LEMMA 4.2
Let **A** be a residuated lattice and let $a_1, \ldots, a_m, b_1, \ldots, b_l \in A$; if $a_i \vee b_j = 1$ for all $i \leq m$ and $j \leq l$, then for all r, s and $\{d_1, \ldots, d_r\} \subseteq \{a_1, \ldots, a_m\}$ and $\{e_1, \ldots, e_s\} \subseteq \{b_1, \ldots, b_l\}$ we have

$$e_1 \ldots e_s \vee d_1 \ldots d_r = 1.$$

The proof is a simple finite induction using the equations holding in residuated lattices (see [25, Lemma 3.2]).

LEMMA 4.3
Let V be a variety of residuated lattices that satisfies $(G_{n,n+1})$. Then every subdirectly irreducible algebra in V is Γ^n-connected.

PROOF. Let **A** be subdirectly irreducible and let $a, b \in A$ with $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$, we will show that then either $a \geq 1$ or $b \geq 1$. Pick any $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$, then if $d \in \mathrm{Fil}_{\mathbf{A}}(\gamma_1(a)) \cap \mathrm{Fil}_{\mathbf{A}}(\gamma_2(b))$ by using the description of the congruence filter generated by an element in Corollary 2.3 we get that

$$\delta_1(\gamma_1(a)) \ldots \delta_l(\gamma_1(a)) \vee \delta'_1(\gamma_2(b)) \ldots \delta'_m(\gamma_2(b)) \leq d$$

for some $\delta_i, \delta'_j \in \Gamma^k$, for $i = 1 \ldots l, j = 1 \ldots m$ and $k \in \mathbb{N}$. Indeed the iterated conjugates can all be considered of the same (maximum) length k. Let $\varepsilon_i = \delta_i \gamma_1$ and $\varepsilon'_j = \delta'_j \gamma_2$, then $\varepsilon_i, \varepsilon_j \in \Gamma^{n+k}$ for $i = 1 \ldots l, j = 1 \ldots m$. As **A** satisfies $(G_{n,n+1})$ by Lemma 3.3, it satisfies $(G_{n,n+k})$ and then

$$\varepsilon_i(a) \vee \varepsilon'_j(b) = 1 \qquad \text{for all } i, j;$$

now we apply Lemma 4.2, and we get

$$\varepsilon_1(a) \ldots \varepsilon_l(a) \vee \varepsilon'_1(b) \ldots \varepsilon'_m(b) = 1$$

so $d \geq 1$. Since d was generic we must have $\mathrm{Fil}_{\mathbf{A}}(\gamma_1(a)) \cap \mathrm{Fil}_{\mathbf{A}}(\gamma_2(b)) = A^+$; but since **A** is subdirectly irreducible the isomorphism between the congruence lattice and the congruence filter lattice of **A** forces either $\mathrm{Fil}_{\mathbf{A}}(\gamma_1(a)) = A^+$ or $\mathrm{Fil}_{\mathbf{A}}(\gamma_2(b)) = A^+$. This works for any choice of $\gamma_1, \gamma_2 \in \Gamma^n$, thus in particular it works for $\gamma_1 = \gamma_2 = l_1 \circ \ldots \circ l_1 = l_1$, thus we get that either $a \wedge 1 \geq 1$ or $b \wedge 1 \geq 1$ i.e. either $a \geq 1$ or $b \geq 1$ as desired. Thus, **A** is Γ^n-connected. □

Finally, we complete the connection with logic. Let $\mathcal{L}$ be a substructural logic over $\mathcal{FL}^+$; given any two axiomatic extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ axiomatized by formulas ϕ and ψ, respectively, for any n Lemma 3.6 implicitly gives a set of formulas $B^n_{\mathcal{L}}(\phi,\psi)$ such that $B_{\mathcal{L}}(\phi,\psi)=\bigcup_{n\in\mathbb{N}} B^n_{\mathcal{L}}(\phi,\psi)$ axiomatizes the intersection $\mathcal{L}_1\cap\mathcal{L}_2$, corresponding to the join of the varieties $\mathsf{V}_{\mathcal{L}_1}\vee\mathsf{V}_{\mathcal{L}_2}$. We say that $\mathcal{L}$ is **Γ^n-complete** if for all formulas ϕ and ψ which have no variables in common, if $\mathcal{L}\vdash B^n_{\mathcal{L}}(\phi,\psi)$ then either $\mathcal{L}\vdash\phi$ or $\mathcal{L}\vdash\psi$.

THEOREM 4.4
Let V be a variety of residuated lattices satisfying $(G_{n,n+1})$ for some $n\in\mathbb{N}$; then the following are equivalent.

1. $\mathcal{L}_{\mathsf{V}}$ is Γ^n-complete;
2. V is join irreducible;
3. $\mathsf{V}=\mathbf{V}(\mathbf{A})$ for some Γ^n-connected algebra $\mathbf{A}$.

PROOF. We show first that (1) implies (2). Assume that (2) fails, i.e. $\mathsf{V}=\mathsf{W}\vee\mathsf{Z}$ and $\mathsf{V}\neq\mathsf{W},\mathsf{Z}$. Let $p\geq 1$ be an equation holding in W but not in Z and $q\geq 1$ an equation holding in Z but not in W, respectively; if W',Z' are the varieties axiomatized relative to V by $p\geq 1, q\geq 1$ respectively, we have

$$\mathsf{V}=\mathsf{W}\vee\mathsf{Z}\subseteq\mathsf{W}'\vee\mathsf{Z}'\subseteq\mathsf{V};$$

so we may assume that W,Z are axiomatized relative to V by $p\geq 1$ and $q\geq 1$. Let $\varphi:=p\geq 1$ and $\psi:=q\geq 1$; then $\mathcal{L}_{\mathsf{W}}\cap\mathcal{L}_{\mathsf{Z}}=\mathcal{L}_{\mathsf{V}}$ is axiomatized by $B_{\mathcal{L}_{\mathsf{V}}}(\varphi,\psi)$. Hence, $\mathcal{L}_{\mathsf{V}}\vdash B_{\mathcal{L}_{\mathsf{V}}}(\phi,\psi)$ but neither $\mathcal{L}_{\mathsf{V}}\vdash\varphi$ nor $\mathcal{L}_{\mathsf{V}}\vdash\psi$. Hence, $\mathcal{L}_{\mathsf{V}}$ does not have the Γ^n-completeness property.

We now show that (2) implies (1). Assume that (1) fails; hence, there exist formulas ϕ and ψ with no common variables such that $\mathcal{L}_{\mathsf{V}}\vdash B^n_{\mathcal{L}}(\phi,\psi)$ but neither $\mathcal{L}_{\mathsf{V}}\vdash\phi$ nor $\mathcal{L}_{\mathsf{V}}\vdash\psi$. Then we can consider the two axiomatic extensions of $\mathcal{L}_{\mathsf{V}}$, $\mathcal{L}_1$ and $\mathcal{L}_2$, axiomatized by ϕ and ψ, respectively. This corresponds to taking two subvarieties of V, W and Z, defined by identities $p\wedge 1\approx 1$ and $q\wedge 1\approx 1$, respectively. Since V satisfies $(G_{n,n+1})$, by Lemma 3.6, $\mathsf{W}\vee\mathsf{Z}$ is finitely axiomatizable relative to V by $B^n(p,q)$ and the corresponding formula axiomatizing $\mathcal{L}_1\cap\mathcal{L}_2$ is $B^n_{\mathcal{L}_{\mathsf{V}}}(\phi,\psi)$. But we are assuming that $\mathcal{L}_{\mathsf{V}}\vdash B^n_{\mathcal{L}}(\phi,\psi)$, which implies that $\mathsf{W}\vee\mathsf{Z}=\mathsf{V}$; thus, V is not join irreducible, a contradiction. Hence, (2) implies (1), and (1) and (2) are equivalent.

We now show the equivalence of (2) and (3). Assume (2) and let $\mathbf{A}$ be any algebra such that $\mathsf{V}=\mathbf{V}(\mathbf{A})$; let $\{\theta_i : i\in I\}$ be the set of all congruences of $\mathbf{A}$ such that $\mathbf{A}/\theta_i$ is subdirectly irreducible. Next for any equation ε in the language of V we define $L_\varepsilon=\{i\in I:\mathbf{A}/\theta_i\not\models\varepsilon\}$; we claim that $\Delta=\{L_\varepsilon:\mathsf{V}\not\models\varepsilon\}$ is a collection of nonempty subsets with the finite intersection property.

First note that if $\mathsf{V}\not\models\varepsilon$ then $\mathbf{A}\not\models\varepsilon$ and hence (by Birkhoff's theorem) $\mathbf{A}/\theta_i\not\models\varepsilon$ for some i; therefore $L_\varepsilon\neq\emptyset$. Next we prove that for any ε,δ such that $\mathsf{V}\not\models\varepsilon,\delta$ there is a γ such that $\mathsf{V}\not\models\gamma$ and $L_\gamma\subseteq L_\varepsilon\cap L_\delta$.

Let W,Z be the subvarieties of V axiomatized by ε and δ, respectively; since $\mathsf{V}\not\models\varepsilon,\delta$ they are both proper subvarieties of V. Let ε be $p(x_1,\dots,x_n)\geq 1$ and δ be $q(y_1,\dots,y_m)\geq 1$; since V satisfies $(G_{n,n+1})$, by Lemma 3.6 $\mathsf{W}\vee\mathsf{Z}$ is axiomatized by the finite set of equations $B^n(p,q)$, which contains in particular the equation $(p\wedge 1)\vee(q\wedge 1)\approx 1$.

Let γ be the inequality equivalent to the conjunction of the equations in $B^n(p,q)$; if $i\notin L_\varepsilon$ then $\mathbf{A}/\theta_i\models p\geq 1$, and thus also $\mathbf{A}/\theta_i\models\gamma_1(p)\geq 1$ for any $\gamma_1\in\Gamma^n(\mathbf{A}/\theta_i)$. Hence, $\mathbf{A}_i/\theta\models\gamma_1(p)\vee\gamma_2(q)\approx 1$ for all $\gamma_1,\gamma_2\in\Gamma^n$, and those are exactly the equations in $B^n(p,q)$, thus $i\notin L_\gamma$. Since the same can be said for δ, it follows by contraposition that $L_\gamma\subseteq L_\varepsilon\cap L_\delta$. If $\mathsf{V}\models\gamma$, then $\mathsf{V}=\mathsf{W}\vee\mathsf{Z}$, contrary to the hypothesis that V is join irreducible. Hence, $\mathsf{V}\not\models\gamma$ and so Δ has the finite intersection property.

So we can take an ultrafilter U on I containing Δ; let $\mathbf{B} = \prod_{i\in I} \mathbf{A}/\theta_i/U$. Since $\mathbf{A}/\theta_i$ is subdirectly irreducible, then by Lemma 4.3 it is Γ^n-connected. But being Γ^n-connected is a first-order property, so $\mathbf{B}$ is Γ^n-connected as well and clearly $\mathbf{B} \in \mathbf{V}(\mathbf{A})$. Moreover, if an equation ε fails in $\mathbf{A}$, then $L_\varepsilon \in \Delta$, so $L_\varepsilon \in U$ and so $\mathbf{B} \not\models \varepsilon$. This proves that $\mathbf{V}(\mathbf{B}) = \mathbf{V}(\mathbf{A}) = \mathsf{V}$ and so (3) holds.

Finally, the proof that (3) implies (2) is similar to the one of Lemma 3.2. Suppose that $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some Γ^n-connected algebra $\mathbf{A}$ and that, by way of contradiction, V is not join irreducible, i.e. $\mathsf{V} = \mathsf{W} \vee \mathsf{Z}$ for some proper subvarieties W and Z. Then there is an equation $p(x_1,\dots,x_n) \geq 1$ holding in W but not in Z and an equation $q(y_1,\dots,y_m) \geq 1$ holding in Z but not in W. Clearly, neither equation can hold in V, so there are $a_1,\dots,a_n,b_1,\dots,b_m \in A$ with $p(a_1,\dots,a_n) \not\geq 1$ and $q(b_1,\dots,b_m) \not\geq 1$; since $\mathbf{A}$ is Γ^n-connected, there must be $\hat{\gamma}_1,\hat{\gamma}_2$, instances in $\mathbf{A}$ of iterated conjugates $\gamma_1,\gamma_2 \in \Gamma^n$, such that

$$\hat{\gamma}_1(p(a_1,\dots,a_n)) \vee \hat{\gamma}_2(q(b_1,\dots,b_m)) \neq 1.$$

hence $\mathsf{V} = \mathbf{V}(\mathbf{A}) \not\models \gamma_1(p(x_1,\dots,x_n)) \vee \gamma_2(q(y_1,\dots,y_m)) \approx 1$. On the other hand, since conjugates of elements in the positive cone of on algebra are also in the positive cone, $\gamma_1(p(x_1,\dots,x_n)) \approx 1$ holds in W and $\gamma_2(q(y_1,\dots,y_m)) \approx 1$ holds in Z. Thus, $\mathsf{W},\mathsf{Z} \models \gamma_1(p(x_1,\dots,x_n)) \vee \gamma_2(q(y_1,\dots,y_m)) \approx 1$ and since $\mathsf{V} = \mathsf{W} \vee \mathsf{Z}$ we must have that $\mathsf{V} \models \gamma_1(p(x_1,\dots,x_n)) \vee \gamma_2(q(y_1,\dots,y_m)) \approx 1$. This is a contradiction, derived from the assumption that V was not join irreducible. Thus, also (2) and (3) are equivalent and the proof is completed. □

Observe that Γ^0-connectedness is weak well-connectedness and Γ^0-completeness is the so-called *weak Halldén completeness* for a substructural logic $\mathcal{L}$: if $\mathcal{L} \vdash (\phi \wedge 1) \vee (\psi \wedge 1)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$. So our result extends both [36, Theorem 2.5 and 2.8], giving an (admittedly less meaningful) logical characterizations of the property of being join irreducible for a variety satisfying $(G_{n,n+1})$ for some $n \in \mathbb{N}$.

5 Join irreducibility and subdirect irreducibility

It is a straightforward consequence of Birkhoff's Theorem that if V is any variety of algebras that is strictly join irreducible, then $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some subdirectly irreducible algebra $\mathbf{A}$. In [36], the authors asked if the converse might hold for varieties of residuated lattices. We will see in Section 7 below that not all the varieties generated by a subdirectly irreducible basic hoop are strictly join irreducible so in general the converse does not hold even in very friendly varieties.

Next we point out a corollary of Lemma 4.3 and Theorem 4.4.

COROLLARY 5.1
Let V be a variety of residuated lattices satisfying $(G_{n,n+1})$ for some $n \in \mathbb{N}$. If there is a subdirectly irreducible algebra $\mathbf{A}$ with $\mathsf{V} = \mathbf{V}(\mathbf{A})$, then V is join irreducible.

This is the analogous of [36, Lemma 2.6(2)] and again the authors asked if it was possible to invert it; we will show that our (more general) version is indeed invertible, thus answering their question as well. First observe that

LEMMA 5.2
Let V be a variety of residuated lattices. If $\mathbf{A} \in \mathsf{V}$ is Γ^n-connected, then there exists a countable Γ^n-connected algebra $\mathbf{B} \in \mathsf{V}$ with $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$.

PROOF. Suppose that $\mathbf{A}$ is Γ^n-connected; it is clear that the class of Γ^n-connected in $\mathbf{V}(\mathbf{A})$ is axiomatized by a set of first-order sentences, namely the axioms defining $\mathbf{V}(\mathbf{A})$ and the universal sentence

$$\forall x \forall y\, [\&B^n(x,y) \Rightarrow ((x \geq 1) \text{ or } (y \geq 1))].$$

Where by $\&B^n(x,y)$, we mean the conjunction of the equations in $B^n(x,y) = \{\gamma_1(x) \vee \gamma_2(y) \approx 1 : \gamma_1, \gamma_2 \in \Gamma^n\}$. The downward version of the Löwenheim–Skolem theorem implies that if $\mathbf{A}$ is Γ^n-connected, then there is also a countable Γ^n-connected algebra $\mathbf{B}$ that is elementarily equivalent to $\mathbf{A}$. In particular, $\mathbf{A}$ and $\mathbf{B}$ must satisfy the same equations, so $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$. □

In [9], the authors proved an interesting result about FL_{ew}-algebras: if a variety is generated by a well-connected algebra, then it is also generated by a subdirectly irreducible algebra, which implies that [36, Lemma 2.6(2)] can indeed be inverted. By looking at their proofs, it is clear that they do not rely on zero-boundedness and commutativity. Generalizing it for non-integral structures takes a little more work and we need to clarify the context. If $\mathbf{A}$ is a residuated lattice, we can define a set of equations $D_{\mathbf{A}}$, called the **diagram** of $\mathbf{A}$ in the following way: for every $a \in A$ we pick a variable v_a and

$$D_{\mathbf{A}} = \{v_a * v_b \leftrightarrow v_{a*b} \geq 1 : a, b \in A, * \in \{\vee, \wedge, \rightarrow, /, \backslash\}\},$$

where $a \leftrightarrow b = (a\backslash b) \wedge (b\backslash a)$. It is clear that the diagram encodes the operation table of $\mathbf{A}$; moreover, if $\mathbf{A}$ is countable then all the terms involved are elements of the free algebra over ω generators $\mathbf{F}_{\mathsf{V}}(\omega)$, of any variety V to which $\mathbf{A}$ belongs.

So if $\mathbf{A} \in \mathsf{V}$, we can use the semantical consequence relation $\vDash$ on V and the expression $D_{\mathbf{A}} \vDash p \geq 1$ makes sense for any term p in the language of residuated lattices. Likewise $\mathbf{B} \in \mathsf{V}$ is a model of $D_{\mathbf{A}} \vDash p \geq 1$ if any homomorphism $h : \mathbf{F}_{\mathsf{V}}(\omega) \longrightarrow \mathbf{B}$, say $h(v_a) = c_a$, such that $c_a * c_b = c_{a*b}$ for all $a, b \in A$, is also such that $h(p) \geq 1$. Now it is straightforward to check that the class of models of $D_{\mathbf{A}} \vDash p \geq 1$ is closed under subalgebras and direct products; it follows that if there is a countermodel in V of $D_{\mathbf{A}} \vDash p \geq 1$, then there is a subdirectly irreducible countermodel. This circle of ideas is the core of the following lemma.

LEMMA 5.3
Let V be any variety of residuated lattices and let $\mathbf{A}$ be a countable Γ^n-connected algebra in V; then there exists a subdirectly irreducible algebra $\mathbf{B} \in \mathsf{V}$ such that $\mathbf{A} \in \mathbf{S}(\mathbf{B})$.

PROOF. Let $\mathbf{A}$ be a countable Γ^n-connected algebra in V and let $V = \{v_a : a \in A\}$ be the set of variables defining the diagram $D_{\mathbf{A}}$.

Now we define a set of equations Ψ not valid in $\mathbf{A}$ because of its Γ^n-connectedness. In particular, in Ψ we write the meets of all possible joins of iterated conjugates of length n of m given variables, for each $m \in \mathbb{N}$. Notice that given $m \in \mathbb{N}$, such joins are in a finite number. Indeed, as argued in [25, (p. 232)], one can enumerate the indices of the iterated conjugates of a certain length and they will form an initial segment of the natural numbers. Let us call Γ^n_m the finite set of all possible m-uples of iterated conjugates of length n and J_m the enumeration of the m-uples of indices of the conjugates in Γ^n_m. That is, each $\mathbf{j} \in J_m$ is of the kind $\mathbf{j} = (j_1, \ldots j_m)$ and identifies an m-uple $(\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^n_m$. Then

$$\Psi = \{ \bigwedge_{\mathbf{j} \in J_m} (\gamma_{j_1}(v_{a_1}) \vee \ldots \vee \gamma_{j_m}(v_{a_m})) \approx 1 : \; m \in \mathbb{N}, (\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^n_m,$$
$$a_i \in A, a_i \not\geq 1, \; \text{for } i = 1 \ldots m\}.$$

Notice that Ψ is *directed* in the following sense:

$$\text{if } \bigwedge_{j\in J} p_j \approx 1 \in \Psi \text{ and } \bigwedge_{k\in K} q_k \approx 1 \in \Psi, \quad \text{then} \quad \bigwedge_{j\in J}\bigwedge_{k\in K} p_j \vee q_k \approx 1 \in \Psi.$$

We call $\bigwedge_{j\in J}\bigwedge_{k\in K} p_j \vee q_k \approx 1$ the *upper bound* of $\bigwedge_{j\in J} p_j \approx 1$ and $\bigwedge_{k\in K} q_k \approx 1$, since both

$$\bigwedge_{j\in J} p_j \leq \bigwedge_{j\in J}\bigwedge_{k\in K} p_j \vee q_k \text{ and } \bigwedge_{k\in K} q_k \leq \bigwedge_{j\in J}\bigwedge_{k\in K} p_j \vee q_k.$$

Moreover, $D_{\mathbf{A}} \not\models p \approx 1$ for each $p \approx 1 \in \Psi$, since $\mathbf{A}$ naturally gives a countermodel. Indeed, with the assignment $g : \mathbf{F}_{\mathsf{V}}(\omega) \longrightarrow \mathbf{A}$ such that $g(v_a) = a$ for all $a \in A$, $\mathbf{A}$ satisfies all inequalities in $D_{\mathbf{A}}$ by construction, but it does not satisfy any of the equations $p \approx 1 \in \Psi$: if $p = \bigwedge_{j\in J}(\gamma_{j_1}(v_{a_1}) \vee \ldots \vee \gamma_{j_m}(v_{a_m})) \approx 1$ is satisfied with respect to the assignment g, then all of the meetands $(\gamma_{j_1}(a_1) \vee \ldots \vee \gamma_{j_m}(a_m)) = 1$, for all $(\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^n_m$, and thus at least one of the elements $a_i \geq 1$, since $\mathbf{A}$ is Γ^n-connected. But by the definitions of Ψ and g, $a_i \not\geq 1$ for all $i = 1 \ldots m$, a contradiction.

Let[1] v be a variable different from all the v_a, and let

$$D' = D_{\mathbf{A}} \cup \{p\backslash v \geq 1 : p \approx 1 \in \Psi\}.$$

We claim that $D' \not\models v \geq 1$. Suppose that $D' \models v \geq 1$, then by compactness there is a finite subset $D'' \subseteq D'$ such that $D'' \models v \geq 1$; let $r \approx 1$ be the upper bound of the finite set of equalities $\{p \approx 1 \in \Psi : p\backslash v \geq 1 \in D''\}$ (such upper bound exists as Ψ is directed). Since $D_{\mathbf{A}} \not\models \psi$ for each $\psi \in \Psi$, in particular $D_{\mathbf{A}} \not\models r \approx 1$; so there is an algebra $\mathbf{C} \in \mathsf{V}$ and a homomorphism $k : \mathbf{F}_{\mathsf{V}}(\omega) \longrightarrow \mathbf{C}$ such that all identities in $D_{\mathbf{A}}$ are satisfied in $\mathbf{C}$ but $k(r) \neq 1$, or equivalently $k(r) \not\geq 1$. Now if we extend k by setting $k(v) = k(r)$ then $\mathbf{C}$ becomes a countermodel of $D' \vDash v \geq 1$ as well, a contradiction. Thus, $D' \not\models v \geq 1$. It follows that there is a subdirectly irreducible algebra $\mathbf{B} \in \mathsf{V}$ that is a countermodel of $D' \vDash v \geq 1$, with respect to a homomorphism $h : \mathbf{F}_{\mathsf{V}}(\omega) \longrightarrow \mathbf{B}$.

Observe that, since $D_{\mathbf{A}} \subseteq D'$, $\mathbf{B}$ also satisfies all inequalities in $D_{\mathbf{A}}$ with respect to h, but $h(p) \neq 1$ for all $p \approx 1 \in \Psi$. Indeed, if it were $h(p) = 1$ it would follow that $h(v) \geq h(p)h(p\backslash v) \geq 1$, a contradiction.

Let us define $f : A \to B$ as $f(a) = h(v_a)$ and show that it is an embedding, which would settle the proof. Since $\mathbf{B}$ satisfies $D_{\mathbf{A}}$ with respect to h, f is a homomorphism. To show that f is injective suppose that $f(a) = f(b)$ given $a, b \in A$; then by residuation $1 \leq f(a\backslash b), 1 \leq f(b\backslash a)$. Note that if $c \not\geq 1$, then $v_c \wedge 1 \approx 1 \in \Psi$, thus $f(c) = h(v_c) \not\geq 1$; in fact, if $h(v_c) \geq 1$ we would have $h(v_c \wedge 1) = h(v_c) \wedge 1 = 1$, a contradiction, as $\mathbf{B}$ satisfies no equality in Ψ with respect to h. This implies that $1 \leq a\backslash b$ and $1 \leq b\backslash a$, thus $a = b$ and the proof is completed. □

THEOREM 5.4
Let V be a variety of residuated lattices that satisfies $(G_{n,n+1})$ for some $n \in \mathbb{N}$; if V is join irreducible, then there is a subdirectly irreducible algebra $\mathbf{B} \in \mathsf{V}$ such that $\mathbf{V}(\mathbf{B}) = \mathsf{V}$.

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[1] This part of the proof is essentially the same as in [18, Lemma 3.4], which we rewrite in this more general setting.

PROOF. Since V is join irreducible and satisfies $(G_{n,n+1})$, by Theorem 4.4, there is a Γ^n-connected algebra $\mathbf{A}$ with $\mathbf{V}(\mathbf{A}) = \mathsf{V}$. By Lemma 5.2, we may assume without loss of generality that $\mathbf{A}$ is countable; hence by Lemma 5.3 there is a subdirectly irreducible $\mathbf{B} \in \mathsf{V}$ with $\mathbf{A} \in \mathbf{S}(\mathbf{B})$. Clearly, $\mathsf{V} = \mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$ and the thesis holds. □

Thus combining Theorem 5.4 and Corollary 5.1 we obtain the following.

COROLLARY 5.5
Let V be a variety of residuated lattices that satisfies $(G_{n,n+1})$ for some $n \in \mathbb{N}$. Then V is join irreducible if and only if there is a subdirectly irreducible algebra $\mathbf{A} \in \mathsf{V}$ such that $\mathbf{V}(\mathbf{A}) = \mathsf{V}$.

Of course the same results we presented in this section hold for varieties of FL-algebras. In particular then, the previous corollary applies to all varieties satisfying (G), thus e.g. normal varieties of RL and FL and thus CRL, CIRL, FL_e and FL_{ew} (answering then to Kihara and Ono question), and also representable subvarieties of RL and FL. Moreover, the theorem also applies to subvarieties of ℓ-groups, since they satisfy $(G_{1,2})$.

6 Representable varieties

A residuated lattice is **representable** if it is a subdirect product of totally ordered residuated lattices; a variety is representable if each of its members is representable, which is equivalent to say that each subdirectly irreducible member of the variety is totally ordered. Being representable for a variety is equationally definable; indeed for any $\mathsf{V} \in \mathsf{RL}$, V is representable if and only if $\mathsf{V} \models u\backslash((x \vee y)\backslash x)u \vee v((x \vee y)\backslash y)/v$ (see [15, 35]).

A residuated lattice is **prelinear** if it satisfies

$$(x/y) \vee (y/x) \geq 1.$$

A variety of residuated lattices is prelinear if each of its members is prelinear. Since any totally ordered residuated lattice is clearly prelinear, any representable variety is prelinear.

It is well known that any representable variety satisfies (G) (see for instance [15, Theorem 6.7]) so we may apply Theorem 4.4 to conclude that each join irreducible subvariety is generated by a single weakly well-connected algebra. Moreover, since clearly the underlying lattice of any representable residuated lattice is distributive (as a subdirect product of chains) the concepts of well-connected and weakly well-connected coincide. Moreover, the following holds.

LEMMA 6.1
Let V be a prelinear variety of residuated lattices and let $\mathbf{A} \in \mathsf{V}$. Then the following are equivalent:

1. $\mathbf{A}$ is well-connected;
2. $\mathbf{A}$ is totally ordered.

PROOF. The implication (2) implies (1) clearly follows from the definition of well-connectedness. Suppose now that $\mathbf{A}$ is prelinear and well-connected. Since it is prelinear for $a, b \in A$, $(a/b) \vee (b/a) \geq 1$. Since it is well-connected, either $a/b \geq 1$ or $b/a \geq 1$; so either $b \leq a$ or $a \leq b$ and $\mathbf{A}$ is totally ordered. □

If the variety we are dealing with is also integral, then we can use ordinal sums, of which we recall the definition for the reader's convenience.

Let $(I, \leq)$ be a totally ordered set. For all $i \in I$, let $\mathbf{A}_i$ be a totally ordered integral residuated ($\wedge$-semi)lattice, such that for $i \neq j$, $A_i \cap A_j = \{1\}$. Then $\bigoplus_{i \in I} \mathbf{A}_i$, the *ordinal sum* of the family

$(\mathbf{A}_i)_{i\in I}$, is the structure whose base set is $\bigcup_{i\in I} A_i$ and the operations are defined as follows:

$$x \cdot y \quad = \quad \begin{cases} x \cdot^{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ with } i > j \\ x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$x \backslash y \quad = \quad \begin{cases} x \backslash^{A_i} y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

$$y / x \quad = \quad \begin{cases} y /^{A_i} x & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j. \end{cases}$$

It is easily seen that join and meet are defined by the ordering obtained by stacking the various algebras one over the other. Notice that the ordinal sum of a family of totally ordered integral residuated lattices can be always defined, by safely assuming that the algebras intersect in their top elements (or, more precisely, by considering their isomorphic copies that do). The ordinal sum construction makes sense also for not totally ordered structures, in which case one needs to be careful about the join operation (see [33]). This is not needed here, since in the present paper we will only use the ordinal sum among totally ordered algebras. The following result however holds in general. We call sum-irreducible an algebra that is not isomorphic to an ordinal sum of nontrivial components. Then

THEOREM 6.2
(Theorem 3.2 of [2]) Any integral residuated semilattice is the ordinal sum of sum-irreducible residuated semilattices.

In particular, if V is integral and representable, then every strictly join irreducible subvariety of V is generated by a single totally ordered algebra $\mathbf{A}$, which is the ordinal sum of sum-irreducible totally ordered algebras in V. Therefore, knowing the sum-irreducible algebras in a variety of integral and representable residuated lattices is paramount. Though this is usually a difficult task there are cases in which it can be achieved.

A **pseudohoop** is an integral and divisible residuated lattice; the variety of representable pseudohoops will be denoted by RPsH. A **Wajsberg pseudohoop** is a representable pseudohoop satisfying

$$(y/x)\backslash y \approx (x/y)\backslash x;$$

$$y/(x\backslash y) \approx x/(y\backslash x).$$

It turns out that totally ordered Wajsberg pseudohoops are precisely the totally ordered sum-irreducible pseudohoops. This implies that

THEOREM 6.3
[21] Let $\mathbf{A}$ be a totally ordered pseudohoop; then $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ where each $\mathbf{A}_i$ is a (totally ordered) Wajsberg pseudohoop (and the decomposition is unique up to isomorphism of the components).

If $\mathbf{A}$ is a totally ordered pseudohoop and $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ is its decomposition into Wajsberg pseudohoops, the index of $\mathbf{A}$ is n if $|I| = n$ and infinite if I is infinite. Let us consider the equation

$$\bigwedge_{i=0}^{n-1} x_i/(x_i/(x_{i+1}\backslash x_i)) \leq \bigvee_{i=0}^{n} x_i. \qquad (\lambda_n)$$

LEMMA 6.4
(Lemma 6.1 in [2]) For any totally ordered pseudohoop $\mathbf{A}$, the index of $\mathbf{A}$ is less or equal to $n \in \mathbb{N}$ if and only if $\mathbf{A} \vDash \lambda_n$.

By Theorem 4.4 and Lemma 6.1, if $\mathbf{A}$ is a totally ordered pseudohoop, $\mathbf{V}(\mathbf{A})$ is join irreducible in $\Lambda(\mathsf{RPsH})$.

THEOREM 6.5
Let $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ be a totally ordered pseudohoop; if the index of $\mathbf{A}$ is infinite, then $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible.

PROOF. Let C be the set of all chains in $\mathbf{V}(\mathbf{A})$ having finite index; by Lemma 6.4, $\mathbf{V}(\mathbf{C}) \neq \mathbf{V}(\mathbf{A})$ for all $\mathbf{C} \in C$. Suppose that an equation $p(x_1,\ldots,x_n) \approx 1$ fails in $\mathbf{A}$; then there are $a_1,\ldots,a_n \in A$ such that $p(a_1,\ldots,a_n) < 1$. If $\mathbf{C}$ is the subalgebra of $\mathbf{A}$ generated by $a_1,\ldots,a_n$, then $\mathbf{C} \nvDash p(x_1,\ldots,x_n) \approx 1$ and $\mathbf{C}$ cannot have index greater then n. Hence, $\mathbf{C} \in C$ and $\bigvee_{\mathbf{C}\in C} \mathbf{V}(\mathbf{C}) = \mathbf{V}(\mathbf{A})$; hence $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible. $\square$

So if $\mathbf{A}$ is a totally ordered pseudohoop and $\mathbf{V}(\mathbf{A})$ is strictly join irreducible in $\Lambda(\mathsf{RPsH})$, then $\mathbf{A}$ must have finite index. Chains of finite index have been studied in [5] in a different context (BL-algebras and basic hoops); we will show that the results therein extend to totally ordered pseudohoops in a straightforward way.

Let $\mathbf{P}$ be a poset; a **join dense completion** of $\mathbf{P}$ is a complete lattice $\mathbf{L}$ with an order embedding $\alpha : \mathbf{P} \longrightarrow \mathbf{L}$ such that $\alpha(\mathbf{P})$ is **join dense** in $\mathbf{L}$, i.e. for every $x \in L$, $x = \bigvee\{\alpha(p) : p \in P, \alpha(p) \leq x\}$. For any subvariety V of RPsH, we define

$$\mathcal{F}_{\mathsf{V}} = \{\mathbf{C} \in \mathsf{V} : \mathbf{C} \text{ is a chain of finite index}\}$$

and

$$\Phi_{\mathsf{V}} = \{\mathbf{V}(\mathbf{A}) : \mathbf{A} \in \mathcal{F}_{\mathsf{V}}\};$$

Φ_{V} is clearly a poset under inclusion and moreover:

THEOREM 6.6
For any subvariety V of RPsH, $\Lambda(\mathsf{V})$ is the join dense completion of Φ_{V}.

PROOF. Let $\mathbf{A}$ be a n-generated totally ordered pseudohoop in V; then, $\mathbf{A}$ has index $\leq n$ and so $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$. Since every subvariety W of V is generated by its totally ordered members and every algebra is in the variety generated by its finitely generated subalgebras, every subvariety W of V is the supremum of all varieties $\mathbf{V}(\mathbf{A})$, where $\mathbf{A} \in \mathcal{F}_{\mathsf{W}}$. We have thus shown that for every $\mathsf{W} \in \Lambda(\mathsf{V})$

$$\mathsf{W} = \bigvee\{\mathbf{V}(\mathbf{A}) : \mathbf{A} \in \mathcal{F}_{\mathsf{W}}\}.$$

This implies that $\Lambda(\mathsf{V})$ is the join dense completion of Φ_{V}. $\square$

COROLLARY 6.7
Let V be any variety of representable pseudohoops; if $\Lambda(\mathsf{V})$ is finite then V satisfies λ_n for some $n \in \mathbb{N}$, and for all $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$, $\mathbf{V}(\mathbf{A})$ is strictly join irreducible in $\Lambda(\mathsf{V})$.

PROOF. If $\Lambda(\mathsf{V})$ is finite then it must satisfy λ_n for some $n \in \mathbb{N}$; otherwise in V there would be chains of arbitrarily large index and they would generate infinitely many distinct varieties. Moreover, if $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$, then by Theorem 4.4 and Lemma 6.1, $\mathbf{V}(\mathbf{A})$ is join irreducible. Thus, since $\Lambda(\mathsf{V})$ is finite, $\mathbf{V}(\mathbf{A})$ is strictly join irreducible in $\Lambda(\mathsf{V})$. □

If $\mathbf{A}, \mathbf{B} \in \mathcal{F}_{\mathsf{RPsH}}$ we define $\mathbf{A} \preceq \mathbf{B}$ iff $\mathbf{HSP}_u(\mathbf{A}) \subseteq \mathbf{HSP}_u(\mathbf{B})$; then $\preceq$ is a preorder with associated equivalence relation $\equiv$ and by Jónsson Lemma $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$ if and only $\mathbf{A} \equiv \mathbf{B}$. It follows that to determine which varieties of representable pseudohoops generated by a chain of finite index are strictly join irreducible we may restrict to a representative of each $\equiv$-class. To do so, we must have some ways of classifying the pseudo Wajsberg components in the decomposition.

THEOREM 6.8
[13, 20] Let $\mathbf{A}$ be any totally ordered Wajsberg pseudohoop. If $\mathbf{A}$ is cancellative, then there is a totally ordered group $\mathbf{G}$ such that $\mathbf{A}$ is isomorphic with $\mathbf{G}^-$, the negative cone of $\mathbf{G}$. If $\mathbf{A}$ is bounded, then there is a totally ordered group $\mathbf{G}$ with strong unit u, such that $\mathbf{A} \cong \Gamma(\mathbf{G}, u)$.

$\Gamma(\mathbf{G}, u)$ is often called the *Mundici's functor*, see [20] for a definition. Since by [1, Lemma 4.5] each Wajsberg pseudohoop is either bounded or cancellative, the theorem gives a complete representation of totally ordered Wajsberg pseudohoops. However, this is less useful than it seems, at least in general. First, totally ordered groups are common (every free group is totally orderable); second, if $\mathbf{G}$ is a totally ordered group we have no idea in general of what $\mathbf{ISP}_u(\mathbf{G}^-)$ is (except for one case, see next section). We believe that a careful analysis of the techniques employed in [22] and [23] could lead to a better understanding of the problem; however, we will not explore this path here, leaving it open for future investigations. From now on, we will deal only with commutative structures, first because it is possible to prove meaningful results about them and second because they are more suited for the applications we have in mind.

7 Strictly join irreducible varieties of basic hoops

From now on, we will stay within the variety CIRL of commutative and integral residuated lattices; in this case $a\backslash b = b/a$ and we will use the symbol $a \to b$ for both. For any commutative residuated lattice, the concepts of prelinear and representable coincide (see for instance [1]). In what follows, we will focus on varieties of algebras that are very relevant in the realm of the algebraic semantics of fuzzy logics. We call a **basic hoop** a commutative and representable pseudohoop and a **Wajsberg hoop** a commutative Wajsberg pseudohoop. The connection with fuzzy logics is given by the fact that basic hoops are the 0-free subreducts of BL-algebras, equivalent algebraic semantics of Hájek Basic Logic, and Wajsberg hoops are the 0-free subreducts of Wajsberg algebras (which are term equivalent to MV-algebras), the equivalent algebraic semantics of Łukasiewicz logic [31].

By Theorem 6.3, any totally ordered basic hoop is the ordinal sum of Wajsberg hoops (and this is how the result was originally formulated in [4]); hence by the same argument as before (Theorem 6.5), a strictly join irreducible variety V of basic hoops must be $\mathbf{V}(\mathbf{A})$ for some chain of finite index $\mathbf{A}$. In the commutative case however we can choose representatives of the $\equiv$-classes in a very nice and uniform way.

We recall that given any totally ordered hoop of finite index $\mathbf{A} = \bigoplus_{i=1}^{n} \mathbf{A}_i$ we have [4]

$$\mathbf{HSP}_u(\mathbf{A}) = \bigcup_{i=1}^{n} (\bigoplus_{j=1}^{i-1} \mathbf{ISP}_u(\mathbf{A}_j) \oplus \mathbf{HSP}_u(\mathbf{A}_i)).$$

Therefore, if $\mathbf{A} = \bigoplus_{i=1}^{n} \mathbf{A}_i$ and $\mathbf{B} = \bigoplus_{j=1}^{m} \mathbf{B}_i$ belong to the class of totally ordered basic hoops of finite index $\mathcal{F}_{\mathsf{BH}}$, then $\mathbf{A} \equiv \mathbf{B}$ if and only if $n = m$, $\mathbf{ISP}_u(\mathbf{A}_i) = \mathbf{ISP}_u(\mathbf{B}_i)$ for $i = 1, \ldots, n-1$ and $\mathbf{HSP}_u(\mathbf{A}_n) = \mathbf{HSP}_u(\mathbf{B}_n)$.

First, we need to straighten out some technical details; we recall again that a **BL-algebra** is a zero-bounded basic hoop, i.e. a hoop with a nullary operation, denoted by 0, satisfying $0 \leq x$; a **Wajsberg algebra** is a zero-bounded Wajsberg hoop. It is clear that any zero-free subreduct of a BL-algebra is a basic hoop and that any bounded (i.e. with a smallest element) basic hoop is a reduct of a BL-algebra; moreover, the variety BH is the class of zero-free subreducts of BL-algebras [3]. The operator $\mathbf{ISP}_u$ on Wajsberg hoops has been extensively studied first in [29] and then in [4]; really the results in [29] are about MV-algebras but we can use them anyway and this is why. MV-algebras are termwise equivalent to Wajsberg algebras which in turn are polynomially equivalent to bounded Wajsberg hoops. The following lemma is easy to show and is often implicitly used in the study of Wajsberg hoops and Wajsberg algebras.

LEMMA 7.1
Let $\mathbf{O}$ be a class operator that is a composition of $\mathbf{I}, \mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_u$. Let $\mathbf{A}, \mathbf{B}$ be totally ordered Wajsberg algebras and $\mathbf{A}_0, \mathbf{B}_0$ be their Wajsberg hoop reducts. Then $\mathbf{O}(\mathbf{A}) \subseteq \mathbf{O}(\mathbf{B})$ if and only if $\mathbf{O}(\mathbf{A}_0) \subseteq \mathbf{O}(\mathbf{B}_0)$.

This allows us to consider totally ordered bounded Wajsberg hoops *as if they were* Wajsberg algebras. In other words, to check that $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{B})$ when they are both bounded totally ordered Wajsberg hoops is enough to check the equality by considering them as Wajsberg algebras. Since a totally ordered Wajsberg hoop is either bounded or cancellative [14], we can use Gispert's results in [29] for Wajsberg algebras and integrate them with the cancellative case.

In [5], an algorithm is given to choose a representative for each $\equiv$-class of finite ordinal sums of BL-algebras; while we reassure that the same argument goes through for basic hoops, in this case it is more useful to present a slightly finer classification, based on some ideas in [11]. Let $\mathbf{G}$ be a lattice ordered abelian group; by [38], if u is a strong unit of $\mathbf{G}$ we can construct a bounded Wajsberg hoop $\Gamma(\mathbf{G}, u) = \langle [0, u], \rightarrow, \cdot, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \rightarrow b = \max\{u - a + b, u\}$. Let now $\mathbb{Z} \times_l \mathbb{Z}$ denote the lexicographic product of two copies of $\mathbb{Z}$. In other words, the universe is the Cartesian product, the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $\mathbb{Z}$); then $\mathbb{Z} \times_l \mathbb{Z}$ is a totally ordered abelian group and we can apply Γ to it. We now define some useful Wajsberg chains:

- the finite Wajsberg chain with $n + 1$ elements $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$;
- the infinite Wajsberg chain $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, 0))$;
- the infinite Wajsberg chain $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, k))$;
- the infinite bounded Wajsberg chain $[0, 1]_{\mathbf{Ł}} = \Gamma(\mathbf{R}, 1)$, i.e. the real interval with operations induced by the *Wajsberg t-norm*, i.e. $xy = \max(x + y - 1, 0)$, $x \rightarrow y = \min(1 + x - y, 1)$;
- the infinite bounded Wajsberg chain $\mathbf{Q} = \Gamma(\mathbb{Q}, 1) = \mathbb{Q} \cap [0, 1]_{\mathbf{Ł}}$;
- if U is a set of positive integers we denote by $\mathbf{Q}(U)$ the subalgebra of $\mathbf{Q}$ generated by $\{\mathbf{Ł}_k : k \in U\}$, where $\mathbf{Ł}_k$ is seen as an algebra with universe $\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$;

- the unbounded Wajsberg chain $\mathbf{C}_\omega$ that has as universe the free group on one generator, where the product is the group product and $a^l \to a^m = a^{\max(m-l,0)}$;
- finally we fix once and for all an irrational number $\alpha \in [0,1]$ and we let X be the totally ordered dense subgroup of $\mathbb{R}$ generated by α and 1; then $\mathbf{S}_n = \Gamma(X, n)$.

The **radical** of a bounded Wajsberg hoop $\mathbf{A}$, in symbols $\mathrm{Rad}(\mathbf{A})$, is the intersection of the maximal congruence filters of $\mathbf{A}$; it is easy to see that $\mathrm{Rad}(\mathbf{A})$ is a cancellative basic subhoop of $\mathbf{A}$. We say that a bounded Wajsberg hoop $\mathbf{A}$ has **rank** n, if $\mathbf{A}/\mathrm{Rad}(\mathbf{A}) \cong \text{Ł}_n$; otherwise, it has infinite rank. For any bounded Wajsberg hoop $\mathbf{A}$, $d_{\mathbf{A}}$, called the **divisibility index**, is the maximum k such that Ł_k is embeddable in $\mathbf{A}$ if any, otherwise $d_{\mathbf{A}} = \infty$.

LEMMA 7.2

1. Ł_n has rank n and divisibility index n.
2. For any $k \geq 0$, $\text{Ł}_{n,k}$ has rank n and $d_{\text{Ł}_{n,k}} = \gcd(n,k)$; in particular $d_{\text{Ł}_n^\infty} = n$.
3. $\mathbf{S}_n$ has infinite rank and $\text{Ł}_k \in \mathbf{S}(\mathbf{S}_n)$ if and only if $k \mid n$; hence $d_{\mathbf{S}_n} = n$.
4. If $\mathbf{A}$ is a nontrivial totally ordered cancellative hoop then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{C}_\omega)$.
5. If $\mathbf{A}$ is a non-simple bounded Wajsberg chain of finite rank n, then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\text{Ł}_n^\infty)$ if and only if $d_{\mathbf{A}} = n$.
6. If $\mathbf{A}$ is a non-simple bounded Wajsberg chain of finite rank k, $d_{\mathbf{A}}$ divides k, and $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\text{Ł}_{k,d_{\mathbf{A}}})$.
7. If $\mathbf{A}$ is a Wajsberg chain of infinite rank then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{S}_n)$ if and only if $d_{\mathbf{A}} = n$.
8. If $\mathbf{A}$ is a bounded Wajsberg chain of infinite rank and $d_{\mathbf{A}} = \infty$, then $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(\mathbf{Q}(U))$ where $U = \{k : \text{Ł}_k \in \mathbf{S}(\mathbf{A})\}$.
9. If $\mathbf{A}$ is an infinite subdirectly irreducible Wajsberg algebra of finite rank k, then $\mathbf{HSP}_u(\mathbf{A}) = \mathbf{HSP}_u(\text{Ł}_{k,k})$.
10. If $\mathbf{A}$ is a subdirectly irreducible Wajsberg algebra of infinite rank, then $\mathbf{HSP}_u(\mathbf{A}) = \mathbf{HSP}_u([0,1]_{\text{Ł}})$.

PROOF. Claims (1) and (3) are obvious, while (2), (4) and (5) are in [4]. The proofs of (6), (7) and (8) are in [29].

For (9), note that both $\mathbf{A}$ and $\text{Ł}_{k,k}$ are subdirectly irreducible and generate the same variety, namely the variety of Wajsberg algebras $\mathbf{A}$ such that $d_{\mathbf{A}}$ divides k (see [4, Lemmas 6.1 and 6.3]). For (10), it is well known that any two algebras of infinite rank generate the entire variety of Wajsberg algebras (see for instance [6, Theorem 2.5]). Since any variety of Wajsberg hoops is congruence distributive, (9) and (10) follow from Jónssons's Lemma. □

THEOREM 7.3
Let $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ be any totally ordered basic hoop of finite index; then there is a canonical way of choosing $\mathbf{B} \in \mathcal{F}_{\mathsf{BH}}$, unique up to $\equiv$, such that $\mathbf{A} \equiv \mathbf{B}$. More precisely, if $\mathbf{B} = \bigoplus_{i=1}^n \mathbf{B}_i$, then

$$\mathbf{B}_i = \begin{cases} \mathbf{C}_\omega, & \text{if } \mathbf{A}_i \text{ is cancellative;} \\ \text{Ł}_n, & \text{if } \mathbf{A}_i \cong \text{Ł}_n; \\ \text{Ł}_{k,d}, & \text{if } \mathbf{A}_i \text{ is infinite, has rank } k \text{ and } d = d_{\mathbf{A}_i}; \\ \mathbf{Q}(U), & \text{if } \mathbf{A}_i \text{ has infinite rank and} \\ & U = \{k : \text{Ł}_k \in \mathbf{S}(\mathbf{A})\} \text{ is infinite;} \\ \mathbf{S}_k, & \text{if } \mathbf{A}_i \text{ has infinite rank and } d_{\mathbf{A}_i} = k. \end{cases}$$

for $i < n$ and

$$\mathbf{B}_n = \begin{cases} \text{Ł}_n, & \text{if } \mathbf{A}_n \cong \text{Ł}_n; \\ \text{Ł}_{k,k}, & \text{if } \mathbf{A}_n \text{ is infinite and has rank } k; \\ \mathbf{C}_\omega, & \text{if } \mathbf{A}_n \text{ is cancellative;} \\ [0,1]_{\text{Ł}}, & \text{if } \mathbf{A}_n \text{ has infinite rank.} \end{cases}$$

PROOF. That $\mathbf{HSP}_u(\mathbf{A}) = \mathbf{HSP}_u(\mathbf{B})$ follows from Lemma 7.2 and the description of $\mathbf{HSP}_u(\mathbf{A})$. To prove uniqueness, let $\mathbf{C} = \bigoplus_{i=1}^m \mathbf{C}_i \in \mathcal{F}_{\mathsf{BH}}$; if $m \neq n$, then $\mathbf{ISP}_u(\mathbf{B}) \neq \mathbf{ISP}_u(\mathbf{C})$. If they have the same index n and they are different, then they must differ on some Wajsberg component. The first possibility is that there exists an $i < n$ such that

- $\mathbf{B}_i$ is cancellative and $\mathbf{C}_i$ is not (or vice versa);
- for some k, $\mathbf{B}_i$ has at most k elements and $\mathbf{C}_i$ has more (or vice versa);
- Ł_k is embeddable in $\mathbf{B}_i$ and it is not embeddable in $\mathbf{C}_i$ (or vice versa);
- $\{k : \text{Ł}_k$ is embeddable in $\mathbf{B}_i\}$ is infinite and $\{k : \text{Ł}_k$ is embeddable in $\mathbf{C}_i\}$ is not (or vice versa);
- $\mathbf{B}_i$ and $\mathbf{C}_i$ have different divisibility index.

But it is easy to see that any of this conditions implies $\mathbf{ISP}_u(\mathbf{B}_i) \neq \mathbf{ISP}_u(\mathbf{C}_i)$, so $\mathbf{HSP}_u(\mathbf{B}) \neq \mathbf{HSP}_u(\mathbf{C})$. If this is not the case, then $\mathbf{B}_n$ and $\mathbf{C}_n$ must differ; a similar argument shows that in this case $\mathbf{HSP}_u(\mathbf{B}_n) \neq \mathbf{HSP}_u(\mathbf{C}_n)$. Hence, $\mathbf{B}$ is unique up to $\equiv$. □

Now we can characterize the subvarieties of BH that are strictly join irreducible (the reader can compare this with the analogous result for BL-algebras proved in [11]).

THEOREM 7.4
A subvariety V of BH is strictly join irreducible if and only if it is equal to $\mathbf{V}(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$ such that its Wajsberg components are either cancellative or are bounded and have finite rank.

PROOF. Let us prove necessity first. We can work modulo $\equiv$, i.e. we may suppose that $\mathbf{A} = \bigoplus_{i=1}^n \mathbf{A}_i$ has the 'canonical form' suggested by Theorem 7.3. We show that if one of the (canonically chosen) components is not either cancellative or bounded with finite rank then $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible. Notice that a Wajsberg chain that is not cancellative nor bounded with finite rank is bounded with infinite rank, since unbounded Wajsberg chains are cancellative. We start from the n-th component, $\mathbf{A}_n$, where the only option of it being bounded with infinite rank is the case where $\mathbf{A}_n = [0,1]_{\text{Ł}}$. Suppose then that $\mathbf{A}_n = [0,1]_{\text{Ł}}$ and for any $k \in \mathbb{N}$ let $\mathbf{C}_k = \bigoplus_{i=1}^{n-1} \mathbf{A}_i \oplus \text{Ł}_k$; then clearly $\mathbf{C}_k \not\equiv \mathbf{A}$ for all k. However, $\mathbf{HSP}_u(\{\mathbf{C}_k : k \in \mathbb{N}\} = \mathbf{HSP}_u(\mathbf{A})$ since it is well known that $\mathbf{HSP}_u(\{\text{Ł}_k : k \in \mathbb{N}\}) = \mathbf{HSP}_u([0,1])$. Hence, $\bigvee_{k \in \mathbb{N}} \mathbf{V}(\mathbf{C}_k) = \mathbf{V}(\mathbf{A})$ and $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible.

Next we suppose that for some $i < n$, $\mathbf{A}_i$ is bounded and has infinite rank; modulo $\equiv$ either $\mathbf{A}_i = \mathbf{Q}(U)$, where $U = \{k : \text{Ł}_k \in \mathbf{S}(\mathbf{A})\}$ is infinite, or else $\mathbf{A} = \mathbf{S}_m$ for some m. In the first case, for $k \in U$ we let $\mathbf{C}_k$ to be the chain in which we have replaced the i-th component of $\mathbf{A}$ with Ł_k; clearly $\mathbf{C}_k \not\equiv \mathbf{A}$ for all k. However, since $\mathbf{Q}(U)$ is generated by $\{\mathbf{L}_k : k \in U\}$ any equation that fails in $\mathbf{A}$ must fail in $\mathbf{C}_k$ for some k; this proves that $\bigvee_{k \in U} \mathbf{V}(\mathbf{C}_k) = \mathbf{V}(\mathbf{A})$ and so $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible.

For the second case, we let $\mathbf{C}_k$ to be the chain in which we have replaced the i-th component of $\mathbf{A}$ with $\text{Ł}_{m,k}$; again clearly $\mathbf{C}_k \not\equiv \mathbf{A}$ for all k and a very similar argument to the one above shows that $\bigvee_k \mathbf{V}(\mathbf{C}_k) = \mathbf{V}(\mathbf{A})$ (but see [11, Proposition 8] for details). Hence also in this case $\mathbf{V}(\mathbf{A})$ is not strictly join irreducible and the proof of necessity is finished.

Let $\mathsf{V} = \mathbf{V}(\mathbf{A})$ where $\mathbf{A}$ satisfies the hypotheses of the theorem; since $\mathbf{A}$ is well-connected by Lemma 6.1, then V is join irreducible in $\Lambda(\mathsf{B}H)$ by Theorem 4.4. Therefore, if we show that $\Lambda(\mathsf{V})$ is

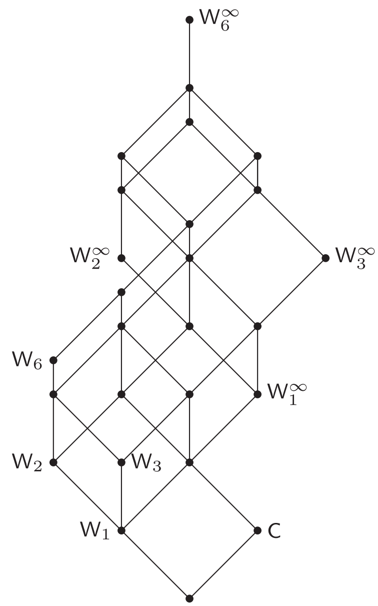

FIGURE 1 $\Lambda(\mathsf{W}_6^\infty)$

finite, then the thesis will hold. Now by Theorem 6.6 it is enough to show that Φ_V is finite, i.e. that $\mathcal{F}_\mathsf{V}/\equiv$ is a finite set. Now any 'canonical' representative in $\mathcal{F}_\mathsf{V}/\equiv$ must be a chain of index $\leq$ than the index of $\mathbf{A}$ and whose components are either $\mathbf{C}_\omega$ or whose rank cannot exceed the maximum rank of the non cancellative components of $\mathbf{A}$. Clearly, there are only finitely many choices, so $\mathcal{F}_\mathsf{V}/\equiv$ is finite as wished. □

COROLLARY 7.5
Let V be any variety of basic hoops; then the following are equivalent:

1. $\Lambda(\mathsf{V})$ is finite;
2. V satisfies λ_n for some $n \in \mathbb{N}$, and for all $\mathbf{A} \in \mathcal{F}_\mathsf{V}$, $\mathbf{V}(\mathbf{A})$ is strictly join irreducible in $\Lambda(\mathsf{V})$.

8 Some examples

The only subvariety of BH satisfying λ_1 is of course the variety WH of Wajsberg hoops; $\Lambda(\mathsf{WH})$ has been totally described in [6]. It turns out that a proper subvariety of $\mathsf{W}H$ is generated by finitely many totally ordered hoops among $\mathbf{C}_\omega$ and $\{\mathbf{Ł}_n, \mathbf{Ł}_n^\infty : n \in \mathbb{N}\}$. This (by Jónnson Lemma) implies that for any proper subvariety V of WH, $\Lambda(\mathsf{V})$ is finite and hence by Corollary 7.5, V is strictly join irreducible if and only if it is join irreducible if and only if it is generated by a single totally ordered Wajsberg hoop that is either cancellative or has finite rank (and so it is either $\mathbf{Ł}_n$ or $\mathbf{Ł}_n^\infty$ for some n).

Let $\mathsf{C} = \mathbf{V}(\mathbf{C}_\omega)$ be the variety of cancellative hoops and let for $n \in \mathbb{N}$, $\mathsf{W}_n = \mathbf{V}(\mathbf{Ł}_n)$ and $\mathsf{W}_n^\infty = \mathbf{V}(\mathbf{Ł}_n^\infty)$. In Figure 1, we see a picture of $\Lambda(\mathsf{W}_6^\infty)$ where we have labelled the subdirectly irreducible elements.

Now we would like to examine a larger lattice of subvarieties and we need some definitions. A **Gödel hoop** is an idempotent basic hoop. Gödel hoops are termwise equivalent to *relative Stone* *lattices* [32] and the variety is denote by GH; it is well known that the only subdirectly irreducible in GH is $\mathbf{2}$ (the two element residuated lattice) and that $\Lambda(\mathsf{GH})$ is a chain of type $\omega+1$. More precisely, if $\mathbf{G}_n = \bigoplus_{i=1}^n \mathbf{2}$, then the proper subvarieties of GH are $\mathsf{G}_n = \mathbf{V}(\mathbf{G}_n)$.

A **product hoop** is a basic hoop satisfying

$$(y \to z) \vee ((y \to xy) \to x) \approx 1.$$

The variety PH of product hoops has been examined in [3]; it turns out that $\Lambda(\mathsf{PH})$ has five elements and the proper nontrivial subvarieties are C, G_1 and $\mathsf{C} \vee \mathsf{G}_1$. Moreover, $\mathsf{PH} = \mathbf{V}(\mathbf{2} \oplus \mathbf{C}_\omega)$ so it is strictly join irreducible in $\Lambda(\mathsf{BH})$. Let $\mathsf{L} = \mathsf{WH} \vee \mathsf{PH} \vee \mathsf{GH}$; we want to describe $\Lambda(\mathsf{L})$.

Since basic hoops are congruence distributive, $\Lambda(\mathsf{BH})$ is distributive and since it is also dually algebraic it satisfies the infinite distributive law

$$x \vee \bigwedge_{i \in I} y_i \approx \bigwedge_{i \in I} (x \vee y_i).$$

Let $\mathbf{L}$ be any lattice; an **interval** is the set $[a,b] = \{c : a \leq c \leq b\}$. An interval $[a,b]$ is an *upper transpose* of an interval $[c,d]$ (and also $[c,d]$ is a *lower transpose* of $[a,b]$) if $a \vee d = b$ and $a \wedge d = c$. Two intervals $[a,b]$, $[c,d]$ are *n-step projective* if there exists intervals $[a,b] = [x_0,y_0],[x_1,y_1],\ldots,[x_n,y_n] = [c,d]$ such that $[x_i,y_i]$ is an upper or lower transpose of $[x_{i+1},y_{i+1}]$. Two intervals are projective if they are n-step projective for some n. It is a nontrivial fact that in a distributive lattice 'projectivity = 2-step projectivity'. Moreover, we say that b is a *cover* of a and we write $a \prec b$ if $[a,b] = \{a,b\}$.

Let us apply these facts to L. First observe that

$$\mathsf{WH} \cap \mathsf{PH} = \mathsf{C} \vee \mathsf{G}_1 = \mathbf{V}(\mathbf{2}, \mathbf{C}_\omega);$$

to simplify the notation we will call that variety CPH. Since $\mathsf{WH} \cap \mathsf{PH} = \mathsf{CPH}$ and $\mathsf{CPH} \prec \mathsf{PH}$, we have that $\mathsf{WH} \prec \mathsf{WH} \vee \mathsf{PH}$ and $[\mathsf{CPH}, \mathsf{WH}] \cong [\mathsf{PH}, \mathsf{WH} \vee \mathsf{PH}]$. The same conclusion holds if we substitute WH with any variety of Wajsberg hoops W.

LEMMA 8.1
Let W be any variety of Wajsberg hoops. If $\mathsf{V} \in [\mathsf{PH}, \mathsf{W} \vee \mathsf{PH}]$ then there exists a variety $\mathsf{W}' \subseteq \mathsf{W}$ such that $\mathsf{W}' \vee \mathsf{PH} = \mathsf{V}$. If $\mathsf{V} \in [\mathsf{CPH}, \mathsf{W} \vee \mathsf{PH}]$, then either $\mathsf{PH} \leq \mathsf{V}$ or $\mathsf{V} \leq \mathsf{WH}$.

PROOF. Let $\mathsf{W}' = \mathsf{WH} \cap \mathsf{V}$; then

$$\begin{aligned} \mathsf{W}' \vee \mathsf{PH} &= (\mathsf{WH} \cap \mathsf{V}) \vee \mathsf{PH} \\ &= (\mathsf{WH} \vee \mathsf{PH}) \cap (\mathsf{V} \vee \mathsf{PH}) = (\mathsf{WH} \vee \mathsf{PH}) \cap \mathsf{V} = \mathsf{V}. \end{aligned}$$

For the second statement, let $\mathsf{W}' = \mathsf{WH} \cap \mathsf{V}$. By distributivity, we see at once that $[\mathsf{W}', \mathsf{W}' \vee \mathsf{PH}]$ and $[\mathsf{CPH}, \mathsf{PH}]$ are isomorphic. It follows that $\mathsf{W}' \prec \mathsf{W}' \vee \mathsf{PH}$ and hence either $\mathsf{V} = \mathsf{W}'$ or $\mathsf{V} = \mathsf{W}' \vee \mathsf{PH}$. The former implies $\mathsf{V} \leq \mathsf{WH}$ and from the latter

$$\begin{aligned} \mathsf{V} = \mathsf{W}' \vee \mathsf{PH} &= (\mathsf{WH} \cap \mathsf{V}) \vee \mathsf{PH} \\ &= (\mathsf{WH} \cap \mathsf{PH}) \vee (\mathsf{V} \vee \mathsf{PH}) = \mathsf{CPH} \vee (\mathsf{V} \vee \mathsf{PH}) = \mathsf{V} \vee \mathsf{PH}. \end{aligned}$$

Thus, $\mathsf{PH} \leq \mathsf{V}$. □

The same argument can be applied as it is to the intervals $[\mathsf{W}_2, \mathsf{GH}]$, $[\mathsf{CPH}, \mathsf{CPH} \vee \mathsf{GH}]$ and $[\mathsf{PH}, \mathsf{PH} \vee \mathsf{GH}]$. Then to $[\mathsf{CPH}, \mathsf{GH} \vee \mathsf{PH}]$ and $[\mathsf{WH}, \mathsf{L}]$, and finally to $[\mathsf{CPH}, \mathsf{WH} \vee \mathsf{PH}]$ and $[\mathsf{GH}, \mathsf{L}]$.

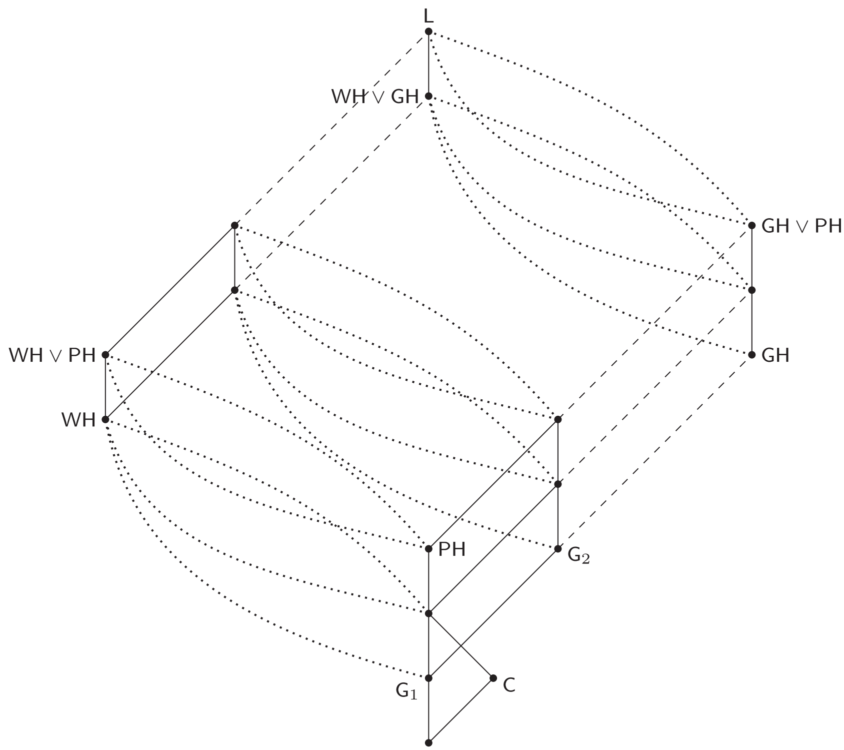

FIGURE 2 The lattice of subvarieties of L

Repeating this process for any variety of Wajsberg hoops or Gödel hoops, we obtain a picture of the lattice of subvarieties of L (Figure 2).

Clearly Figure 2 is rather sketchy, since all the equal ovals are isomorphic intervals and they are all connected. However, using the same techniques as in [6], it is not hard to prove the following:

LEMMA 8.2
Let V be a subvariety of L not above WH or GH. Then there are finite $I, J \subseteq \mathbb{N}$ (possibly empty) and an $n \in \mathbb{N}$ (possibly 0) such that

$$\mathsf{V} = \mathbf{V}(\{\mathbf{G}_n, \mathbf{2} \oplus \mathbf{C}_\omega, \mathbf{C}_i, \mathbf{C}_j^\infty : i \in I, j \in J\}).$$

It follows that if $\mathsf{V} \subseteq \mathsf{L}$ and V is not above WH or GH, then $\Lambda(\mathsf{V})$ is finite, so Corollary 7.5 applies and then all the subvarieties generated by a chain of finite index are strictly join irreducible. Note that GH and WH are join irreducible but not strictly join irreducible in $\Lambda(\mathsf{V})$, since WH is generated by any Wajsberg chain of infinite rank and GH is generated by the chain $[0,1]_\mathsf{G}$, where $xy = x \wedge y$ in the usual ordering.

9 Generalized rotations

In this section, we are going to use the results contained in [7] to lift information about strictly join irreducible varieties of basic hoops to some subvarieties of MTL-algebras. In particular, starting from a variety of basic hoops (or more in general, of representable CIRLs), we can use their *generalized rotation* (introduced in [16]) to generate a variety of MTL-algebras, and we are able to describe the relations between the respective lattices of subvarieties. Let us be more precise.

Consider a class of representable commutative residuated integral lattices (RCIRLs for short) K, a natural number $n \geq 2$, and a **term defined rotation**, i.e. a unary term δ in the language of residuated lattices that on every algebra in $\mathbf{A} \in \mathsf{K}$ defines a lattice homomorphism that is also a nucleus (a closure operator satisfying $\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$ for all $x, y \in A$). It is easy to see that $\delta(x) = x$ and $\delta(x) = 1$ are both term defined rotations on any class of CIRLs. Then for every algebra $\mathbf{A} \in \mathsf{K}$ we can consider its **generalized n-rotation**, whose definition we write here for the reader's convenience (for more details see [16]).

The lattice structure is given by the disjoint union of $\mathbf{A}$ and $\delta[A]' = \{(\delta(a))' : a \in A\}$ with the dualized order:

$$(1)\ a' < b, \text{ and } (2)\ a' \leq b' \text{ iff } b \leq a$$

to which we add a chain of $n-2$ elements strictly between $\mathbf{A}$ and $\delta[A]'$. Intuitively, the resulting algebra has as skeleton a Łukasiewicz chain of n elements, $\mathbf{Ł}_{n-1}$, where the top is 1 and the bottom is $0 = 1'$. Let us call the elements of the Łukasiewicz chain:

$$0 = l_0 < l_1 < \ldots < l_{n-2} < l_{n-1} = 1.$$

$\mathbf{A}$ is a substructure, the products in $\delta[A]'$ are all defined to be the bottom element $0 = 1'$. Moreover,

$$a \cdot b' = (a \to b)', \quad a \to b' = (\delta(b \cdot a))', \quad a' \to b' = b \to a$$
$$al_i = l_i = l_i a, \quad b' l_i = 0 = l_i b' \quad (\text{for } i \notin \{0, n-1\}).$$

The obtained structure, which we shall denote with $\mathbf{A}^{\delta_n}$, is an MTL-algebra. It is relevant to notice that the term defined rotation δ coincides exactly with the double negation of $\mathbf{A}^{\delta_n}$ when restricted to $\mathbf{A}$. Moreover, $\mathbf{A}$ is the radical of $\mathbf{A}^{\delta_n}$ (and its only maximal congruence filter).

We can now define subvarieties of MTL in the following way: let K be any class of representable CIRLs, $n \geq 2$, δ a term defined rotation for K, and we define

$$\mathsf{K}^{\delta_n} = \{\mathbf{A}^{\delta_m} : m-1 \mid n-1, \mathbf{A} \in \mathsf{K}\}.$$

From now on, we will write δ for δ_2 (this coincides with the construction in [12]). Whenever we consider a variety of CIRLs V, we will write V^{δ_n} for the generated variety $\mathbf{V}(\mathsf{V}^{\delta_n})$. Moreover, we will write V_t for the class of totally ordered members in V.

When we use δ as the identity map, we find interesting examples where the algebras obtained by rotations are involutive. We shall call this particular instance of the construction a **generalized disconnected n-rotation**. For instance, with V the variety of Gödel hoops, its generalized disconnected 3-rotation V^{δ_3} is the variety of nilpotent minimum algebras NM. Starting with a variety of basic hoops V, the only disconnected n-rotation that is a subvariety of BL-algebras is the variety generated by perfect MV-algebras, given by 2-rotations of cancellative hoops [7, Proposition 3.4]. Thus with this particular construction we are able to move from basic hoops to MTL-algebras. Notice that if instead we use the term defined rotation $\delta(x) = 1$ on varieties of basic hoops, then we obtain varieties of BL-algebras. In particular, for $n = 2$, we get the variety of SBL of pseudocomplemented (or, equivalently, Stonean) BL-algebras. Let us recall some results about the lattice of subvarieties of term-defined rotations obtained in [7].

LEMMA 9.1 ([7]).
Let K be a class of RCIRLs, δ a term-defined rotation for K and $n \geq 2$. Then the following are equivalent:

1. $\mathbf{A} \in \mathbf{HSP}_u(\mathsf{K})$
2. $\mathbf{A}^{\delta_m} \in \mathbf{HSP}_u(\mathsf{K}^{\delta_n})$ for every $m : m-1 \mid n-1$.

In the case where K consists of a single algebra, we can actually obtain some extra information. In what follows, when we consider a term-defined rotation δ, we assume it to be a term-defined rotation for the algebras involved.

LEMMA 9.2
Let δ be a term-defined rotation, $\mathbf{B} \in \mathsf{RCIRL}$, $n \geq 2$ and $m-1 \mid n-1$. Then $\mathbf{D}^{\delta_m} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_n})$ if and only if $\mathbf{D}^{\delta_n} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_n})$.

PROOF. The right-to-left direction is a consequence of Lemma 9.1. Now suppose $\mathbf{D}^{\delta_m} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_n})$ with $m-1 \mid n-1$; we want to show that also $\mathbf{D}^{\delta_n} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_n})$. Notice that $\mathbf{B}^{\delta_n}$ is directly indecomposable (see [16]), which is a first-order property in $\mathsf{F}L_{ew}$-algebras, since it corresponds to the fact that the only complemented elements are 0 and 1, which can clearly be expressed with a first order sentence. Thus, all ultrapowers of $\mathbf{B}^{\delta_n}$ are also directly indecomposable. All directly indecomposable algebras in a variety generated by generalized rotations are generalized rotations themselves [16, Theorem 4.8]. Moreover, having an MV-skeleton of n elements can also be expressed as a first-order property, using the term γ_n defined in [16]. In generalized rotations, the term γ_n is the identity map only on the elements of the MV-skeleton, evaluates to 1 in the elements of the radical and evaluates to 0 in the elements of the coradical. Therefore, if one wishes to describe with a first-order sentence that a generalized rotation has an MV-skeleton of n elements, one can say that there exist n elements $x_1 \dots x_n$, pairwise different, such that $\gamma_n(x_i) = x_i$ for $i = 1 \dots k$, and that are the only ones with such property. So we can conclude that any ultrapower of $\mathbf{B}^{\delta_n}$ is a generalized rotation $\mathbf{U}^{\delta_n}$, for some particular $\mathbf{U} \in \mathsf{RCIRL}$. Thus, $\mathbf{D}^{\delta_m} \in \mathbf{HS}(\mathbf{U}^{\delta_n})$ and of course $\mathbf{D}^{\delta_m} \in \mathbf{HS}(\mathbf{U}^{\delta_m})$. Thus, via Lemma 9.1, $\mathbf{D} \in \mathbf{HS}(\mathbf{U})$ and then $\mathbf{D}^{\delta_n} \in \mathbf{HS}(\mathbf{U}^{\delta_n})$; this implies $\mathbf{D}^{\delta_n} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_n})$ and the proof is completed. □

The following is a consequence of Lemma 9.1.

PROPOSITION 9.3
[7] Let V, W be subvarieties of RCIRL, δ a term-defined rotation and $n \geq 2$. Then $\mathsf{V}^{\delta_m} \subseteq \mathsf{W}^{\delta_n}$ iff $m-1 \mid n-1$ and $\mathsf{V} \subseteq \mathsf{W}$.

As a special case notice that, given a subvariety K of RCIRL and δ a term defined rotation, the lattice of subvarieties of K and the lattice of subvarieties of K^{δ} are isomorphic. In the case of varieties obtained as rotations of basic hoops, we can also say something more.

LEMMA 9.4
For any subvariety V of BH, $\Lambda(\mathsf{V}^{\delta_n})$ is the join dense completion of $\Phi_{\mathsf{V}}^{\delta_n} = \{\mathbf{V}(\mathbf{A}^{\delta_m}) : \mathbf{A} \in \mathcal{F}_{\mathsf{V}}, m-1 \mid n-1\}$.

PROOF. Let $\mathbf{B}$ be a totally ordered algebra in V^{δ_n}, then $\mathbf{B}$ is isomorphic to a generalized rotation $\mathbf{A}^{\delta_m}$ with $m-1 \mid n-1$. Moreover, if $\mathbf{B}$ is finitely generated, then $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$, since $\mathbf{B}$ is completely determined by its MV-skeleton $\mathbf{Ł}_{m-1}$ and the totally ordered basic hoop that is its radical $\mathbf{A}$ that cannot have infinitely many sum-irreducible components if it is finitely generated. Then the proof continues as in Lemma 6.6, and we can conclude that $\Lambda(\mathsf{V}^{\delta_n})$ is the join dense completion of $\Phi_{\mathsf{V}}^{\delta_n}$. □

LEMMA 9.5
Let $\mathsf{V}, \mathsf{W}_1 \dots \mathsf{W}_k$ be subvarieties of RCIRL, δ a term-defined rotation and $n \geq 2$. Then

$$\mathsf{V} = \bigvee_{i \in I} \mathsf{W}_i \quad \text{if and only if} \quad \mathsf{V}^{\delta_n} = \bigvee_{i \in I} \mathsf{W}_i^{\delta_n}.$$

PROOF. First, notice that given a RCIRL $\mathbf{A}$, $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}^{\delta_m}$ is subdirectly irreducible. Indeed, in the generalized rotation construction $\mathbf{A}$ is a congruence filter of $\mathbf{A}^{\delta_m}$; thus, the former has a unique atomic congruence filter if and only if the latter does.

Let us show the left-to-right direction first, thus we suppose that $\mathsf{V} = \bigvee_{i \in I} \mathsf{W}_i$. Now, as a consequence of Lemma 9.1, V^{δ_n} is the variety generated by algebras in the set $\{\mathbf{A}^{\delta_n} : \mathbf{A} \in \mathsf{V}, \mathbf{A}$ subdirectly irreducible$\}$. Notice that in particular all the algebras $\mathbf{A}^{\delta_m}$ for $m-1 \mid n-1$ are subalgebras of $\mathbf{A}^{\delta_n}$. Now we consider a subdirectly irreducible algebra $\mathbf{A} \in \mathsf{V}$, and we call $\mathsf{siW}_i^{\delta_n}$ and siW_i the class of subdirectly irreducible algebras in $\mathsf{W}_i^{\delta_n}$ and W_i, respectively. We get

$$\begin{aligned}
\mathbf{A}^{\delta_n} \in \mathsf{V}^{\delta_n} \quad & \Leftrightarrow \quad \mathbf{A}^{\delta_m} \in \mathsf{V}^{\delta_n} \text{ for all } m-1 \mid n-1 \\
& \Leftrightarrow \quad \mathbf{A} \in \mathsf{V} \\
& \Leftrightarrow \quad \mathbf{A} \in \bigvee_{i \in I} \mathsf{W}_i \\
& \Leftrightarrow \quad \mathbf{A} \in \mathbf{HSP}_u(\bigcup_{i \in I} \mathsf{siW}_i) \\
& \Leftrightarrow \quad \mathbf{A}^{\delta_m} \in \mathbf{HSP}_u(\bigcup_{i \in I} \mathsf{siW}_i^{\delta_n}) \text{ for all } m-1 \mid n-1 \\
& \Leftrightarrow \quad \mathbf{A}^{\delta_m} \in \bigvee_{i \in I} \mathsf{W}_i^{\delta_n} \text{ for all } m-1 \mid n-1 \\
& \Leftrightarrow \quad \mathbf{A}^{\delta_n} \in \bigvee_{i \in I} \mathsf{W}_i^{\delta_n}.
\end{aligned}$$

The first and last equivalence follow from the fact that $\mathbf{A}^{\delta_m}$ is a subalgebra of $\mathbf{A}^{\delta_n}$ iff $m-1 \mid n-1$; the second and the third to last equivalences follow from Lemma 9.1; the third one follows by hypothesis. So it remains to show the fourth equivalence (the proof of the second to last one being analogous). Since $\mathbf{A}$ is subdirectly irreducible and residuated lattices are congruence distributive, by Jònnson's Lemma $\mathbf{A} \in \mathbf{V}(\mathsf{K})$ iff $\mathbf{A} \in \mathbf{HSP}_u(\mathsf{K})$ for any class of algebras K. The equivalence then follows from the fact that $\bigcup_{i \in I} \mathsf{siW}_i$ is a set of generators for $\bigvee_{i \in I} \mathsf{W}_i$. Indeed, clearly $\mathbf{V}(\bigcup_{i \in I} \mathsf{siW}_i) \subseteq \bigvee_{i \in I} \mathsf{W}_i$. Moreover, for all $i \in I$, $\mathsf{W}_i \subseteq \mathbf{V}(\bigcup_{i=1\dots k} \mathsf{siW}_i)$ thus $\mathbf{V}(\bigcup_{i \in I} \mathsf{siW}_i) = \bigvee_{i \in I} \mathsf{W}_i$.

We have showed that $\mathsf{V}^{\delta_n} = \bigvee_{i \in I} \mathsf{W}_i^{\delta_n}$. The proof of the right-to-left direction uses the very same equivalences and is left to the reader. □

Now we can prove the following.

PROPOSITION 9.6
Let $\mathsf{V} \subseteq \mathsf{W}$ be subvarieties of BH, δ a term defined rotation and $m, n \geq 2$. Then V^{δ_m} is strictly join irreducible in the lattice of subvarieties of W^{δ_n} iff V is strictly join irreducible in the lattice of subvarieties of W and $m-1 \mid n-1$.

PROOF. The left-to-right direction is relatively easy to see by contraposition. Indeed, first of all if $m-1$ does not divide $n-1$ then V^{δ_m} is not a subvariety of W^{δ_n} by Proposition 9.3. Moreover, if $m-1$ divides $n-1$ but V is not strictly join irreducible, then $\mathsf{V} = \bigvee_{i \in I} \mathsf{W}_i$ for some proper subvarieties $\mathsf{W}_i, i \in I$, but then $\mathsf{V}^{\delta_m} = \bigvee_{i \in I} \mathsf{W}_i^{\delta_m}$ by Lemma 9.5. Next $\mathsf{W}_i^{\delta_m}$ is a subvariety of V^{δ_m} by

Proposition 9.3. It is also proper subvariety: let $\mathbf{A} \in \mathsf{V}, \mathbf{A} \notin \mathsf{W}_i$. Then by Lemmas 9.1 and 9.2, $\mathbf{A}^{\delta_m} \in \mathsf{V}^{\delta_m}$ but $\mathbf{A}^{\delta_m} \notin \mathsf{W}_i^{\delta_m}$. Thus, V^{δ_m} is not strictly join irreducible.

Suppose now that V is strictly join irreducible and $m-1 \mid n-1$. Then $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some chain $\mathbf{A}$ of finite order whose Wajsberg components are either cancellative or are bounded and have finite rank (Theorem 7.4). We can show that $\mathsf{V}^{\delta_m} = \mathbf{V}(\mathbf{A}^{\delta_m})$. One inclusion is obvious, while for the other one we use the fact that subdirectly irreducible algebras in V^{δ_m} are generalized rotations of subdirectly irreducible algebras in V. So let us take a subdirectly irreducible algebra $\mathbf{C}^{\delta_k} \in \mathsf{V}^{\delta_m}$, with $(k-1) \mid (m-1)$. Then $\mathbf{C}$ is a subdirectly irreducible algebra in $\mathsf{V} = \mathbf{V}(\mathbf{A})$, which implies that $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{A})$. Via Lemma 9.1, this implies that $\mathbf{C}^{\delta_k} \in \mathbf{HSP}_u(\mathbf{A}^{\delta_m})$, and thus every subdirectly irreducible in V^{δ_m} belongs to $\mathbf{V}(\mathbf{A}^{\delta_m})$, which completes the proof of the equality $\mathsf{V}^{\delta_m} = \mathbf{V}(\mathbf{A}^{\delta_m})$.

We have shown that V^{δ_m} is generated by a totally ordered algebra, which by Corollary 6.1 is also well-connected; by applying Theorem 4.4, we get that V^{δ_m} is join irreducible. We now show that $\Lambda(\mathsf{V}^{\delta_m})$ is finite, which will settle the proof. In particular, given Lemma 9.4, it suffices to show that $\Phi_{\mathsf{V}}^{\delta_n}$ is finite. In order to prove this, we will show that given $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{F}_{\mathsf{V}}$, for all $k-1 \mid m-1$ we get

$$\text{if} \quad \mathbf{HSP}_u(\mathbf{B}) = \mathbf{HSP}_u(\mathbf{C}) \quad \text{then} \quad \mathbf{HSP}_u(\mathbf{B}^{\delta_k}) = \mathbf{HSP}_u(\mathbf{C}^{\delta_k}).$$

Since V is strictly join irreducible, Φ_{V} is finite as shown in the proof of Theorem 7.4, which then implies that also $\Phi_{\mathsf{V}}^{\delta_n}$ is finite.

So let now $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{F}_{\mathsf{V}}$ such that $\mathbf{HSP}_u(\mathbf{B}) = \mathbf{HSP}_u(\mathbf{C})$. We show that for any k such that $k-1 \mid m-1$, $\mathbf{HSP}_u(\mathbf{B}^{\delta_k}) = \mathbf{HSP}_u(\mathbf{C}^{\delta_k})$.

Indeed consider an algebra in $\mathbf{HSP}_u(\mathbf{B}^{\delta_k})$; then it is equal to $\mathbf{D}^{\delta_j}$ for some totally ordered basic hoop $\mathbf{D}$ and $j-1 \mid k-1$. Thus by Lemma 9.2, $\mathbf{D}^{\delta_k} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_k})$, and then $\mathbf{D}^{\delta_l} \in \mathbf{HSP}_u(\mathbf{B}^{\delta_k})$ for all $l-1 \mid k-1$ (since they are subalgebras of $\mathbf{D}^{\delta_k}$); hence by Lemma 9.1, $\mathbf{D} \in \mathbf{HSP}_u(\mathbf{B}) = \mathbf{HSP}_u(\mathbf{C})$ by hypothesis, and then (using again Lemma 9.1), $\mathbf{D}^{\delta_j} \in \mathbf{HSP}_u(\mathbf{C}^{\delta_k})$, which implies that $\mathbf{HSP}_u(\mathbf{B}^{\delta_k}) \subseteq \mathbf{HSP}_u(\mathbf{C}^{\delta_k})$. The equality follows from the fact that we can use the same reasoning starting with an algebra in $\mathbf{HSP}_u(\mathbf{C}^{\delta_k})$. Therefore, the proof is completed. □

Moreover, we can also use the following result to obtain information on splitting algebras in varieties generated by generalized rotations. We recall that an algebra $\mathbf{A} \in \mathsf{V}$ is splitting in the lattice of subvarieties of V if there is a subvariety $\mathsf{W}_{\mathbf{A}}$ of V (the conjugate variety of $\mathbf{A}$) such that for any variety $\mathsf{U} \subseteq \mathsf{V}$ either $\mathsf{U} \subseteq \mathsf{W}_{\mathbf{A}}$ or $\mathbf{A} \in \mathsf{U}$.

PROPOSITION 9.7

Let $\mathbf{A}$ be a totally ordered basic hoop of finite index such that its Wajsberg components are either cancellative or bounded with finite rank, and let $\mathsf{V} = \mathbf{V}(\mathbf{A})$. Then for all suitable term defined rotations δ, $\mathbf{A}^{\delta_m}$ is splitting in V^{δ_n} for any m such that $m-1 \mid n-1$.

PROOF. It follows from Theorem 7.4 that V is strictly join irreducible in BH, then the result follows by applying [7, Lemma 3.3]. □

9.1 Some examples

We shall now give some specific applications of the theorems of this section. For details about the subvarieties mentioned here and their axiomatization, we refer the reader to [7]. Let us start considering the generalized rotations obtained with the term defined rotation $\delta(x) = \ell(x) = 1$. Then BH^{ℓ_n} is the subvariety of BL generated by all the so-called n-liftings of basic hoops: ordinal sums of the kind $\mathbf{Ł}_n \oplus \mathbf{B}$, for a basic hoop $\mathbf{B}$. If $t_n(x) \approx 1$ is the equation axiomatizing $\mathsf{V}(\mathbf{Ł}_n)$, then the variety BH^{ℓ_n} is axiomatized by $t_n(\neg\neg x) \approx 1$. Clearly, for $n = 2$ we have the variety of Stonean BL-algebras.

Then Proposition 9.6 applies to this variety, and we can say that if V is a variety of basic hoops, then V^{ℓ_m} is strictly join irreducible in the lattice of subvarieties of BH^{ℓ_n} iff V is strictly join irreducible in the lattice of subvarieties of basic hoops and $m-1 \mid n-1$.

More in general, the following characterization is a direct consequence of Proposition 9.6 and Theorem 7.4.

COROLLARY 9.8
Let V be a subvariety of BH, δ a term-defined rotation and $m, n \geq 2$. V^{δ_m} is strictly join irreducible in the lattice of subvarieties of BH^{δ_n} if and only if $m-1 \mid n-1$ and $\mathsf{V} = \mathbf{V}(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{F}_{\mathsf{V}}$ such that its Wajsberg components are either cancellative or are bounded and have finite rank.

For the rest of the section, we are going to focus on the term defined rotation given by the identity map, $\delta = id$. Thus, the resulting varieties will be involutive subvarieties of MTL.

9.1.1 Nilpotent minimum varieties Let us first consider involutive varieties generated by generalized disconnected rotations of Gödel hoops, which we called in [7] *nilpotent minimum varieties*, in similarity to the specific case of nilpotent minimum algebras $\mathsf{NM} = \mathsf{GH}^{id_3}$.

Every proper subvariety $\mathsf{G}_k = \mathbf{V}(\mathbf{G}_k)$ of Gödel hoops is strictly join irreducible. Thus, $\mathbf{G}_k^{id_m}$ is strictly join irreducible in the lattice of subvarieties of GH^{id_n} whenever $m-1 \mid n-1$.

Moreover, all algebras $\mathbf{G}_k$ satisfy the hypothesis of Proposition 9.6; thus, all rotations $\mathbf{G}_k^{id_m}$ are splitting in $\mathbf{G}_k^{id_n}$ whenever $m-1 \mid n-1$.

9.1.2 Nilpotent Łukasiewicz varieties We call *nilpotent Łukasiewicz varieties* all subvarieties of BH^{id_n} generated by disconnected n-rotations of Wajsberg hoops. WH^{id_n} can be axiomatized by

$$(\nabla_n(x) \wedge \nabla_n(y)) \to (((x \to y) \to y) \to ((y \to x) \to x)) \approx 1. \tag{1}$$

Proper subvarieties of WH are all generated by finitely many chains [6] so their lattice of subvarieties is finite. Any proper variety of Wajsberg hoops V is axiomatized (modulo basic hoops) by a single equation in one variable of the form $t_{\mathsf{V}}(x) \approx 1$, and V^{id_n} is axiomatized by $\neg x^n \vee t_{\mathsf{V}}(x) \approx 1$ (modulo BH^{id_n}).

Now, the proper subvarieties of WH that are strictly join irreducible are the varieties generated by a single Wajsberg chain that is either cancellative, or with finite rank, i.e. either $\mathbf{Ł}_n$ or $\mathbf{Ł}_n^\infty$ for some n. Calling W_n and W_n^∞ the varieties generated by, respectively, $\mathbf{Ł}_n$ or $\mathbf{Ł}_n^\infty$, for all $m \geq 2$, $\mathsf{W}_n^{id_m}$ and $(\mathsf{W}_n^\infty)^{id_m}$ are strictly join irreducible in W^{id_n} whenever $m-1 \mid n-1$.

9.1.3 Nilpotent product varieties We refer to the varieties C^{id_n} for $n \geq 2$ as **nilpotent product varieties**. Since cancellative hoops are axiomatized relative to Wajsberg hoop by $(x \to x^2) \to x \approx 1$, the variety C^{id_n} is axiomatized by

$$\neg x^n \to ((x \to x^2) \to x) \approx 1.$$

The lattice of subvarieties can be easily described knowing that $\mathsf{C}^{id_n} = \mathbf{V}(\mathbf{C}_\omega^{id_n})$. For each C^{id_n}, the varieties C^{id_m} where $m-1 \mid n-1$ are strictly join irreducible.

10 Linear varieties

A variety V of residuated lattices is **linear** if $\Lambda(\mathsf{V})$ is totally ordered. Clearly, each subvariety of a linear variety is join irreducible; via Theorem 4.4 each subvariety is then generated by a single

Γ^n-connected algebra. Since obviously if every subvariety of a variety is join irreducible then the variety is linear, we have

LEMMA 10.1
A variety V is linear if and only if each subvariety of V is generated by a single Γ^n-connected algebra. Moreover, if the order type of the chain is $\omega + 1$, then every subvariety is generated by a single subdirectly irreducible algebra.

If the variety is also representable, then each subvariety is generated by a single chain. The prototype of a variety satisfying this is the variety GH of Gödel hoops; $\Lambda(\mathsf{GH})$ is the chain $\mathsf{G}_1 \subseteq \mathsf{G}_2 \subseteq \cdots \subseteq \mathsf{GH}$ and each G_n is generated by $\bigoplus_{i=1}^n \mathbf{2}$ which is of course subdirectly irreducible. Therefore, GH is generated by any infinite set of chains of finite index or else by a single chain of infinite index. This is no coincidence. First, we need a simple lemma:

LEMMA 10.2
Let $\mathbf{A}$ be totally ordered residuated lattice. Then $\mathbf{A}$ is finite if and only if $\mathbf{A} \vDash \bigvee_{i<n} x_{i+1}/x_i \geq 1$ for some n.

PROOF. Let $\mathbf{A}$ be finite, say $|A| = n - 1$. Therefore, if $a_1, \dots, a_n \in A$ there must be some $i < n$ with $a_i \leq a_{i+1}$; this implies $\bigvee_{i<n} a_{i+1}/a_i \geq 1$ and so $\mathbf{A}$ satisfies the desired equation.

Conversely suppose that $\mathbf{A}$ is infinite; then for any n we can find $a_1, \dots, a_n \in A$ with $a_n < \cdots < a_1$, so that $a_{i+1}/a_i < 1$ for all $i < n$. As $\mathbf{A}$ is a chain 1 is join prime, so $\bigvee_{i<n} a_{i+1}/a_i < 1$. Therefore, $\mathbf{A}$ does not satisfy any of the desired equations. □

LEMMA 10.3
Let V be any linear variety of representable residuated lattices.

1. Given any infinite chain $\mathbf{A}$, $\mathbf{V}(\mathbf{A})$ contains the variety V_c generated by all finite chains in V.
2. If there is an infinite chain in V, V has the finite model property (FMP for short) if and only if $\mathsf{V}_c = \mathsf{V}(\mathbf{A})$ for all infinite chains $\mathbf{A} \in \mathsf{V}$.
3. For any fixed n, all totally ordered members of V of cardinality n are isomorphic.

PROOF. Let us prove (1) first. Let $\mathbf{A}$ be any infinite chain in V and let $\mathbf{B}$ be a finite chain in V. Then since V is linear, either $\mathbf{V}(\mathbf{A}) \subseteq \mathbf{V}(\mathbf{B})$ or $\mathbf{V}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{A})$. But since $\mathbf{B}$ is finite, by Lemma 10.2, it satisfies an identity that $\mathbf{A}$ does not, thus $\mathbf{V}(\mathbf{A}) \not\subseteq \mathbf{V}(\mathbf{B})$ and then $\mathbf{V}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{A})$. Thus, $\mathsf{V}(\mathbf{A})$ contains all finite chains in V, which proves (1).

We now show (2). We have already shown that $\mathsf{V}_c \subseteq V(\mathbf{A})$ for all infinite chains $\mathbf{A} \in \mathsf{V}$. We recall that a variety has the finite model property iff it is generated by its finite members. So if the FMP holds for V, then $\mathsf{V}_c = \mathsf{V}$ and then $\mathsf{V}_c = \mathsf{V}(\mathbf{A}) = \mathsf{V}$. Vice versa, suppose $\mathsf{V}_c = \mathsf{V}(\mathbf{A})$ for all infinite chains $\mathbf{A} \in \mathsf{V}$. If an equation fails in V, it fails in some chain, possibly infinite, $\mathbf{B}$. But $\mathsf{V}(\mathbf{B}) = \mathsf{V}_c$; thus, the equation must fail in a finite chain. This means that the FMP holds for V.

Finally, we prove (3). Let $\mathbf{A}, \mathbf{B} \in \mathsf{V}$ be two chains having the same cardinality with $\mathbf{A} \not\cong \mathbf{B}$; as they are finite and totally ordered they are subdirectly irreducible and Jónnson's Lemma implies that $\mathbf{B} \notin \mathbf{V}(\mathbf{A})$ and $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$. Since V is linear, this is a contradiction and (2) holds. □

COROLLARY 10.4
Let V be a linear variety of representable residuated lattices with the FMP (i.e. V is generated by its finite algebras).

1. Every infinite chain generates V, and the only proper subvarieties of V are generated by a single finite chain (up to isomorphism).
2. If V is generated by a finite chain, then $\Lambda(\mathsf{V})$ is finite.
3. If V is generated by an infinite chain, $\Lambda(\mathsf{V})$ has order type $\omega + 1$.

PROOF. Since V has the finite model property, by Lemma 10.3, if V has an infinite chain it generates the whole variety since it generates all finite chains. Therefore, all the proper subvarieties must be generated by a finite chain. If V is generated by a finite chain $\mathbf{A}$ then by Jónnson's lemma any totally ordered member of V must be in $\mathbf{HS}(\mathbf{A})$; hence, they are all finite and (1) holds.

If V is generated by a finite chain of cardinality n, then in V there are only finite chains of cardinality at most n (by Lemma 10.2). Since finite chains of equal cardinality are isomorphic (by Lemma 10.3(3)) $\Lambda(\mathsf{V})$ must be finite.

Finally, if V is generated by an infinite chain, then the only proper subvarieties are generated by finite chains; by Lemma 10.3(3), we can pick a representative chain $\mathbf{A}_{n+1}$ with n-elements for any $n \in \mathbb{N}$ and $\Lambda(\mathsf{V})$ is

$$\mathbf{V}(\mathbf{A}_0) < \mathbf{V}(\mathbf{A}_1) < \cdots < \mathbf{V}(\mathbf{A}_{n+1}) < \cdots < \mathsf{V}.$$

Therefore, (3) holds. □

Note that the proofs of Lemma 10.3 and Corollary 10.4 depend basically on the fact that we can distinguish equationally finite chains of different cardinalities and finite chains from infinite chains.

If V is a subvariety of RPsH we can rephrase the previous results substituting the finiteness of the chain with the finiteness of the index. In other words

COROLLARY 10.5
Let V be a linear variety contained in RPsH with the finite model property.

1. V is generated by each chain of infinite index $\mathbf{A}$ (if any); hence, the only proper subvarieties of V are those generated by chains of finite index.
2. $\Lambda(\mathsf{V})$ is either finite or it is a chain of type $\omega + 1$.

Now we can use the classification of chains of finite index in BH to describe all linear varieties of basic hoops. First, we observe that the linear varieties of Wajsberg hoops are easy to classify simply by inspection using the description of $\Lambda(\mathsf{WH})$ in [6]. We recall that we are denoting by W_n the variety generated by the Wajsberg hoop with $n + 1$-elements, $\mathbf{Ł}_n$.

THEOREM 10.6
The only linear varieties of Wajsberg hoops are the variety C of cancellative hoops and W_n where either $n = 1$ or n is a prime power.

Observe that if $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a totally ordered hoop, then $\mathbf{V}(\mathbf{A}_i) \subseteq \mathbf{V}(\mathbf{A})$ for all $i \in I$. Hence, if $\mathbf{V}(\mathbf{A})$ happens to be linear, then by Theorem 10.6, each $\mathbf{A}_i$ must be either $\mathbf{C}_\omega$ or $\mathbf{L}_n$ where n is a prime power.

LEMMA 10.7
Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$, with $|I| > 1$, be a totally ordered basic hoop such that $\mathbf{V}(\mathbf{A})$ is linear. Then

1. either $\mathbf{A}_i = \mathbf{2}$ for $i \in I$
2. or $\mathbf{A}_i = \mathbf{C}_\omega$ for $i \in I$.

PROOF. It follows from Theorem 10.6 that the only possible components are either $\mathbf{C}_\omega$, $\mathbf{2}$ or $\mathbf{L}_n$ for some prime power n. Note that since $\mathbf{V}(\mathbf{2})$ and $\mathbf{V}(\mathbf{C}_\omega)$ are incomparable varieties of Wajsberg hoops, $\mathbf{2}$ and $\mathbf{C}_\omega$ cannot appear both in the decomposition of $\mathbf{A}$. Since $\mathbf{2}$ is a subalgebra of each $\mathbf{Ł}_n$, also $\mathbf{Ł}_n$ (for any n) and $\mathbf{C}_\omega$ cannot appear both in the decomposition of $\mathbf{A}$. Thus, if there is i such that $\mathbf{A}_i = \mathbf{C}_\omega$, then $\mathbf{A}_i = \mathbf{C}_\omega$ for all $i \in I$ and (2) holds.

Suppose now that $\mathbf{C}_\omega$ is not in the decomposition of $\mathbf{A}$, and suppose that for some i, $\mathbf{A}_i = \mathbf{2}$. If there is a $j \neq i$ such that $\mathbf{A}_j = \mathbf{Ł}_n$ where n is a prime power, $n > 1$, it follows immediately that $\mathbf{V}(\mathbf{Ł}_n)$ and $\mathbf{V}(\mathbf{2} \oplus \mathbf{2})$ are incomparable subvarieties of $\mathbf{V}(\mathbf{2} \oplus \mathbf{Ł}_n)$ so $\mathbf{V}(\mathbf{A})$ is not linear. The same holds if there are two components of the kind $\mathbf{Ł}_n$ and $\mathbf{Ł}_m$, with m, n (possibly equal) prime powers. Thus, the only other possibility is that (1) holds. □

Now we can characterize all the linear varieties of basic hoops. Let us call $\Omega(\mathsf{C})$ the variety generated by arbitrary ordinal sums of $\mathbf{C}_\omega$.

THEOREM 10.8
A variety V of basic hoops is linear if and only if

1. V is a linear variety of Wajsberg hoops or
2. V is a subvariety of GH or
3. V is a subvariety of the variety $\Omega(\mathsf{C})$ of hoops satisfying $x \to x^2 \le x$.

PROOF. We have already seen that GH is linear. For the variety $\Omega(\mathsf{C})$, we simply note that any chain of finite index has only cancellative hoops as components, since the equation $x \to x^2 \le x$, in conjunction with Tanaka's equation $(x \to y) \to y = (y \to x) \to x$, implies cancellativity. Therefore, modulo $\equiv$, any chain of index n is of the form $\bigoplus_{i<n} \mathbf{C}_\omega$; it is obvious now that $\Omega(\mathsf{C})$ is linear and generated by the chain $\bigoplus_{n\in\omega} \mathbf{C}_\omega$.

Conversely assume that V is a linear variety of basic hoops that does not consist entirely of Wajsberg hoops. Then V is generated by a chain $\mathbf{A}$ of index > 1 (possibly infinite). Hence, by Lemma 10.7, the components of $\mathbf{A}$ are all equal to $\mathbf{2}$ or to $\mathbf{C}_\omega$ and the conclusion follows. □

Moreover, similarly to the case of Wajsberg hoops the varieties of Wajsberg algebras that are linear can be found by simply inspecting the description of the lattice of subvarieties in [19]. Let us call Wa_n the variety generated by the $n + 1$ elements MV-chain.

LEMMA 10.9
Let V be a variety of Wajsberg algebras; then V is linear if and only if it is either Wa_n where either $n = 1$ or n is a prime power, or else it is generated by the Chang algebra (i.e. the zero-bounded version of $\mathbf{Ł}_2^\infty$).

We can now apply our results to varieties obtained by generalized n-rotations. In particular, if we pick δ to be any term-defined rotation on BH and $n = 2$, we can use the lattice isomorphism between $\Lambda(\mathsf{BH})$ and $\Lambda(\mathsf{BH}^{\delta_2})$ described after Proposition 9.3. In particular, the following result applies if $\delta = 1$ or $\delta = id$, that is to say, to SBL and the variety generated by disconnected rotations of basic hoops that we shall call bIDL.

THEOREM 10.10
Let δ be a term defined rotation, then a variety V contained in BH^{δ_2} is linear if and only if $\mathsf{V} = \mathsf{W}^{\delta_2}$ where W is a linear variety of basic hoops.

Thus, in particular, a subvariety of SBL is linear if and only if it is either

- a subvariety of the variety of Gödel algebras (bounded Gödel hoops), or

- the variety generated by $\mathbf{2} \oplus \mathbf{Wa}_n$ where n is a prime power or
- the variety generated by $\mathbf{2} \oplus \mathbf{A}$, where $\mathbf{A}$ is a totally ordered member of $\Omega(\mathsf{C})$.

What about BH^{δ_n} for $n > 2$? It turns out that the only linear varieties are those already described in Theorem 10.10.

THEOREM 10.11
Let δ be a term defined rotation, then a variety V contained in BH^{δ_n} is linear if and only if either

1. $\mathsf{V} = \mathsf{Wa}_n$ with $n = 1$ or n a prime power, or
2. $\mathsf{V} = \mathsf{W}^{\delta_2}$ where W is a linear variety of basic hoops.

PROOF. The right-to-left direction is easy to see and is a consequence of Theorem 10.10 and Proposition 9.3. Suppose now that V contained in BH^{δ_n} is linear, and it is not a linear variety of Wajsberg algebras. Since it is linear it is also join irreducible, which means it is generated by a single subdirectly irreducible algebra (Theorem 5.4), say $\mathbf{A}^{\delta_m} \neq \mathbf{Ł}_{m-1}$, with $m-1 \mid n-1$. If $m > 2$, then both the MV-skeleton $\mathbf{Ł}_{m-1}$ and $\mathbf{A}^{\delta_2}$ are subalgebras of $\mathbf{A}^{\delta_m}$, and they generate incomparable subvarieties of V. Thus, necessarily $m = 2$. Thus, the result follows from Theorem 10.10. □

Finally, we show how we can get the classification of all linear varieties of BL-algebras (already obtained in [10]) using our techniques.

THEOREM 10.12
A variety V of BL-algebras is linear if and only if

1. V is a linear variety of Wajsberg algebras or
2. V is a linear variety contained in SBL.

PROOF. Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ be a BL-algebra such that $\mathbf{V}(\mathbf{A})$ is linear; if $\mathbf{A}$ is not a Wajsberg algebra, then $|I| > 1$. Take $k \in I$, $k \neq 0$; then $\mathbf{V}(\mathbf{A}_0 \oplus \mathbf{A}_k)$ must be a linear subvariety of $\mathbf{V}(\mathbf{A})$. If $\mathbf{A}_0 \neq \mathbf{2}$ then $\mathbf{A}_0 \notin \mathbf{HSP}_u(\mathbf{2})$; hence, $\mathbf{2} \oplus \mathbf{A}_k$ and $\mathbf{A}_0$ generate incomparable subvarieties of $\mathbf{V}(\mathbf{A}_0 \oplus \mathbf{A}_k)$. But this is a contradiction so $\mathbf{A}_0 = \mathbf{2}$ and $\mathbf{A} \in \mathsf{SBL}$. □

11 Conclusions

We have investigated in a very general setting the properties of being, respectively, join irreducible and strictly join irreducible in subvarieties of residuated lattices. Moreover, we have applied the results found to representable varieties and obtained a precise characterization of (strictly) join irreducible varieties and linear varieties in the case of basic hoops. Finally, we made use of the generalized rotation construction to lift some of these results to subvarieties of MTL-algebras.

As for future work, it is worth mentioning that a study of the techniques employed in [22] and [23] could lead to an analogue characterization for varieties of representable pseudohoops.

Finally, we believe that the results and techniques developed in the first sections of this work can pave the way for many other possible applications than those considered here.

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