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MANUSCRIPT

Localized Mixing Zone for Muskat Bubbles and Turned Interfaces

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Abstract
We construct mixing solutions to the incompressible porous media equation starting from Muskat type data in the partially unstable regime. In particular, we consider bubble and turned type interfaces with Sobolev regularity. As a by-product, we prove the continuation of the evolution of IPM after the Rayleigh–Taylor and smoothness breakdown exhibited in (Castro et al. in Arch Ration Mech Anal 208(3):805–909, 2013, Castro et al. in Ann Math. (2) 175(2):909–948, 2012). At each time slice the space is split into three evolving domains: two non-mixing zones and a mixing zone which is localized in a neighborhood of the unstable region. In this way, we show the compatibility between the classical Muskat problem and the convex integration method.

1 Introduction and Main Results

We consider two incompressible fluids with different constant densities ρ_-, ρ_+ and equal viscosity μ, separated by a connected curve $z^\circ = (z_1^\circ, z_2^\circ)$ inside a 2D porous medium with constant permeability κ (or Hele-Shaw cell [67]) and under the action of gravity $-g(0, 1)$. As we deal with closed and open curves, it is convenient to fix an orientation for z°. For closed curves we fix the clockwise orientation ($\circlearrowright$) and for open curves the orientation from $x_1 = -\infty$ to $+\infty$. Then, we denote Ω_-° (Ω_+°) by the domain to the left (right) side of z°. Thus, the initial density will be written as

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$$\rho^\circ(x) := \begin{cases} \rho_-, \ x \in \Omega_-^\circ, \\ \rho_+, \ x \in \Omega_+^\circ, \end{cases} \tag{1.1}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. It is widely accepted that the dynamic of this two-phase flow can be modelled by the Incompressible Porous Media (IPM) system

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.2}$$

$$\nabla \cdot v = 0, \tag{1.3}$$

$$\tfrac{\mu}{\kappa} v = -\nabla p - \rho g(0, 1), \tag{1.4}$$

where $\rho(t, x) \equiv$ density, $v(t, x) \equiv$ velocity field, $p(t, x) \equiv$ pressure. By normalizing, we may assume w.l.o.g. that $|\rho_\pm| = \mu = \kappa = g = 1$.

The investigations on the Muskat problem ([62]) which deals with the interface evolution under the assumption of immiscibility, have been very intense both in the applied community due to the many applications (see e.g. [48, 53, 72, 73]) and in the theoretical side as this constitutes a challenging free boundary problem.

Mathematically, the theory has bifurcated into two regimes, the so-called stable regime and unstable regime. This division arises from the linear stability analysis of the equation for the interface evolution. It is classical (see e.g. [28]) that such linear stability is characterized by the sign of the Rayleigh–Taylor function $\sigma := (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$ as follows:

$$\text{stable} \quad \text{on} \quad \sigma(\alpha) > 0, \tag{1.5a}$$

$$\text{unstable} \quad \text{on} \quad \sigma(\alpha) \leq 0. \tag{1.5b}$$

This simply classifies whether the heavier fluid remains (locally) below the lighter one or not. If the initial interface is a graph, $z^\circ(\alpha) = (\alpha, f^\circ(\alpha))$, the interface evolution is governed by a a nonlinear parabolic equation, which can be linearized as $\partial_t f = (\rho_+ - \rho_-)(-\Delta)^{1/2} f$. Hence, the stability simply depends on the sign of the density jump $\rho_+ - \rho_-$. Therefore, for $\rho_+ > \rho_-$ (i.e. the heavier fluid is below z°) what is called the fully stable regime, the analogy with the heat equation gives hope of well-posedness theory in a suitable Sobolev space H^k. We refer to the corresponding weak solutions to IPM as non-mixing solutions (see [6, 28, 31, 69, 74] for initial results). In the last years there have taken extensive steps to reduce the initial k (see [2, 20, 26, 56, 63]). The current world record is the result of Alazard and Nguyen [3] where they have proved the critical case $k = 3/2$ (see also [4, 5]). For small enough initial data these solutions are global-in-time. Additional results of global well-posedness for medium size initial data can be found in [24, 25] and global solutions with large initial slope in [15, 32, 39].

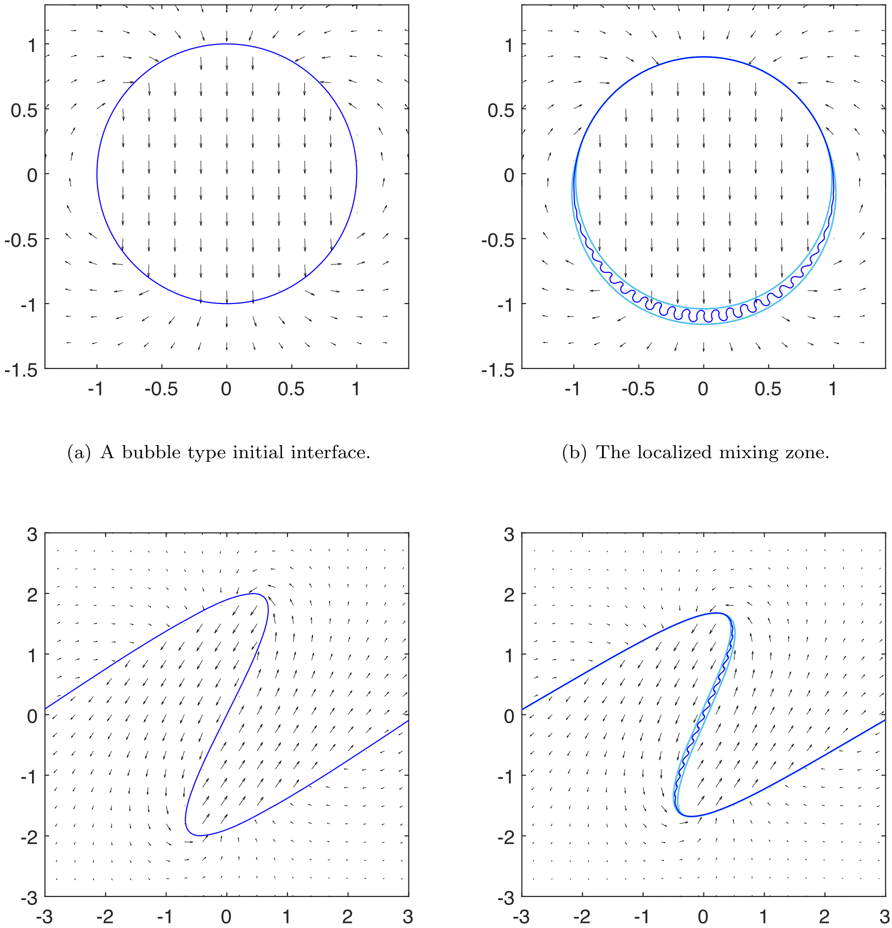
The instability in the linearization is called Rayleigh–Taylor (or Saffman–Taylor [67]) for the Muskat problem. In the graph case, it corresponds to $\rho_+ < \rho_-$ (i.e. the heavier fluid is above z°) what is called the fully unstable regime, and the analogy is now with the backwards heat equation. Therefore, it is to be expected that the problem is ill-posed unless the initial data is real-analytic C^ω. As a matter of fact, all the techniques available in the stable case catastrophic fail in this situation. Indeed, it can be proved that in the fully unstable regime, $\sigma(\alpha) \leq 0$ for every α, the Cauchy problem for f is ill-posed in Sobolev spaces (see e.g. [69]). However, practical and numerical experiments show the existence of the so-called mixing solutions, solutions in which there exist a mixing zone where the two fluids mix stochastically (see e.g. [48, 73]). Numerically, it can be seen that small disturbances of an analytic initial interface increases rapidly creating finger patterns at different scales in the unstable region (see e.g. [53, 72] and Fig. 1).

In spite of the fact that the linearized problem is ill-posed and in accordance with what is observed in the experiments, weak solutions to IPM, in the fully unstable case, have been constructed in the last years by replacing the continuum free boundary assumption with the opening of a mixing zone Ω_{mix} where the fluids begin to mix indistinguishably. These mixing solutions (ρ, v) are recovered by the convex integration method applied in Ω_{mix} to a so-called "subsolution" $(\bar{\rho}, \bar{v}, \bar{m})$ (cf. Sect. 2). These subsolutions are intended to be a kind of coarse-grained solutions to IPM, with $\bar{m}$ representing the relaxation of the momentum $\bar{\rho}\bar{v}$. The subsolutions are very related to the relaxed solutions appearing in the Lagrangian relaxation approach of Otto [65, 66] (see also [51]).

In the context of large data, an striking result from [17, 18] shows that there exist analytic initial interfaces in the fully stable regime (i.e. a graph) such that part of the curve turns to the unstable regime (i.e. no longer a graph) and later, at some $T_* > 0$, the interface $z(T_*)$ is analytic but at a point in the unstable region where it is not C^4. The argument in [17] could be adapted to prove weaker singularities in C^k where $k \geq 5$ (i.e. the interface leaves to be C^k but is still C^{k-1}). Thus, the Rayleigh–Taylor instability can arise spontaneously and the regularity might break down. After the blow-up time T_* it is to be expected that the Muskat problem is ill-posed.

Note that at this point of the theory h-principles are available and the task is to find a suitable subsolution. Even for the most simple subsolutions (null density in the mixing zone) the structure of the relaxation of IPM constrains the growth-rate $c(\alpha) \geq 0$ and the shape of the mixing zone (see Sect. 2.4). This constraint prevents the two fluids from mixing ($c(\alpha) = 0$) around fully stable points (stable points with zero slope) and it implies an equation of Muskat type in $\{c(\alpha) = 0\}$. In the unstable region, where the interface dynamic is expected to be ill-posed (analogous to the fully unstable case), the creation of mixing ($c(\alpha) > 0$) is the only mechanism we know to solve IPM. This is in stark contrast with previous constructions of mixing solutions available in the literature: the mixing zone can not include the fully stable points and has to include the unstable part completely. Therefore, a new method is needed to find compatibility between the parabolic analysis in the stable regime and the relaxation approach in the unstable case. We remark that all previous approaches modelling instabilities blow up at some point when $c(\alpha) = 0$ ([16, 58, 64]). Indeed, gluing a solution of IPM in $\{c(\alpha) = 0\}$ with a subsolution in $\{c(\alpha) > 0\}$ becomes really subtle on the boundary of the support of $c(\alpha)$. Thus, apart from the applications, these issues make the problem challenging. At the end of the intro we explain a number of novelties which allow us to bypass those difficulties.

The original motivation of this work was to continue the solutions after the breakdown ([17, 18]) described before. However, there are numerous scenarios



(a) A bubble type initial interface. (b) The localized mixing zone.

(c) A turned type initial interface. (d) The localized mixing zone.

Fig. 1 **a**, **c** The initial interface $z^\circ(\alpha)$ separating two fluids with different constant densities $\rho_\pm = \pm 1$ as in (1.6) (1.7) respectively. **b, d** At some $t > 0$, the two boundaries of the non-mixing zones $z_\pm(t,\alpha) = z(t,\alpha) \mp tc(\alpha)\tau(\alpha)^\perp$ (light blue) for some pseudo-interface $z(t,\alpha)$ and growth-rate $c(\alpha)$, with $\tau(\alpha) = \frac{\partial_\alpha z^\circ(\alpha)}{|\partial_\alpha z^\circ(\alpha)|}$. Inside the mixing zone $\Omega_{\mathrm{mix}}(t)$ we plot the Rayleigh–Taylor curve $z_{\mathrm{per}}(t)$ (dark blue) which starts from a tiny perturbation of z° (via the vortex-blob method). In all the figures we have added the coarse-grained velocity field $\bar{v}(t,x)$ outside Ω_{mix}

which are partially unstable. In this work we will concentrate on two of them: The so-called **bubble interfaces** where the two fluids are separated by a closed chord-arc curve (see [44] for the case with surface tension) and the **turned interfaces** where the interface is an open chord-arc curve which cannot be parametrized as a graph. We describe both scenarios readily, prior to the statement of the theorems.

The bubble type initial interfaces are described by

$$\begin{aligned} \Omega_-^\circ &\equiv \text{ exterior domain of } z^\circ, \\ \Omega_+^\circ &\equiv \text{ interior domain of } z^\circ, \end{aligned} \tag{1.6}$$

with $\rho_\pm = \pm 1$, for some closed chord-arc curve $z^\circ \in H^k(\mathbb{T};\mathbb{R}^2)$ with k big enough (cf. Fig. 1a). Recall that we have taken z° clockwise oriented (↻) to be consistent with the notation in (1.1).

The turned type initial interfaces are described by

$$\begin{aligned} \Omega_-^\circ &\equiv \text{ upper domain of } z^\circ, \\ \Omega_+^\circ &\equiv \text{ lower domain of } z^\circ, \end{aligned} \tag{1.7}$$

with $\rho_\pm = \pm 1$, for some open chord-arc curve z° whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \le 0\}$ has positive measure. Here we consider both the x_1-periodic case $z^\circ - (\alpha, 0) \in H^k(\mathbb{T};\mathbb{R}^2)$ and the asymptotically flat case $z^\circ - (\alpha, 0) \in H^k(\mathbb{R};\mathbb{R}^2)$ with k big enough (cf. Fig. 1b).

Now we are ready to state our two main theorems.

Theorem 1.1 *For every closed chord-arc curve $z^\circ \in H^6(\mathbb{T};\mathbb{R}^2)$ there exist infinitely many mixing solutions to IPM starting from* (1.1) (1.6) *with $\rho_\pm = \pm 1$.*

Theorem 1.2 *For every open chord-arc curve z°, either x_1-periodic $z^\circ - (\alpha, 0) \in H^6(\mathbb{T};\mathbb{R}^2)$ or asymptotically flat $z^\circ - (\alpha, 0) \in H^6(\mathbb{R};\mathbb{R}^2)$, whose turned region $\{\partial_\alpha z_1^\circ(\alpha) \le 0\}$ has positive measure there exist infinitely many mixing solutions to IPM starting from* (1.1) (1.7) *with $\rho_\pm = \pm 1$.*

The definition of mixing solutions is by now classical and will be rigorously defined in Sect. 2 where the reader is exposed to the convex integration framework.

Remark 1.1 Theorem 1.2 is the first result proving the continuation of the evolution of IPM after the breakdown exhibited in [17, 18].

Remark 1.2 As mentioned above, the h-principle applied to a coarse-grained solution, a subsolution, yields infinitely many weak solutions. This path goes in both directions as, by taking suitable averages of the solutions, the subsolution is essentially recovered [19]. Thus, the relevant macroscopic properties of the solutions are described by the subsolution. Our construction yields piecewise constant subsolutions as in [43] and it is still open whether a continuous subsolution (similar to that in [16]) might be built in the partially unstable regime.

Remark 1.3 As in [16, 43, 64, 71], our mixing zone grows linearly in time around an evolving pseudo-interface. However, in Theorems 1.1 and 1.2 the mixing region must be localized in a neighborhood of the unstable region. Furthermore, this approach reveals the admissible regime for the growth-rate $c(\alpha)$ of the mixing zone compatible with the relaxation of IPM. This is

$$\left| c(\alpha) + \frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2 + \varpi(\alpha)^2}} \right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{1.8}$$

which is characterized by the **Rayleigh–Taylor** function $\sigma := (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$ and the **vorticity** strength $\varpi := -(\rho_+ - \rho_-)\partial_\alpha z_2^\circ$ along z° (cf. Sect. 2). Observe that (1.8) prevents the two fluids from mixing wherever the initial interface is stable ($\sigma(\alpha) > 0$) and there is not vorticity ($\varpi(\alpha) = 0$).

The proof of the theorems relies on the pioneering adaptation of the convex integration method to Hydrodynamics by De Lellis and Székelyhidi ([35, 36]). The method has turned out to be very robust and flexible and the research on it has been extremely intense in the last decade. We contempt ourselves with describing a few landmarks: It has successfully described several problems related to turbulence as the Onsager's conjecture (see e.g. [10, 34, 49]), the evolution of active scalars ([11, 30, 47, 50, 52, 68]) and transport equations ([33, 59–61]), the compressible Euler equations (see e.g. [1, 21, 22, 42, 54, 55]), the Navier-Stokes equations (see e.g. [9, 13, 14, 23]) and Magnetohydrodynamics ([8, 40, 41]) (see also the surveys [12, 37, 38] and the references therein).

In the context of modeling instabilities in Fluid Dynamics via convex integration, the first result in the IPM context (see also [30]) was proved in [71] where Székelyhidi constructed infinitely many weak solutions to IPM starting from the unstable planar interface. Remarkably, the coarse-grained density (the subsolution in the convex integration jargon) agrees with the Otto's Lagrangian relaxation of IPM (cf. [65] and also [57]). In [16] the first two authors and Córdoba constructed mixing solutions starting with a non-flat interface. In this work and all the subsequent ones, the mixing zone is described as an envelop of size $tc(\alpha)$ of a curve $z(t, \alpha)$ whose evolution is dictated by an operator which is an average of the classical Muskat operator. In [16] the coarse-grained density $\bar{\rho}$ is a continuous interpolation between the two fluids, which induces through an adapted h-principle a degraded mixing property ([19]). As a by-product of this version of the h-principle [19], one shows that the subsolution is recovered from the solution by taking suitable averages. Remarkably, if one considers instead piecewise constant coarse-grained densities, the evolution of the pseudo-interface greatly simplifies as was shown in [43] by Förster and Székelyhidi. See also [7, 64] for possible choices of the speed of opening of the mixing zone $c(\alpha)$.

After the works in IPM, instabilities for the incompressible Euler equations have been successfully modeled with related strategies, e.g. the Rayleigh–Taylor ([45, 46]) and the Kelvin-Helmholtz ([58, 70]) instabilities.

All the previous works deal either with the fully stable or fully unstable regime of the various instabilities and hence new twists should be added to the theory to deal with the partially unstable case. We finish the introduction with some comments on the natural obstructions and a non-technical description of the new view points needed to address them. We believe that it is likely that the ideas from this paper can be adapted and extended to consider different partially unstable scenarios in various problems concerning instabilities in Fluid Dynamics.

Since it is to be expected that the classical Muskat problem is ill-posed in this partially unstable situation, we need to see a way to find compatibility between the parabolic analysis for the stable case and the relaxation approach for the unstable case. In particular, the mixing region needs to envelope the unstable region. That is (recall $\sigma = (\rho_+ - \rho_-)\partial_\alpha z_1^\circ$)

$$\{\sigma(\alpha) \leq 0\} \subset \{c(\alpha) > 0\}. \tag{1.9}$$

As h-principles are by now standard [16, 19, 71], the main issue of the proof relies on building a mixing zone which admits a suitable subsolution $(\bar{\rho}, \bar{v}, \bar{m})$. We will follow [43] and declare $\bar{\rho}$ piecewise constant in the mixing zone. In fact, for the sake of simplicity during the introduction we will assume the simplest case, $\bar{\rho} = 0$ in Ω_{mix}.

At each time slice $0 < t \leq T \ll 1$, the mixing zone is the open set in $\mathbb{R}^2$ given by

$$\Omega_{\text{mix}}(t) := \{z_\lambda(t, \alpha) \,:\, c(\alpha) > 0,\ \lambda \in (-1, 1)\}, \tag{1.10}$$

parametrized by the map

$$z_\lambda(t, \alpha) := z(t, \alpha) - \lambda t c(\alpha)\tau(\alpha)^\perp, \tag{1.11}$$

where $\tau(\alpha)$ is an unitary vector field, $c(\alpha)$ is the growth-rate of the mixing zone and $z(t, \alpha)$ is the pseudo-interface evolving from $z^\circ(\alpha)$, that we have to determine.

In order to optimize the speed of opening of the mixing zone, it is convenient to take τ as the tangential vector field to z°

$$\tau(\alpha) = \operatorname{sgn}(\rho_+ - \rho_-)\frac{\partial_\alpha z^\circ(\alpha)}{|\partial_\alpha z^\circ(\alpha)|}. \tag{1.12}$$

With our ansatz for $\bar{\rho}$ as in [43] and this optimal choice for τ, the admissible regime for $c(\alpha)$ compatible with the relaxation of IPM becomes

$$\left|2c(\alpha) + \frac{\sigma(\alpha)}{\sqrt{\sigma(\alpha)^2 + \varpi(\alpha)^2}}\right| < 1 \quad \text{on} \quad c(\alpha) > 0. \tag{1.13}$$

We remark in passing that $2c(\alpha)$ above can be replaced by $\frac{2N}{2N-1}c(\alpha)$ for any $N \geq 1$ as in [43, 64], which yields (1.8) as $N \to \infty$ (cf. Sect. 6.2). Observe that this inequality requires $c(\alpha) = 0$ if $\sigma(\alpha) = |(\sigma(\alpha), \varpi(\alpha))|$, or equivalently $\partial_\alpha z_1^\circ(\alpha) = \operatorname{sgn}(\rho_+ - \rho_-)|\partial_\alpha z^\circ(\alpha)|$ (cf. Remark 1.3). Since in the regimes we are considering there are always such points, we are forced to treat the case where there is no opening in some region, i.e. $c(\alpha) = 0$. An extra difficulty at this level is that our estimates need certain smoothness in c (i.e. the very definition of the velocity) which necessarily creates cusp singularities on Ω_{mix}. We deal with this problem by interpreting the mixing zone as a superposition of regular domains (cf. Fig. 2 and Lemma 2.1).

Next we turn to the coarse-grained velocity and the associated Muskat type operator. Here we start from [43] as we have chosen the same ansatz for the coarse-grained density and then explain the new idea. The Förster-Székelyhidi's velocity is also an average of the classical Muskat velocity as in [16] but only between the two boundaries of the non-mixing zones $z_\pm = z \mp tc\tau^\perp$. The associated Muskat type operator is (cf. Sect. 2.1)

$$B := \frac{1}{2}\sum_{a=\pm} B_a, \qquad B_a := \sum_{b=\pm} B_{a,b}, \tag{1.14}$$

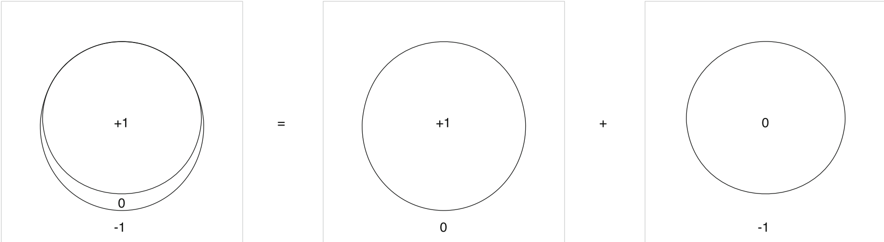


Fig. 2 The macroscopic density $\bar{\rho}$ (2.26) can be decomposed as the sum of the contribution of ρ_+ on $\Omega_+ \cup \Omega_{\mathrm{mix}}$ and ρ_- on $\Omega_- \cup \Omega_{\mathrm{mix}}$. In this way, the cusp singularities in Ω_{mix} can be understood as the superposition of regular domains

where

$$B_{a,b}(t,\alpha) := \frac{\rho_+ - \rho_-}{4\pi} \int \left(\frac{1}{z_a(t,\alpha) - z_b(t,\beta)} \right)_1 (\partial_\alpha z_a(t,\alpha) - \partial_\alpha z_b(t,\beta))\, \mathrm{d}\beta. \tag{1.15}$$

We remark in passing that, for open curves as in Theorem 1.2, all these integrals are taken with the Cauchy's principal value at infinity. However, we will focus on the closed case until Sect. 6 for clarity of exposition.

The evolution of z is driven by the operator B. On the one hand, as it is explained in the discussion after (1.13), in the partially unstable case there is always a non-mixing region where we must solve a classical Muskat equation exactly

$$\partial_t z = B \quad \text{on} \quad c(\alpha) = 0. \tag{1.16}$$

On the other hand, the flexibility of the notion of subsolution gives some space to define the pseudo-interface ([43, 64]). Namely, in the mixing region it is enough to solve (1.16) approximately

$$\partial_t z = B + \text{error} \quad \text{on} \quad c(\alpha) > 0,$$

where the error must be small in some sense that shall be specified in Sects. 2.1 and 2.4. Due to the Rayleigh–Taylor instability, it is to be expected that the choice error $= 0$ above yields an ill-posed equation as in the fully unstable regime. In spite of this, following another clever idea from [43], in the fully unstable regime it is possible to take error $= B^{1)} - B +$ error, where $B^{1)}$ denotes the first order expansion in time of B. This choice yields the following well-defined evolution for z

$$\partial_t z = B^{1)} + \text{error} \quad \text{on} \quad c(\alpha) > 0. \tag{1.17}$$

We remark that, if the error in (1.17) was zero, then the Eqs. (1.16) and (1.17) do not match at $c(\alpha) = 0$. In order to glue these equations we first introduce a partition of the unity $\{\psi_0, \psi_1\}$ which, as required in (1.9), allows also to open the mixing zone slightly inside the stable region, namely $\operatorname{supp}\psi_0 \subset \{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\operatorname{supp}\psi_1 = \operatorname{supp} c$. That is, we bypass the gluing problem by writing

$$\partial_t z = \psi_0 B + \psi_1 B^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

where the error is supported on $\{c(\alpha) > 0\}$. Yet the energy inequalities that we obtain for the operator $\psi_0 B$ (or other modifications) yields a factor $1/c$ which blows up in the region where $c(\alpha)$ tends to zero. The way out of this vicious circle is to treat the interaction between separate boundaries as a perturbation. In this way, one can write $B = E + \text{error}$ in such a way that E yields good energy inequalities and the error is small in the supremum norm and supported on $\{c(\alpha) > 0\}$. Thus, the perturbation can be absorbed in the relaxation even if its derivatives are badly behaving. Hence, we will solve

$$\partial_t z = \psi_0 E + \psi_1 E^{1)} + \text{error} \quad \text{on} \quad \mathbb{T},$$

for some error term supported on $\{c(\alpha) > 0\}$, where $E^{1)}$ denotes the first order expansion in time of E. Essentially, $E = B_{+,+} + B_{-,-}$ as the factor $1/c$ comes from the terms with $a \neq b$ in (1.15).

Organization of the Paper We start Sect. 2 by recalling briefly the Classical and the Mixing Muskat problem. After this, we recall also the concepts of mixing solution and subsolution, as well as the h-principle in IPM. Then, we define our ansatz for the subsolution in terms of the mixing zone and derive the conditions for the growth-rate c and the pseudo-interface z under which such subsolution truly exists. The construction of a pair (c, z) satisfying such requirements appears in Sects. 3–5. Finally, we prove in Sect. 6 the Theorems 1.1, 1.2 and the optimal regime for c given in (1.8).

Notation

- (Complex coordinates) It is convenient to identify the Euclidean space $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ as usual, $z = (z_1, z_2) = z_1 + iz_2$. Therefore, along the whole paper we will use complex coordinates and subindexes 1, 2 indicate real and imaginary parts for a complex number.
  Thus, $i \equiv (0, 1)$ plays the roll both of the standard vertical vector and the imaginary unit. We will denote $z^* := (z_1, -z_2) = z_1 - iz_2$, $z^\perp := (-z_2, z_1) = iz$ and $z \cdot w := z_1 w_1 + z_2 w_2 = (zw^*)_1$. In this regard, we also have $\nabla = (\partial_1, \partial_2) = \partial_1 + i\partial_2$, and so $\nabla^* = \partial_1 - i\partial_2$ and $\nabla^\perp = i\nabla$.
- (Function spaces) We will consider the usual Hölder spaces $C^{k,\delta}$ with norm

$$\|f\|_{C^{k,\delta}} := \sup_{j \le k} \|\partial^j f\|_{L^\infty} + |\partial^k f|_{C^\delta} \quad \text{with} \quad |g|_{C^\delta} := \sup_{\alpha,\beta} \frac{|g(\alpha) - g(\alpha - \beta)|}{|\beta|^\delta},$$

  and also the Sobolev spaces H^k with

$$\|f\|_{H^k} := \left(\sum_{j=0}^{k} \|\partial^j f\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

- (Increments and different quotients) Given a function $f = f(\alpha)$ and another parameter β, it will be handy to use the expressions $f' = f(\alpha - \beta)$, $\delta_\beta f = f - f'$ and $\Delta_\beta = \frac{\delta_\beta}{\beta}$ as in [29, 32].

2 The Mixing Zone and the Subsolution

2.1 The Muskat Problem

The Muskat problem describes IPM under the assumption that there is a time-dependent oriented curve $z(t, \alpha)$ separating $\mathbb{R}^2$ into two complementary open domains

$$\begin{aligned} \Omega_-(t) &\equiv \text{ domain to the left side of } z(t), \\ \Omega_+(t) &\equiv \text{ domain to the right side of } z(t), \end{aligned} \tag{2.1}$$

each one occupied by a fluid with different constant densities ρ_- and ρ_+ respectively.

The incompressibility condition (1.3) implies that $v = \nabla^\perp \psi$ for some stream function $\psi(t, x)$. Hence, the Darcy's law (1.4) can be written in complex coordinates as $\nabla(p + i\psi) = -i\rho$, which yields the following Poisson equation ($\nabla^*\nabla = \Delta$)

$$\Delta(p + i\psi) = -i\nabla^*\rho.$$

In view of (2.1), the density jump along z implies that

$$\nabla\rho = -(\rho_+ - \rho_-)\partial_\alpha z^\perp \delta_z,$$

in the sense of distributions. Hence, p and ψ are recovered from the Poisson equation through the Newtonian potential

$$(p + i\psi)(t, x) = \frac{\rho_+ - \rho_-}{2\pi} \int \log|x - z(t, \beta)| \partial_\alpha z(t, \beta)^* \, \mathrm{d}\beta, \qquad x \neq z(t, \beta).$$

Then, p and ψ are continuous but have discontinuous gradients along z, and indeed $\Delta(p + i\psi) = (\sigma + i\varpi)\delta_z$ where $\sigma \equiv$ Rayleigh–Taylor and $\varpi \equiv$ vorticity strength ($\Delta\psi = \nabla^\perp \cdot v = \omega$), which satisfy

$$\sigma + i\varpi = (\rho_+ - \rho_-)\partial_\alpha z^*. \tag{2.2}$$

The velocity v is recovered from the vorticity through the Biot-Savart law

$$\begin{aligned} v(t, x) &= \left(\frac{1}{2\pi i} \int \frac{\varpi(t, \beta)}{x - z(t, \beta)} \, \mathrm{d}\beta \right)^* \\ &= -\frac{\rho_+ - \rho_-}{2\pi} \int \left(\frac{1}{x - z(t, \beta)} \right)_1 \partial_\alpha z(t, \beta) \, \mathrm{d}\beta, \qquad x \neq z(t, \beta), \end{aligned} \tag{2.3}$$

where we have applied that $\varpi = -(\rho_+ - \rho_-)\partial_\alpha z_2$ and the Cauchy's argument principle in the last equality

$$\left(\int \frac{\partial_\alpha z(t,\beta)}{x - z(t,\beta)}\,\mathrm{d}\beta\right)_1 = 0, \qquad x \neq z(t,\beta). \tag{2.4}$$

It is easy to see that v is bounded, smooth outside z but with tangential discontinuities along z. Its normal component is well-defined and satisfies

$$\lim_{\Omega_\pm(t)\ni x\to z(t,\alpha)} (v(t,x) - B(t,\alpha))\cdot \partial_\alpha z(t,\alpha)^\perp = 0,$$

where

$$B(t,\alpha) := \frac{\rho_+ - \rho_-}{2\pi}\int \left(\frac{1}{z(t,\alpha) - z(t,\beta)}\right)_1 (\partial_\alpha z(t,\alpha) - \partial_\alpha z(t,\beta))\,\mathrm{d}\beta. \tag{2.5}$$

Observe that the operator B is obtained by adding a suitable tangential term to the velocity (2.3) and then taking the limit $\Omega_\pm(t)\ni x \to z(t,\alpha)$. We refer to (2.5) as the classical Muskat operator. Let us remark that this operator (2.5) coincides with (1.14) when tc is identically zero. Since it only appears in this Sect. 2.1 and the notation of the paper is heavy enough, we do not give it another name.

Finally, it is easy to check that the conservation of mass equation (1.2) is equivalent to find z satisfying

$$(\partial_t z - B)\cdot \partial_\alpha z^\perp = 0. \tag{2.6}$$

Thus, the Muskat problem is equivalent to solve this Cauchy problem for the interface z starting from z° given in (1.16). We remark that because of (2.6), one may add any tangential term to (1.16). This only changes the parametrization and does not modify the geometric evolution of the curve. We refer to (1.16) as the Classical Muskat problem.

Assuming that the interface can be parametrized as a graph, $z(t,\alpha) = \alpha + if(t,\alpha)$ in complex coordinates, the Eq. (1.16) reads as

$$\partial_t f = \frac{\rho_+ - \rho_-}{2\pi}\mathrm{pv}\int_{\mathbb{R}} \left(\frac{1}{1 + i\triangle_\beta f}\right)_1 \partial_\alpha \triangle_\beta f\,\mathrm{d}\beta,$$

which can be linearized as $\partial_t f = (\rho_+ - \rho_-)(-\Delta)^{1/2} f$. In analogy with the heat equation, the fully stable regime ($\rho_+ > \rho_-$) admits a parabolic analysis through energy estimates.

However, the same strategy for the fully unstable regime ($\rho_+ < \rho_-$) is not viable. Despite this, mixing solutions to IPM starting from fully unstable Muskat inital data have been constructed in the last years through the convex integration method [16, 43, 64, 71]. In these works, the mixing zone is given as in (1.10) (1.11) but with

$\tau = (-1, 0)$ instead of (1.12). More generally, we may consider any unitary vector field $\tau(\alpha)$ satisfying

$$(\rho_+ - \rho_-)\partial_\alpha z^\circ(\alpha) \cdot \tau(\alpha) > 0 \quad \text{on} \quad c(\alpha) > 0.$$

Thus, the triplet (τ, c, z) parametrizes the mixing zone, which does not exist when $tc(\alpha) = 0$. Here we follow [43, 64], where $\bar{\rho} = 0$ on Ω_{mix}. In this case, the coarse-grained velocity becomes

$$\bar{v}(t, x) = -\frac{\rho_+ - \rho_-}{4\pi} \sum_{b=\pm} \int \left(\frac{1}{x - z_b(t, \beta)} \right)_1 \partial_\alpha z_b(t, \beta)\, \mathrm{d}\beta, \qquad x \neq z_b(t, \beta),$$

where $z_\pm(t, \alpha) = z(t, \alpha) \mp tc(\alpha)\tau(\alpha)^\perp$ are the two boundaries of the non-mixing zones. The admissible regime for $c(\alpha)$ compatible with the relaxation of IPM is

$$\left| 2c(\alpha) + \frac{\sigma(\alpha)}{(\rho_+ - \rho_-)\partial_\alpha z^\circ(\alpha) \cdot \tau(\alpha)} \right| < 1 \quad \text{on} \quad c(\alpha) > 0, \tag{2.7}$$

which agrees with [43, 64] as in this case $\rho_\pm = \mp 1$, $\partial_\alpha z^\circ = (1, \partial_\alpha f^\circ)$ and $\tau = (-1, 0)$ (cf. Rem. 2.4). Observe that (2.7) requires $c(\alpha) = 0$ if $\partial_\alpha z_1^\circ(\alpha) = \partial_\alpha z^\circ(\alpha) \cdot \tau(\alpha)$. In view of (1.9), this prevents some choices for $\tau(\alpha)$ as for instance the one from [16, 19, 43]. Thus, we really need to optimize by opening the mixing zone perpendicularly to the curve (this is also the case in [58]). This is why we have chosen τ as in (1.12). With such optimal choice for τ, (2.7) reads as (1.13) (recall (2.2)).

Once $\tau(\alpha)$ and $c(\alpha)$ are fixed, we must determine the time-dependent pseudo-interface $z(t, \alpha)$. Modulo technical details which will be explained in Sect. 2.4, the existence of a relaxed momentum $\bar{m}(t, x)$ is reduced to find z satisfying

$$\int_0^\alpha ((\partial_t z - B) \cdot \partial_\alpha z^\perp + tD \cdot \partial_\alpha(c\tau))\, \mathrm{d}\alpha' = o(t)c(\alpha), \tag{2.8}$$

uniformly in α as $t \to 0$, where

$$D(t, \alpha) := -\frac{1}{2} \sum_{a=\pm} aB_a - i(c\tau + \tfrac{1}{2}), \tag{2.9}$$

with τ, c, B and B_a given in (1.10)–(1.15). Observe that $D \cdot \partial_\alpha(c\tau) = 0$ for $\partial_\alpha z = (1, \partial_\alpha f)$ and $\tau = (-1, 0)$, and thus it does not appear in [43, 64]. Hence, the Eq. (2.8) generalizes both (1.16) and (1.17). As we mentioned in the introduction, we cannot simply glue these evolution equations because they do not match at $c(\alpha) = 0$. In order to interpolate between the two regions, we introduce a partition of the unity $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively. Then, we consider (cf. (4.1))

$$\partial_t z = \psi_0 E + \psi_1 E^{1)} + \text{error}.$$

Here, E should be an extension of B and good for energy inequalities, which justify its name twice.

In view of (1.16) and (1.17), one would be initially tempted to take $E = B$. However, the terms with $a \neq b$ in (1.14) introduce a factor $\partial_\alpha \log c(\alpha)$ in the energy estimates which we did not see how to compensate. Thus, we will declare

$$E = \sum_{b=\pm} B_{b,b}, \tag{2.10}$$

which equals B on $tc(\alpha) = 0$ and only includes interaction of stable Muskat type.

The error term is localized on the mixing region with order t. This is

$$\text{error} = -(t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

where $D^{(0)} = D|_{t=0}$, $\kappa = \partial_t(E - B)|_{t=0}$ depends on the initial curvature and $h = O(t^2)$ is a time-dependent average. As a result, $D^{(0)}$ and κ only depends on z° while $h(t)$ depends on $z(t)$ but not on α. This allows to treat the error as a harmless term in the energy estimates.

2.2 Weak Solutions, Subsolutions and the Mixing Zone

Let us start by recalling the rigorous definition of weak solutions, mixing solutions and subsolutions in the IPM context.

Given $T > 0$ and ρ° as in (1.1), a **weak solution** to IPM

$$(\rho, v) \in C([0, T]; L^\infty_{w*}(\mathbb{R}^2; [-1, 1] \times \mathbb{R}^2))$$

satisfies that, for every test function $\phi \in C^1_c(\mathbb{R}^3)$ with $\phi^\circ := \phi|_{t=0}$ and $0 < t \leq T$:

$$\int_0^t \int_{\mathbb{R}^2} \rho(\partial_t \phi + v \cdot \nabla\phi)\, \mathrm{d}x\, \mathrm{d}s = \int_{\mathbb{R}^2} \rho(t)\phi(t)\, \mathrm{d}x - \int_{\mathbb{R}^2} \rho^\circ \phi^\circ\, \mathrm{d}x, \tag{2.11a}$$

$$\int_0^t \int_{\mathbb{R}^2} v \cdot \nabla\phi\, \mathrm{d}x\, \mathrm{d}s = 0, \tag{2.11b}$$

$$\int_0^t \int_{\mathbb{R}^2} (v + \rho i) \cdot \nabla^\perp \phi\, \mathrm{d}x\, \mathrm{d}s = 0. \tag{2.11c}$$

In addition, a weak solution is a **mixing solution** if, at each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$, satisfying that (ρ, v) is continuous on the non-mixing zones $\Omega_\pm$:

$$\rho = \pm 1 \quad \text{on} \quad \Omega_\pm, \tag{2.12}$$

while it behaves wildly inside the mixing zone Ω_{mix}:

$$\int_\Omega (1 - \rho^2)\, \mathrm{d}x = 0 < \int_\Omega (1 - \rho)\, \mathrm{d}x \int_\Omega (1 + \rho)\, \mathrm{d}x, \tag{2.13}$$

for every open $\emptyset \neq \Omega \subset \Omega_{\mathrm{mix}}(t)$.

Conversely, we say that (ρ, v) is a **non-mixing solution** if $\Omega_{\mathrm{mix}} = \emptyset$.

In convex integration, a subsolution (a macroscopic solution) is defined in term of a conservation law and a relaxed constitutive relation, which is typically given by the Λ-convex hull. In the IPM context, the hull was computed in [71] (see also [57] for related computations).

Given $T > 0$ and ρ° as in (1.1), a **subsolution** to IPM

$$(\bar{\rho}, \bar{v}, \bar{m}) \in C([0,T]; L^\infty_{w^*}(\mathbb{R}^2; [-1,1] \times \mathbb{R}^2 \times \mathbb{R}^2))$$

satisfies that, for every test function $\phi \in C^1_c(\mathbb{R}^3)$ with $\phi^\circ := \phi|_{t=0}$ and $0 < t \leq T$:

$$\int_0^t \int_{\mathbb{R}^2} (\bar{\rho}\partial_t \phi + \bar{m} \cdot \nabla\phi)\, \mathrm{d}x\, \mathrm{d}s = \int_{\mathbb{R}^2} \bar{\rho}(t)\phi(t)\, \mathrm{d}x - \int_{\mathbb{R}^2} \rho^\circ \phi^\circ\, \mathrm{d}x, \tag{2.14a}$$

$$\int_0^t \int_{\mathbb{R}^2} \bar{v} \cdot \nabla\phi\, \mathrm{d}x\, \mathrm{d}s = 0, \tag{2.14b}$$

$$\int_0^t \int_{\mathbb{R}^2} (\bar{v} + \bar{\rho} i) \cdot \nabla^\perp \phi\, \mathrm{d}x\, \mathrm{d}s = 0, \tag{2.14c}$$

such that, at each $0 < t \leq T$, the space $\mathbb{R}^2$ is split into three complementary open domains, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$ satisfying that

$$\bar{\rho} = \pm 1, \quad \bar{m} = \bar{\rho}\bar{v} \quad \text{on} \quad \Omega_\pm, \tag{2.15a}$$

$$|2(\bar{m} - \bar{\rho}\bar{v}) + (1 - \bar{\rho}^2) i| < (1 - \bar{\rho}^2) \quad \text{on} \quad \Omega_{\mathrm{mix}}. \tag{2.15b}$$

In addition, it is required that

$$\sup_{0 \leq t \leq T} \|\bar{v}(t)\|_{L^\infty} < \infty. \tag{2.16}$$

Remark 2.1 Notice that the pressure does not appear in (2.11c) (2.14c). For completeness we will show in Lemma A.1 how $p \in C([0,T] \times \mathbb{R}^2)$ is recovered and its relation with $\bar{p}$. We are not aware of similar computations for the IPM pressure in the convex integration framework.

Theorem 2.1 (H-principle in IPM) *Assume that there exists a subsolution $(\bar{\rho}, \bar{v}, \bar{m})$ to IPM starting from ρ°, for some $T > 0$, $\Omega_\pm$ and Ω_{mix}. Then, there exist infinitely many mixing solutions (ρ, v) to IPM starting from ρ°, for the same $T > 0$, $\Omega_\pm$ and Ω_{mix}, and satisfying $(\rho, v) = (\bar{\rho}, \bar{v})$ outside Ω_{mix}.*

The proof of this h-principle for the $L^\infty_{t,x}$ case can be found in [71], and the generalization to $C_t L^\infty_{w^*}$ in [19]. As noticed in [71], the inequality (2.15b) only provides solutions in L^2. Remarkably, Székelyhidi computed in [71,Prop. 2.4] the additional inequalities which yield solutions in L^∞. In [57,Lemma 4.3] it is checked that, if $\bar{v}$ is controlled as in (2.16), then these additional inequalities are automatically satisfied. By Theorem 2.1, the construction of mixing solutions as stated in Theorems 1.1 and 1.2 is reduced to constructing suitable subsolutions $(\bar{\rho}, \bar{v}, \bar{m})$ adapted to Ω_{mix}.

As described in the intro, it follows from (1.10)-(1.12) that the mixing zone is prescribed by the **growth-rate** c and the **pseudo-interface** z. With this terminology, for bubble interfaces (1.6) we define

$$\begin{aligned} \Omega_-(t) &\equiv \text{ exterior domain of } z_-(t), \\ \Omega_+(t) &\equiv \text{ interior domain of } z_+(t), \end{aligned} \tag{2.17}$$

and for turned interfaces (1.7)

$$\begin{aligned} \Omega_-(t) &\equiv \text{ upper domain of } z_-(t), \\ \Omega_+(t) &\equiv \text{ lower domain of } z_+(t). \end{aligned} \tag{2.18}$$

Thus, $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$ are complementary open domains in $\mathbb{R}^2$. For each region $r = +, -, \mathrm{mix}$, we denote $\Omega_r := \{(t, x) : x \in \Omega_r(t),\ 0 \le t \le T\}$.

Remark 2.2 For the sake of simplicity we will consider from now on the closed case (2.17) and go back at Sect. 6 with the open case (2.18). Recall that for closed interfaces we have assumed that z° is clockwise oriented ($\circlearrowright$). In addition, we may assume w.l.o.g. that z° is the arc-length ($|\partial_\alpha z^\circ| = 1$) parametrization, although $z(t)$ will not be it in general. Thus, we fix $\mathbb{T} = [-\ell_\circ/2, \ell_\circ/2]$ where $\ell_\circ := \mathrm{length}(z^\circ)$.

For a general pair (c, z) we construct in the next Sect. 2.3 a suitable triplet $(\bar\rho, \bar v, \bar m)$ adapted to Ω_{mix}. After this, we derive in Sect. 2.4 conditions for (c, z) under which this $(\bar\rho, \bar v, \bar m)$ becomes a subsolution. Finally, we will prove in Sects. 3–5 the existence of a pair (c, z) satisfying such requirements.
Hipothesis on (c, z). Apart from regularity assumptions, the curve needs to satisfy an angle and a chord-arc condition in a uniform manner. Prior to state the assumptions, let us introduce the angle constant of z w.r.t. τ

$$\mathcal{A}(z) := \inf\left\{ \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|} \cdot \tau(\alpha) \ :\ \alpha \in \mathbb{T} \right\}, \tag{2.19}$$

and recall the definition of a chord-arc curve.

Definition 2.1 A curve $z \in C(\mathbb{T}; \mathbb{R}^2)$ is **chord-arc** if

$$\mathcal{C}(z) := \sup\left\{ \left| \frac{\beta}{z(\alpha) - z(\alpha - \beta)} \right| \ :\ \alpha, \beta \in \mathbb{T} \right\} < \infty. \tag{2.20}$$

Along the rest of this Sect. 2 we will assume the existence of $\delta, T > 0$ such that

$$c \in C^{1,\delta}(\mathbb{T}), \qquad c \ge 0, \tag{2.21}$$

and

$$z \in C^1([0, T]; C^{1,\delta}(\mathbb{T}; \mathbb{R}^2)), \qquad z|_{t=0} = z^\circ \in C^{2,\delta}(\mathbb{T}; \mathbb{R}^2), \tag{2.22}$$

satisfying the following equi-angle condition

$$\mathcal{A}(c,z) := \inf\left\{\frac{\partial_\alpha z_\lambda(t,\alpha)}{|\partial_\alpha z_\lambda(t,\alpha)|}\cdot\tau(\alpha)\,:\,\alpha\in\mathbb{T},\ \lambda\in[-1,1],\ 0\le t\le T\right\}>0, \tag{2.23}$$

and the following equi-chord-arc condition

$$\mathcal{C}(c,z) := \sup\left\{\frac{\sqrt{\beta^2+((\lambda-\mu)tc(\alpha))^2}}{|z_\lambda(t,\alpha)-z_\mu(t,\alpha-\beta)|}\,:\,\alpha,\beta\in\mathbb{T},\ \lambda,\mu\in[-1,1],\ 0\le t\le T\right\}<\infty, \tag{2.24}$$

where we recall that $z_\lambda = z-\lambda tc\tau^\perp$ with $\tau=\partial_\alpha z^\circ$.
The condition (2.23) controls the angle between the family of curves z_λ w.r.t. τ. The equi-chord-arc condition (2.24) bounds the singularity due to the denominator of the operators $B_{a,b}$ (1.15), while the numerator justifies the regularity assumptions (2.21) (2.22). In addition, all they have the following useful consequence.

Remark 2.3 The conditions (2.21)-(2.24) imply that map $(\alpha,\lambda)\mapsto z_\lambda(t,\alpha)$ is a diffeomorphism from $\{c(\alpha)>0\}\times(-1,1)$ to $\Omega_{\mathrm{mix}}(t)$ with Jacobian $tc(\partial_\alpha z_\lambda\cdot\tau)>0$.

In Sect. 3 we will construct a suitable smooth growth-rate c. Once c is fixed, we will still assume (2.22)-(2.24) in Sect. 4. Finally, we will construct a time-dependent pseudo-interface z satisfying such conditions in Sect. 5.
We conclude this subsection by proving an auxiliary lemma which allows to integrate by parts, under certain conditions, on the domain with cusp singularities Ω_{mix}.

Lemma 2.1 *Fix $0\le t\le T$. Let $f\in L^\infty(\mathbb{R}^2)$ satisfying that $f\in C^1$ with $\nabla\cdot f=0$ outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$ and with well-defined continuous limits*

$$\begin{aligned} f_a^a(\alpha) &:= \lim_{\Omega_a(t)\ni x\to z_a(t,\alpha)} f(x),\\ f_a^{\mathrm{mix}}(\alpha) &:= \lim_{\Omega_{\mathrm{mix}}(t)\ni x\to z_a(t,\alpha)} f(x), \qquad tc(\alpha)>0, \end{aligned} \tag{2.25}$$

whenever $z_a(t,\alpha)\in\partial\Omega_r(t)$ for $a=\pm$ and $r=+,-,\mathrm{mix}$. Then, for every $\phi\in C_c^1(\mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}^2} f\cdot\nabla\phi\,\mathrm{d}x &= \int_{tc(\alpha)=0}(f_+^+-f_-^-)\cdot\partial_\alpha z^\perp(\phi\circ z)\,\mathrm{d}\alpha\\ &\quad+\sum_{a=\pm}a\int_{tc(\alpha)>0}(f_a^a-f_a^{\mathrm{mix}})\cdot\partial_\alpha z_a^\perp(\phi\circ z_a)\,\mathrm{d}\alpha. \end{aligned}$$

Proof First of all we split the integral over $\mathbb{R}^2$ into $\Omega_+(t)$, $\Omega_-(t)$ and $\Omega_{\mathrm{mix}}(t)$. On the one hand, by applying the Gauss divergence theorem on the regular domains $\Omega_a(t)$ for $a=\pm$, we get

$$\int_{\Omega_a(t)} f\cdot\nabla\phi\,\mathrm{d}x = a\int_{\mathbb{T}} f_a^a\cdot\partial_\alpha z_a^\perp(\phi\circ z_a)\,\mathrm{d}\alpha,$$

where we have applied that $\nabla \cdot f = 0$ outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$ and that the normal vector to $\partial\Omega_a(t)$ pointing outward is $a\partial_\alpha z_a(t)^\perp$. This concludes the proof for $t = 0$. Now we pay special attention to the cusp singularities in $\Omega_{\mathrm{mix}}(t)$ for $0 < t \leq T$. For any $\varepsilon > 0$ we define

$$\Omega^\varepsilon_{\mathrm{mix}}(t) := \{z_\lambda(t,\alpha) \,:\, c(\alpha) > \varepsilon,\ \lambda \in (-1,1)\},$$

which forms an exhaustion by Lipschitz domains of $\Omega_{\mathrm{mix}}(t)$. Hence, we can apply first the dominated convergence and then the Gauss divergence theorem on $\Omega^\varepsilon_{\mathrm{mix}}(t)$ to obtain

$$\int_{\Omega_{\mathrm{mix}}(t)} f \cdot \nabla\phi \,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\Omega^\varepsilon_{\mathrm{mix}}(t)} f \cdot \nabla\phi \,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\partial\Omega^\varepsilon_{\mathrm{mix}}(t)} (f^{\mathrm{mix}} \cdot n^\varepsilon)\phi \,\mathrm{d}\sigma,$$

where f^{mix} denotes the limit of f on $\partial\Omega^\varepsilon_{\mathrm{mix}}(t)$ and n^ε is the unit normal vector to $\partial\Omega^\varepsilon_{\mathrm{mix}}(t)$ pointing outward. Since $f \in L^\infty$ it follows that the contribution of the boundary integral on $c(\alpha) = \varepsilon$, $\lambda \in (-1,1)$ is zero in the limit $\varepsilon \to 0$. Therefore, we deduce that

$$\int_{\Omega_{\mathrm{mix}}(t)} f \cdot \nabla\phi \,\mathrm{d}x = -\sum_{a=\pm} a \int_{tc(\alpha)>0} f^{\mathrm{mix}}_a \cdot \partial_\alpha z_a^\perp (\phi \circ z_a) \,\mathrm{d}\alpha.$$

This concludes the proof. □

2.3 The Subsolution

2.3.1 The Density

Following [43, 64] we declare $\bar{\rho} = 0$ on Ω_{mix}:

$$\bar{\rho}(t,x) := \mathbb{1}_{\Omega_+(t)}(x) - \mathbb{1}_{\Omega_-(t)}(x), \tag{2.26}$$

with $\Omega_\pm(t)$ given in (2.17). As a result, $\partial_1\bar{\rho}(t)$ is a Dirac measure supported on $\partial\Omega_+(t) \cup \partial\Omega_-(t)$ with density $(\partial_\alpha z_a(t))_2$ on each $\partial\Omega_a(t)$ for $a = \pm$.

Lemma 2.2 *For every $\phi \in C^1_c(\mathbb{R}^2)$ and $0 \leq t \leq T$,*

$$\int_{\mathbb{R}^2} \bar{\rho}(t)\partial_1\phi \,\mathrm{d}x = -\sum_{a=\pm} \int_{\mathbb{T}} (\partial_\alpha z_a(t,\alpha))_2 \phi(z_a(t,\alpha)) \,\mathrm{d}\alpha.$$

Proof It follows from Lemma 2.1 applied to $f = \bar{\rho}$ because, in this case, we have $1 \cdot \partial_\alpha z_a^\perp = -(\partial_\alpha z_a)_2$, $\bar{\rho}^a_a = a$ and $\bar{\rho}^{\mathrm{mix}}_a = 0$. □

2.3.2 The Velocity

In view of Lemma 2.2 we define $\bar{v}$ by means of the Biot-Savart law (cf. (2.3))

$$\bar{v}(t,x)^* = -\frac{1}{2\pi i}\sum_{b=\pm}\int_{\mathbb{T}} \frac{(\partial_\alpha z_b(t,\beta))_2}{x - z_b(t,\beta)}\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta). \tag{2.27}$$

Observe that $\bar{v}$ is continuous (indeed $C_t^1 C^\omega$) outside $\partial\Omega_+ \cup \partial\Omega_-$. Proposition 2.1 shows hat $\bar{v}$ satisfies the Eqs. (2.14b) (2.14c), the boundedness condition (2.16) and that has well-defined continuous limits (2.25). Proposition 2.2 shows that the normal component of $\bar{v}$ on $\partial\Omega_+ \cup \partial\Omega_-$ is well-defined and continuous. Figure 2 explains why this is not surprising.

Proposition 2.1 *Let $\bar{\rho}$ be as in (2.26). The unique velocity satisfying (2.14b) (2.14c) which additionally vanishes as $|x| \to \infty$ is precisely (2.27). Moreover, $\bar{v}$ is uniformly bounded on $[0,T]\times\mathbb{R}^2$ and has well-defined continuous limits (2.25).*

Proof *Step 1 $\bar{v}$ in (2.27) is uniformly bounded and has well-defined continuous limits* (2.25). First of all notice that $\bar{v}$ is continuous (indeed $C_t^1 C^\omega$) outside $\partial\Omega_+ \cup \partial\Omega_-$. Moreover, for any $(t,x) \notin \partial\Omega_+ \cup \partial\Omega_-$ it holds that

$$\begin{aligned}
|\bar{v}(t,x)| &= \frac{1}{2\pi}\left|\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{x - z_b(t,\beta)} - \frac{1}{x - z_b(t,0)}\right)(\partial_\alpha z_b(t,\beta))_2\,\mathrm{d}\beta\right| \\
&\leq \frac{\ell_\circ^2}{8\pi}\sum_{b=\pm}\frac{\|\partial_\alpha z_b\|_{C_t C^0}^2}{\mathrm{dist}(x,\partial\Omega_b(t))^2},
\end{aligned}$$

that is, $\bar{v}$ decays as $|x|^{-2}$ when $|x| \to \infty$.

Let us manipulate the expression (2.27) to help better understand the behavior of $\bar{v}$ near the boundary $\partial\Omega_+ \cup \partial\Omega_-$. We will use de index w.r.t. $z_b(t)$ of points x outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$:

$$\mathrm{Ind}_{z_b(t)}(x) := \frac{1}{2\pi i}\int_{z_b(t)}\frac{\mathrm{d}z}{x - z} = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\beta)}{x - z_b(t,\beta)}\,\mathrm{d}\beta.$$

Recall that $z(t)$ is clockwise oriented ($\circlearrowright$). Hence, the Cauchy's argument principle yields

$$\begin{aligned}
\mathrm{Ind}_{z_+(t)}(x) &= \mathbb{1}_{\Omega_+(t)}(x), \\
\mathrm{Ind}_{z_-(t)}(x) &= 1 - \mathbb{1}_{\Omega_-(t)}(x).
\end{aligned} \tag{2.28}$$

In order to compute the limits (2.25), for any x outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$ but close enough and $a = \pm$, we take $\alpha(x,a) \in \mathbb{T}$ minimizing $|x - z_a(t,\alpha)|$. Then, a straightforward identity of complex numbers yields

$$
\begin{aligned}
\bar{v}(t,x)^* &= \sum_{b=\pm}\left(\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(\partial_\alpha z_b(t,\beta))_2}{x-z_b(t,\beta)}\left(\frac{\partial_\alpha z_b(t,\beta)}{(\partial_\alpha z_b(t,\beta))_2}\frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)}-1\right)\mathrm{d}\beta\right.\\
&\quad\left.-\frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)}\mathrm{Ind}_{z_b(t)}(x)\right)\\
&=\sum_{b=\pm}\left(\frac{1}{2\pi i}\frac{1}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)\cdot\partial_\alpha z_b(t,\beta)^\perp}{x-z_b(t,\beta)}\mathrm{d}\beta-\frac{(\partial_\alpha z_b(t,\alpha))_2}{\partial_\alpha z_b(t,\alpha)}\mathrm{Ind}_{z_b(t)}(x)\right).
\end{aligned}
\tag{2.29}
$$

On the one hand, the regularity conditions (2.21)–(2.24) allow to apply the dominated convergence theorem on the first term in (2.29) as $x\to z_a(t,\alpha)$. On the other hand, the limits $\Omega_r(t)\ni x\to z_a(t,\alpha)$ in the second term in (2.29) change depending on the region $r=+,-,\text{mix}$ where x is coming from due to (2.28). Therefore, $\bar{v}$ has well-defined continuous limits (2.25)

$$
\begin{aligned}
\bar{v}_+^+ &= V_+ - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*} - \frac{(\partial_\alpha z_+)_2}{(\partial_\alpha z_+)^*},\\
\bar{v}_+^{\mathrm{mix}} &= V_+ - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*},\\
\bar{v}_-^{\mathrm{mix}} &= V_- - \frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*},\\
\bar{v}_-^- &= V_-,
\end{aligned}
\tag{2.30}
$$

where

$$
V_a := \sum_{b=\pm} V_{a,b} \quad\text{with}\quad V_{a,b}(t,\alpha)^* := \frac{1}{2\pi i}\frac{1}{\partial_\alpha z_b(t,\alpha)}\int_{\mathbb{T}}\frac{\partial_\alpha z_b(t,\alpha)\cdot\partial_\alpha z_b(t,\beta)^\perp}{z_a(t,\alpha)-z_b(t,\beta)}\mathrm{d}\beta,
$$

for $(t,\alpha)\in[0,T]\times\mathbb{T}$ and $a,b=\pm$. Finally, it follows that $\bar{v}$ is uniformly bounded on $[0,T]\times\mathbb{R}^2$.

Step 2 $\bar{v}$ satisfies (2.14b) (2.14c). Observe that $\bar{v}(t)^*$ is holomorphic outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$ for all $0\le t\le T$. Thus, the Cauchy–Riemann equations imply that $\nabla\cdot\bar{v}(t)=\nabla^\perp\cdot\bar{v}(t)=0$ outside $\partial\Omega_+(t)\cup\partial\Omega_-(t)$. Notice also that $z_+=z_-$ and $V_+=V_-$ on $tc(\alpha)=0$. In particular,

$$
\begin{aligned}
\bar{v}_+^+ - \bar{v}_-^- &= -\left(\frac{(\partial_\alpha z_+)_2}{(\partial_\alpha z_+)^*}+\frac{(\partial_\alpha z_-)_2}{(\partial_\alpha z_-)^*}\right) \quad \text{on} \quad tc(\alpha)=0,\\
\bar{v}_a^a - \bar{v}_a^{\mathrm{mix}} &= -a\frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*} \quad \text{on} \quad tc(\alpha)>0.
\end{aligned}
$$

Let $\phi\in C_c^1(\mathbb{R}^2)$ and $0<t\le T$. Then, by applying Lemma 2.1 to $f=\bar{v}$ and $\bar{v}^\perp$, we deduce that

$$
\int_{\mathbb{R}^2}\bar{v}\cdot\nabla\phi\,\mathrm{d}x = -\sum_{a=\pm}\int_{\mathbb{T}}\frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*}\cdot\partial_\alpha z_a^\perp(\phi\circ z_a)\,\mathrm{d}\alpha = 0,
$$

$$\int_{\mathbb{R}^2} \bar{v} \cdot \nabla^{\perp}\phi \, \mathrm{d}x = \sum_{a=\pm} \int_{\mathbb{T}} \frac{(\partial_\alpha z_a)_2}{(\partial_\alpha z_a)^*} \cdot \partial_\alpha z_a (\phi \circ z_a) \, \mathrm{d}\alpha = \sum_{a=\pm} \int_{\mathbb{T}} (\partial_\alpha z_a)_2 (\phi \circ z_a) \, \mathrm{d}\alpha.$$

These identities jointly with Lemma 2.2 imply that $\bar{v}$ satisfies (2.14b) (2.14c).

Step 3 Uniqueness Finally, it is easy to check that any solution to (2.14b) (2.14c) has the form $\bar{u} = \bar{v} + f^*$ for some (time-dependent) entire function f. Thus, if $\bar{u}$ vanishes as $|x| \to \infty$ too, the Liouville's theorem implies that $f = 0$. □

In the next lemma we deal with the normal component of $\bar{v}$ at the boundary of the mixing zone $\partial\Omega_+(t) \cup \partial\Omega_-(t)$. We will use the notation from Lemma 2.1 for the outer an inner limits and the operators B, B_a defined in the intro (1.14), (1.15).

Proposition 2.2 *Let $a = \pm$ and $r = +, -\mathrm{mix}$. Then, it holds that*

$$(\bar{v}_a^r - B_a) \cdot \partial_\alpha z_a^{\perp} = 0,$$

on $[0, T] \times \mathbb{T}$. *In particular,*

$$(\bar{v}_a^r - B) \cdot \partial_\alpha z^{\perp} = 0 \quad on \quad tc(\alpha) = 0.$$

Proof Let x be outside $\partial\Omega_+(t) \cup \partial\Omega_-(t)$ but close enough. Firstly, using that $(\mathrm{Ind}_{z_b(t)}(x))_2 = 0$, it follows that the velocity (2.27) can be written as (cf. (2.3) (2.4))

$$\bar{v}(t, x) = -\frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{x - z_b(t, \beta)} \right)_1 \partial_\alpha z_b(t, \beta) \, \mathrm{d}\beta.$$

In particular,

$$\begin{aligned} &\bar{v}(t, x) \cdot \partial_\alpha z_a(t, \alpha)^{\perp} \\ &= \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{x - z_b(t, \beta)} \right)_1 (\partial_\alpha z_a(t, \alpha) - \partial_\alpha z_b(t, \beta)) \, \mathrm{d}\beta \cdot \partial_\alpha z_a(t, \alpha)^{\perp}, \end{aligned}$$

where we take $\alpha \in \mathbb{T}$ as in (2.29). Thus, it remains to show that we can take the limit $x \to z_a(t, \alpha)$ in the r.h.s. above. By writing $z_a = z_b - i(a - b)tc\tau$, we split the above integrals into

$$\begin{aligned} &\int_{\mathbb{T}} \left(\frac{1}{x - z_b(t, \beta)} \right)_1 (\partial_\alpha z_b(t, \alpha) - \partial_\alpha z_b(t, \beta)) \, \mathrm{d}\beta - i(a - b)t\partial_\alpha (c(\alpha)\tau(\alpha)) \\ &\quad \left(\int_{\mathbb{T}} \frac{\mathrm{d}\beta}{x - z_b(t, \beta)} \right)_1. \end{aligned}$$

Analogously to (2.29), the regularity conditions (2.21)–(2.24) allow to apply the dominated convergence theorem on the first term as $x \to z_a(t, \alpha)$. Notice that the second term vanishes for $(a-b)tc(\alpha) = 0$. Otherwise, we can consider directly $x \to z_a(t, \alpha)$. This implies the first statement. As a by-product, we have seen that $B_{a,b}$ is split into

$$\begin{aligned} B_{a,b}(t,\alpha) = \frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{1}{z_a(t,\alpha) - z_b(t,\beta)} \right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta)) \, \mathrm{d}\beta \\ - i(a-b)t\partial_\alpha(c(\alpha)\tau(\alpha)) \frac{1}{2\pi} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\beta}{z_a(t,\alpha) - z_b(t,\beta)} \right)_1 . \end{aligned} \tag{2.31}$$

Finally, the second statement follows from the fact that $z_+ = z_-$ and $B_+ = B_- = B$ on $tc(\alpha) = 0$. □

2.3.3 The Relaxed Momentum

In view of the inequality (2.15), it seems suitable to define $\bar{m}$ as

$$\bar{m} := \bar{\rho}\bar{v} - (1 - \bar{\rho}^2)(\gamma + \tfrac{1}{2} i), \tag{2.32}$$

in terms of some $\gamma \in C(\Omega_{\mathrm{mix}}; \mathbb{R}^2)$ to be determined ([16, 43, 64, 71]). Moreover, for any $0 < t \leq T$ we assume that $\gamma(t) \in C^1(\Omega_{\mathrm{mix}}(t))$ with continuous limits

$$\gamma_a(t,\alpha) := \lim_{\Omega_{\mathrm{mix}}(t) \ni x \to z_a(t,\alpha)} \gamma(t,x),$$

whenever $z_a(t,\alpha) \in \partial\Omega_{\mathrm{mix}}(t)$ for $a = \pm$ and $c(\alpha) > 0$.

2.4 Compatibility Between the Mixing Zone and the Subsolution

In the next proposition we derive the conditions for (c, z, γ) under which the corresponding $(\bar{\rho}, \bar{v}, \bar{m})$ given in (2.26),(2.27) and(2.32) becomes a subsolution.

Proposition 2.3 *Assume that* (c, z) *satisfies* (2.21)–(2.24) *for some* $T > 0$. *The triplet* $(\bar{\rho}, \bar{v}, \bar{m})$ *given in* (2.26), (2.27) *and* (2.32) *defines a subsolution to IPM if and only if the triplet* (c, z, γ) *satisfies the following equations on* $\partial\Omega_a$ *for* $a = \pm$

$$(\partial_t z - B) \cdot \partial_\alpha z^\perp = 0 \quad \text{on} \quad tc(\alpha) = 0, \tag{2.33a}$$

$$(\partial_t z_a - B_a - a(\gamma_a + \tfrac{1}{2} i)) \cdot \partial_\alpha z_a^\perp = 0 \quad \text{on} \quad tc(\alpha) > 0, \tag{2.33b}$$

and the following conditions on Ω_{mix}

$$\nabla \cdot \gamma = 0, \tag{2.34a}$$

$$|\gamma| < \tfrac{1}{2}. \tag{2.34b}$$

Proof Recall that $(\bar{\rho}, \bar{v}, \bar{m})$ already satisfies the Eqs. (2.14b) (2.14c) and the conditions (2.15a) (2.16) (see Prop. 2.1). Moreover, it is clear that $(\bar{\rho}, \bar{v}, \bar{m})$ satisfies the Eq.

(2.14a) on Ω_{mix} and the inequality (2.15b) if and only if (2.34) holds. Thus, it remains to analyze (2.14a) outside Ω_{mix}.

Let $\phi \in C_c^1(\mathbb{R}^3)$ and $0 < t \leq T$. On the one hand, since $\bar{\rho} = 0$ on Ω_{mix} and $\bar{\rho} = \pm 1$ on the regular domains $\Omega_{\pm}(t)$, an integration by parts yields

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} \bar{\rho} \partial_t \phi \,\mathrm{d}x \,\mathrm{d}s - \int_{\mathbb{R}^2} \bar{\rho}(t)\phi(t)\,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^\circ \phi^\circ \,\mathrm{d}x \\
&\quad = -2 \int_0^t \int_{c(\alpha)=0} \partial_t z \cdot \partial_\alpha z^\perp (\phi \circ Z)\,\mathrm{d}\alpha \,\mathrm{d}s \\
&\quad\quad - \sum_{a=\pm} \int_0^t \int_{c(\alpha)>0} \partial_t z_a \cdot \partial_\alpha z_a^\perp (\phi \circ Z_a)\,\mathrm{d}\alpha \,\mathrm{d}s,
\end{aligned}$$

where $Z(t,\alpha) := (t, z(t,\alpha))$ and $Z_a(t,\alpha) := (t, z_a(t,\alpha))$. On the other hand, notice (2.32) reads as

$$\bar{m}(t,x) = \begin{cases} \pm \bar{v}(t,x), & x \in \Omega_{\pm}(t), \\ -(\gamma(t,x) + \frac{1}{2} i), & x \in \Omega_{\mathrm{mix}}(t). \end{cases}$$

In particular, $\bar{m}$ satisfies the assumptions in Lemma 2.1. Therefore, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} \bar{m} \cdot \nabla \phi \,\mathrm{d}x = 2 &\int_{c(\alpha)=0} B \cdot \partial_\alpha z^\perp (\phi \circ Z)\,\mathrm{d}\alpha \\
&+ \sum_{a=\pm} \int_{c(\alpha)>0} B_a \cdot \partial_\alpha z_a^\perp (\phi \circ Z_a)\,\mathrm{d}\alpha \\
&+ \sum_{a=\pm} a \int_{c(\alpha)>0} (\gamma_a + \tfrac{1}{2} i) \cdot \partial_\alpha z_a^\perp (\phi \circ Z_a)\,\mathrm{d}\alpha,
\end{aligned}$$

where we have applied Proposition 2.2 in the first two lines above. In summary, we have seen that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} (\bar{\rho} \partial_t \phi + \bar{m} \cdot \nabla \phi)\,\mathrm{d}x \,\mathrm{d}s - \int_{\mathbb{R}^2} \bar{\rho}(t)\phi(t)\,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^\circ \phi^\circ \,\mathrm{d}x \\
&= -2 \int_0^t \int_{c(\alpha)=0} (\partial_t z - B) \cdot \partial_\alpha z^\perp (\phi \circ Z)\,\mathrm{d}\alpha \,\mathrm{d}s \\
&\quad - \sum_{a=\pm} \int_0^t \int_{c(\alpha)>0} ((\partial_t z_a - B_a - a(\gamma_a + \tfrac{1}{2} i)) \cdot \partial_\alpha z_a^\perp)(\phi \circ Z_a)\,\mathrm{d}\alpha \,\mathrm{d}s.
\end{aligned}$$

This concludes the proof. □

We conclude this section by showing that we can construct $\gamma(t,x)$ satisfying the requirements in Proposition 2.3 provided that (c,z) satisfies certain conditions. Observe that $\{c(\alpha) > 0\}$ is open and thus a (countable) union of disjoint intervals (α_1, α_2). Recall the definition of B and D from (1.14) and (2.9).

Lemma 2.3 *Assume that* (c, z) *satisfies* (2.21)–(2.24) *for some* $T > 0$*. Assume further that the following conditions hold uniformly on* $\{c(\alpha) > 0\}$

$$|2c(\alpha) + \partial_\alpha z_1^\circ(\alpha)| < 1, \tag{2.35}$$

and

$$\partial_t z - B_a = o(1), \tag{2.36a}$$

$$\frac{1}{tc(\alpha)} \int_{\alpha_1}^{\alpha} ((\partial_t z - B) \cdot \partial_\alpha z^\perp + tD \cdot \partial_\alpha(c\tau))\, \mathrm{d}\alpha' = o(1), \tag{2.36b}$$

as $t \to 0$*, for* $a = \pm$ *and* $\alpha \in (\alpha_1, \alpha_2)$ *connected component of* $\{c(\alpha) > 0\}$*. Then, there exists* $0 < T' \leq T$ *and* $\gamma(t, \alpha)$ *satisfying* (2.33b)–(2.34) *as long as* $0 < t \leq T'$*.*

Proof *Step 1 Analysis of* (2.33b)–(2.34). For simplicity we may assume w.l.o.g. that there is one connected component $(\alpha_1, \alpha_2) = \{c(\alpha) > 0\}$. Recall that $(\alpha, \lambda) \mapsto z_\lambda(t, \alpha)$ is a diffeomorphism from $(\alpha_1, \alpha_2) \times (-1, 1)$ to $\Omega_{\mathrm{mix}}(t)$ (cf. Rem. 2.3). In particular, since $\Omega_{\mathrm{mix}}(t)$ is simply-connected, (2.34a) implies that $\gamma(t) = \nabla^\perp g(t)$ for some $g(t) \in C^1(\Omega_{\mathrm{mix}}(t))$ to be determined. Moreover, g can be defined in terms of some G in (α, λ)-coordinates as

$$g(Z(t, \alpha, \lambda)) := G(t, \alpha, \lambda),$$

where $Z(t, \alpha, \lambda) := (t, z_\lambda(t, \alpha))$. Notice that (recall $z_\lambda = z - \lambda tc\tau^\perp$)

$$\partial_\alpha G = (\nabla g \circ Z) \cdot \partial_\alpha z_\lambda, \qquad \partial_\lambda G = -tc(\nabla g \circ Z) \cdot \tau^\perp.$$

On the one hand, the boundary conditions (2.33b) for γ read as

$$\partial_\alpha G(t, \alpha, a) = (a(\partial_t z - B_a) - i(c\tau + \tfrac{1}{2})) \cdot \partial_\alpha z_a^\perp, \qquad a = \pm. \tag{2.37}$$

On the other hand, for (2.34b) notice that

$$\nabla g \circ Z = \frac{1}{\partial_\alpha z_\lambda \cdot \tau} \left(\partial_\alpha G \tau - \frac{1}{tc} \partial_\lambda G \partial_\alpha z_\lambda^\perp \right). \tag{2.38}$$

In particular,

$$\partial_\alpha z_\lambda \cdot \tau = \partial_\alpha z^\circ \cdot \tau + t \left(\fint_0^t \partial_t z \, \mathrm{d}s - \lambda \partial_\alpha (c\tau^\perp) \right) \cdot \tau = \partial_\alpha z^\circ \cdot \tau + O(t),$$

with $\partial_\alpha z^\circ \cdot \tau > 0$ uniformly on $c(\alpha) > 0$ by (2.23). Therefore, assuming that c satisfies (2.35), it is enough to find G satisfying (2.37) and the following growth conditions

$$\partial_\alpha G = o(1) - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda, \qquad \frac{1}{tc} \partial_\lambda G = o(1), \tag{2.39}$$

uniformly on $c(\alpha) > 0$ as $t \to 0$, because in this case (recall $\tau = \partial_\alpha z^\circ$ with $|\partial_\alpha z^\circ| = 1$)

$$|\gamma| = |\nabla g| \leq \left| c + \tfrac{1}{2} \tfrac{\partial_\alpha z_1^\circ}{\partial_\alpha z^\circ \cdot \tau} \right| + o(1) < \tfrac{1}{2}. \tag{2.40}$$

Step 2 Ansatz for G. We declare

$$G(t, \alpha, \lambda) := \int_{\alpha_1}^{\alpha} \left(\sum_{a=\pm} \frac{\lambda + a}{2} (\partial_t z - B_a) \cdot \partial_\alpha z_a^\perp - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda \right) \mathrm{d}\alpha'. \tag{2.41}$$

Hence, it follows that

$$\begin{aligned} \partial_\alpha G &= \sum_{a=\pm} \frac{\lambda + a}{2} (\partial_t z - B_a) \cdot \partial_\alpha z_a^\perp - (c\tau + \tfrac{1}{2}) \cdot \partial_\alpha z_\lambda, \\ \partial_\lambda G &= \int_{\alpha_1}^{\alpha} \Big((\partial_t z - B) \cdot \partial_\alpha z^\perp + t D \cdot \partial_\alpha (c\tau) \Big) \mathrm{d}\alpha'. \end{aligned}$$

Notice that (2.37) is satisfied. Finally, assuming that z satisfies (2.36), then (2.39) holds. □

Remark 2.4 Following the previous proof, notice that for $z(t, \alpha) = (\alpha, f(t, \alpha))$ and $\tau = (-1, 0)$ as in [43, 64], the admissible regime for $c(\alpha)$ reads as

$$|2c(\alpha) - 1| < 1,$$

which clear is incompatibly with $c(\alpha) = 0$. Remarkably, the authors in [64] achieved that $c(\alpha) \to 0$ in the limiting case $|\alpha| \to \infty$ for $\alpha \in \mathbb{R}$.

In view of Proposition 2.3 and Lemma 2.3, we need to find a growth-rate $c(\alpha)$ with certain regularity (2.21) and satisfying the inequality (2.35), and a time-dependent pseudo-interface $z(t, \alpha)$ satisfying the regularity assumptions (2.22)-(2.24) and the relations (2.33a) (2.36).

3 The Growth-Rate

In this section we declare a suitable growth-rate c, and also a partition of the unity $\{\psi_0, \psi_1\}$ as discussed in the intro, in terms of $z^\circ \in C^{1,\delta}(\mathbb{T}; \mathbb{R}^2)$. Let us recall what we need. On the one hand, we must construct a regular enough c (2.21) satisfying the inequality (2.35) uniformly on $\{c(\alpha) > 0\}$. At the same time, for the relation (2.36b) it is convenient to control the monotonicity of c near the boundary of $\{c(\alpha) > 0\}$. On the other hand, we shall construct $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively, and satisfying $\partial_\alpha z_1^\circ > 0$ uniformly on supp ψ_0.

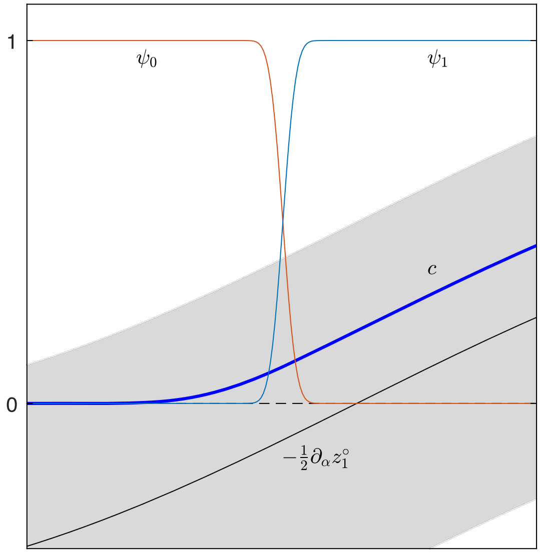


Fig. 3 A regular growth-rate $c(\alpha) \geq 0$ (blue) satisfying the inequality $|2c(\alpha) + \partial_\alpha z_1^\circ(\alpha)| < 1$ (gray zone) uniformly on $\{c(\alpha) > 0\}$. A partition of the unity $\{\psi_0, \psi_1\}$ subordinated to $\{\partial_\alpha z_1^\circ(\alpha) > 0\}$ and $\{c(\alpha) > 0\}$ respectively, and satisfying $\partial_\alpha z_1^\circ > 0$ uniformly on supp ψ_0

In view of Fig. 3, it seems clear that we have enough flexibility to construct such functions. The aim of this section is to make it quantitatively.

Let us introduce the following sets I_η which will be very useful for these purposes.

Lemma 3.1 *Given* $-1 \leq \eta \leq 1$ *we denote*

$$I_\eta := \{\alpha \in \mathbb{T} \,:\, \partial_\alpha z_1^\circ(\alpha) < \eta\}. \tag{3.1}$$

This forms an ascending chain of open subsets of $\mathbb{T}$. *Furthermore, for any* $-1 < \eta_1 < \eta_2 < 1$, *we have* $I_{\eta_1} \subset\subset I_{\eta_2}$ *with*

$$\operatorname{dist}(\partial I_{\eta_1}, \partial I_{\eta_2}) \geq \left(\frac{\eta_2 - \eta_1}{|\partial_\alpha z_1^\circ|_{C^\delta}}\right)^{1/\delta}.$$

Proof First of all notice that $\partial_\alpha z_1^\circ(\mathbb{T}) = [-1, 1]$. Hence, there is $\alpha_j \in \partial I_{\eta_j}$ for $j = 1, 2$, and thus

$$\eta_2 - \eta_1 = \partial_\alpha z_1^\circ(\alpha_2) - \partial_\alpha z_1^\circ(\alpha_1) \leq |\partial_\alpha z_1^\circ|_{C^\delta} \operatorname{dist}(\partial I_{\eta_1}, \partial I_{\eta_2})^\delta,$$

as we wanted to prove. □

In order to interpolate between the Classical and the Mixing Muskat problem, we take a partition of the unity $\{\psi_0, \psi_1\} \subset C_c^\infty(\mathbb{T}; [0, 1])$ as follows. Firstly, we define the indicator function

$$\chi_{\eta,s}(\alpha) := \begin{cases} 1, & \operatorname{dist}(\alpha, \mathbb{T} \setminus I_\eta) > r, \\ 0, & \text{otherwise}, \end{cases} \quad \text{with} \quad r := s\left(\frac{\eta}{|\partial_\alpha z_1^\circ|_{C^\delta}}\right)^{1/\delta}, \tag{3.2}$$

in terms of some parameters $0 < \eta, s < 1$ to be determined. With $\chi_{\eta,s}$ we declare

$$\psi_1 := \phi_r * \chi_{\eta,s}, \qquad \psi_0 := 1 - \psi_1, \tag{3.3}$$

where ϕ_r is a standard mollifier, namely $\phi_r(\alpha) := \frac{1}{r}\phi(\frac{\alpha}{r})$ for some fixed $\phi \in C_c^\infty(-1, 1)$ satisfying $\phi \geq 0$ and $\int \phi = 1$, and $r > 0$ is given in (3.2). Hence,

$$\|\partial_\alpha^k \psi_j\|_{L^\infty} \leq \|\partial_\alpha^k \phi\|_{L^1} r^{-k}, \qquad k \geq 0, \ \ j = 0, 1.$$

Among all the possible choices for c, we declare

$$c := \tfrac{1}{2}\phi_r * (\eta\chi_{\eta,s} + (\partial_\alpha z_1^\circ)_-), \tag{3.4}$$

where $(\partial_\alpha z_1^\circ)_- := -\min(\partial_\alpha z_1^\circ, 0)$. This c is smooth with

$$\|\partial_\alpha^k c\|_{L^\infty} \leq \tfrac{1}{2}(\eta + 1)\|\partial_\alpha^k \phi\|_{L^1} r^{-k}, \qquad k \geq 0,$$

and satisfies $c \geq \psi_1\eta/2$ with $\operatorname{supp} c = \operatorname{supp}\psi_1 \subset \bar{I}_\eta$. In the next lemma we show that we can take η and s in such a way that the inequality (2.35) holds.

Lemma 3.2 *The growth-rate* (3.4) *satisfies*

$$|2c + \partial_\alpha z_1^\circ| \leq \eta(2 + s^\delta) \quad on \quad c(\alpha) > 0.$$

Proof By writing $f = f_+ - f_-$ for $f = \partial_\alpha z_1^\circ$, we split

$$2c + \partial_\alpha z_1^\circ = \eta\psi_1 + f_+ + (\phi_r * f_- - f_-).$$

On the one hand, $\eta\psi_1 \leq \eta$ and also $f_+ \leq \eta$ on $\bar{I}_\eta$ ($\supset \operatorname{supp} c$) by (3.1). On the other hand, by (3.2)

$$|(\phi_r * f_- - f_-)(\alpha)| = \left|\int_{-r}^{r} (f_-(\alpha - \beta) - f_-(\alpha))\phi_r(\beta)\,\mathrm{d}\beta\right| \leq |f_-|_{C^\delta} r^\delta \leq s^\delta \eta.$$

This concludes the proof. □

Let us assume from now on that $s < 1/3$. In the next lemma we show that $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$ ($\supset\supset \mathbb{T} \setminus \operatorname{supp} c$), which will be crucial for the energy estimates in Sect. 5.

Lemma 3.3 *It holds that*

$$\partial_\alpha z_1^\circ \geq \eta(1 - (2s)^\delta) \quad on \quad \operatorname{supp}\psi_0.$$

Proof Let $\alpha \in \operatorname{supp}\psi_0$. If $\alpha \notin I_\eta$, simply $\partial_\alpha z_1^\circ(\alpha) \geq \eta$ by (3.1). Assume now that $\alpha \in I_\eta$. Since $\psi_1 \equiv 1$ on the open set $I_\eta \setminus \bar{B}_{2r}(\partial I_\eta)$ by (3.2) (3.3), necessarily $\alpha \in \bar{B}_{2r}(\partial I_\eta)$. Hence, there is $\alpha_\eta \in \partial I_\eta$ satisfying $|\alpha - \alpha_\eta| \leq 2r$, and thus

$$\partial_\alpha z_1^\circ(\alpha) = \underbrace{\partial_\alpha z_1^\circ(\alpha_\eta)}_{=\eta} + \underbrace{\partial_\alpha z_1^\circ(\alpha) - \partial_\alpha z_1^\circ(\alpha_\eta)}_{\geq -|\partial_\alpha z_1^\circ|_{C^\delta}(2r)^\delta} \geq \eta(1-(2s)^\delta),$$

where we have applied (3.2). □

Next we turn to the behavior of c inside $\{c(\alpha) > 0\}$. Of course c is monotone in a neighborhood of $c(\alpha) = 0$ and away from zero outside it, but we give bounds for these properties that only depends on s, η, δ.

Lemma 3.4 *Let $I = (\alpha_1, \alpha_2)$ be a connected component of $\{c(\alpha) > 0\}$ and denote $\bar{\alpha} := \frac{1}{2}(\alpha_1 + \alpha_2)$. If $|I| < 2r(1/s - 1)$, then c is monotone on $[\alpha_1, \bar{\alpha}]$ and $[\bar{\alpha}, \alpha_2]$. Otherwise, c is monotone on each connected component of $I \cap B_{2r}(\partial I)$, while $c \geq \eta/2$ on $I \setminus B_{2r}(\partial I)$. Moreover $\psi_1\eta/2 < c$ everywhere.*

Proof First of all observe that $\phi_r * (\partial_\alpha z_1^\circ)_- = 0$ (and so $c = \psi_1\eta/2$) outside $B_r(I_0)$.

Case $|I| < 2r(1/s - 1)$. Given $\alpha \in B_r(I_0)$, Lemma 3.1 and (3.2) imply that

$$\operatorname{dist}(\alpha, \partial I) \geq \operatorname{dist}(I_0, \partial I) - r \geq r(1/s - 1).$$

Thus, necessarily $\alpha \notin I$. Then, $I \cap B_r(I_0) = \emptyset$ and so $c = \psi_1\eta/2$ with ψ_1 monotone on $[\alpha_1, \bar{\alpha}]$ and $[\bar{\alpha}, \alpha_2]$ by construction (3.2) (3.3).

Case $|I| \geq 2r(1/s - 1)$. Given $\alpha \in B_{2r}(I_0)$ and $\beta \in B_r(\partial I)$, Lemma 3.1 and (3.2) imply that

$$|\alpha - \beta| \geq \operatorname{dist}(I_0, \partial I) - 3r \geq r(1/s - 3) > 0.$$

Hence, $B_{2r}(\partial I) \cap B_r(I_0) = \emptyset$ and so $c = \psi_1\eta/2$ on $I \cap B_{2r}(\partial I)$ with ψ_1 monotone on each connected component of $I \cap B_{2r}(\partial I)$. Finally, $c \geq \psi_1\eta/2$ with $\psi_1 \equiv 1$ on $I \setminus B_{2r}(\partial I)$. □

Finally, Lemma 3.4 implies the following estimates which will be useful to control (2.36b).

Corollary 3.1 *In the context of Lemma 3.4, let us denote*

$$I(\alpha) := \begin{cases} [\alpha_1, \alpha], \ \alpha_1 \leq \alpha \leq \bar{\alpha}, \\ [\alpha, \alpha_2], \ \bar{\alpha} < \alpha \leq \alpha_2. \end{cases}$$

Then, for $k = 0, 1$, we have

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')| \, \mathrm{d}\alpha' \lesssim c(\alpha),$$

in terms of $(\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$, $1/\eta$ and $\|\partial_\alpha^k c\|_{L^1}$.

Proof *Case* $|I| < 2r(1/s-1)$. For all $\alpha \in I$, the monotonicity of c on $I(\alpha)$ implies

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')|\,\mathrm{d}\alpha' \le |I(\alpha)|^{1-k} c(\alpha),$$

with $|I(\alpha)| \le r(1/s-1) = (1-s)(\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta} \le (\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$.

Case $|I| \ge 2r(1/s-1)$. For all $\alpha \in I \cap B_{2r}(\partial I)$, the previous argument works with $|I(\alpha)| \le 2r \le (\eta/|\partial_\alpha z_1^\circ|_{C^\delta})^{1/\delta}$. Finally, for all $\alpha \in I \setminus B_{2r}(\partial I)$, simply

$$\int_{I(\alpha)} |\partial_\alpha^k c(\alpha')|\,\mathrm{d}\alpha' \le \|\partial_\alpha^k c\|_{L^1} \frac{c(\alpha)}{\eta/2},$$

because $c(\alpha) \ge \eta/2$. □

From now on we fix the parameters η and s satisfying $\eta(2+s^\delta) < 1$ with $s < 1/3$. We remark that they are not necessarily very small (e.g. we can take $\eta = s = \frac{1}{4}$ for $\delta = 1$).

4 The Pseudo-Interface

We define our pseudo-interface z as the solution of the integro-differential equation

$$\begin{aligned} \partial_t z &= F(t, z^\circ, z), \\ z|_{t=0} &= z^\circ, \end{aligned} \tag{4.1}$$

given by the operator

$$F := \psi_0 E + \psi_1 E^{1)} - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

where τ is given in (1.12), $\{\psi_0, \psi_1\}$ is the partition of the unity we fixed in (3.3) and c the growth-rate (3.4). The operator $E(t, z^\circ, z)$ extending B outside $tc(\alpha) = 0$ was already introduced in (2.10),

$$E := \sum_{b=\pm} B_{b,b}.$$

Thus, it expands as

$$E(t,\alpha) := \frac{1}{2\pi} \sum_{b=\pm} \int_{\mathbb{T}} \left(\frac{1}{z_b(t,\alpha) - z_b(t,\beta)} \right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta.$$

The term $E^{1)}(t, z^\circ)$ is

$$E^{1)} := E^{(0)} + tE^{(1)},$$

where $E^{(0)}(z^\circ)$ and $E^{(1)}(z^\circ)$ are

$$
\begin{aligned}
E^{(0)}(\alpha) &:= \frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{z^\circ(\alpha)-z^\circ(\beta)}\right)_1(\partial_\alpha z^\circ(\alpha)-\partial_\alpha z^\circ(\beta))\,\mathrm{d}\beta,\\
E^{(1)}(\alpha) &:= \frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{z^\circ(\alpha)-z^\circ(\beta)}\right)_1(\partial_\alpha z_b^{(1)}(\alpha)-\partial_\alpha z_b^{(1)}(\beta))\,\mathrm{d}\beta\\
&\quad -\frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{z_b^{(1)}(\alpha)-z_b^{(1)}(\beta)}{(z^\circ(\alpha)-z^\circ(\beta))^2}\right)_1(\partial_\alpha z^\circ(\alpha)-\partial_\alpha z^\circ(\beta))\,\mathrm{d}\beta, \qquad (4.2)
\end{aligned}
$$

with

$$
z_b^{(1)} := E^{(0)} - bc\tau^\perp.
$$

The terms $\kappa(z^\circ)$ and $D^{(0)}(z^\circ)$ are

$$
\begin{aligned}
\kappa(\alpha) &:= 2\left(\partial_\alpha^2 z^\circ\left(\frac{c\tau}{(\partial_\alpha z^\circ)^2}\right)_1 + i\left(\frac{1}{\partial_\alpha z^\circ}\right)_2\partial_\alpha(c\tau)\right),\\
D^{(0)}(\alpha) &:= -i(c\tau+\tfrac{1}{2}).
\end{aligned} \qquad (4.3)
$$

The term $h(t, z^\circ, z)$ is the time-dependent average defined on each connected component (α_1, α_2) of $\{c(\alpha) > 0\}$ as

$$
h(t) := \frac{\displaystyle\int_{\alpha_1}^{\alpha_2} H\,\mathrm{d}\alpha}{\displaystyle\int_{\alpha_1}^{\alpha_2}\psi_1\partial_\alpha z\cdot\partial_\alpha z^\circ\,\mathrm{d}\alpha}, \qquad (4.4)
$$

where

$$
\begin{aligned}
H(t,\alpha) := (E - B - t\kappa)\cdot\partial_\alpha z^\perp + \psi_1(E^{1)} - E)\cdot\partial_\alpha z^\perp\\
+t(D - D^{(0)}\partial_\alpha z\cdot\partial_\alpha z^\circ)\cdot\partial_\alpha(c\tau).
\end{aligned}
$$

We notice that, although h is piecewise constant, $h\psi_1$ is smooth in α (recall $\psi_1 \lesssim c$ by Lemma 3.1).

As we will see in the next lemmas, $E^{(0)} = E|_{t=0}$, $E^{(1)} = \partial_t E|_{t=0}$, $z_b^{(1)} = \partial_t z_b|_{t=0}$ and $D^{(0)} = D|_{t=0}$. Thus, $E^{1)}$ equals to the first order expansion in time of E. In addition, we will see that $\kappa = \partial_t(E - B)|_{t=0}$.

In the rest of this section we will assume that there exists a solution z of Eq. (4.1) satisfying (2.22)–(2.24) and show that this implies that z satisfies (2.33a) (2.36) as well.

Theorem 4.1 *Let $z^\circ \in H^{k_\circ}(\mathbb{T};\mathbb{R}^2)$ be a closed chord-arc curve with $k_\circ \geq 6$. Assuming that, for some $T > 0$, there exists $z \in C_t H^{k_\circ-2}$ with $\partial_t z \in C_t H^{k_\circ-3}$ solving (4.1) and satisfying (2.23) (2.24), then z satisfies (2.33a) (2.36).*

The proof of this theorem relies on the forthcoming Lemmas 4.1–4.5. We start by rewriting some of the terms suitably. Let us observe that, assuming (4.1), the l.h.s. of (2.36a) reads, for $a = \pm$, as

$$\partial_t z - B_a = (E - B_a) + \psi_1(E^{1)} - E) - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ), \tag{4.5}$$

and for (2.36b) we have

$$\begin{aligned}(\partial_t z - B) \cdot \partial_\alpha z^\perp + tD \cdot \partial_\alpha(c\tau) \\ = (E - B - t\kappa) \cdot \partial_\alpha z^\perp + \psi_1(E^{1)} - E) \cdot \partial_\alpha z^\perp + t(D - D^{(0)}\partial_\alpha z \cdot \partial_\alpha z^\circ) \cdot \partial_\alpha(c\tau) \\ - h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ.\end{aligned} \tag{4.6}$$

The core of the section is to prove that E remains close to B in L^∞. In Lemma 4.2 we show that $E - B_a = O(t(c + |\partial_\alpha c|))$. This is sufficient for (2.36a). Indeed, we show in Lemma 4.3 that $E - B - t\kappa = O(t^2(c + |\partial_\alpha c|))$, which is sufficient for (2.36b). In particular, this implies that $\kappa = \partial_t(E - B)|_{t=0}$ (a similar term appears in [43, 64]). Both estimates are based on a nice technical consequence of the argument principle Lemma 4.1 (a related argument appears in [58,Lemma 4.2]). In Lemma 4.4 we show that $D - D^{(0)}\partial_\alpha z \cdot \partial_\alpha z^\circ = O(t)$. The term h has been introduced because (2.36b) reads as

$$\int_{\alpha_1}^{\alpha} (H - h\psi_1 \partial_\alpha z \cdot \partial_\alpha z^\circ)\, \mathrm{d}\alpha' = c(\alpha) o(t).$$

This requires to obtain a cancellation for $\alpha = \alpha_2$, which is equivalent to (4.4). Finally, we show in Lemma 4.5 that h, $E - E^{1)} = O(t^2)$. All the ingredients are ready to prove Theorem 4.1.

In the proof of Lemmas 4.1–4.5 all the assumptions in Theorem 5.1 are valid. We start with the technical lemma.

Lemma 4.1 *For all $k \in \mathbb{N}$, the function*

$$C^k_{\lambda,\mu}(t,\alpha) := \int_{\mathbb{T}} \frac{\beta^{k-1}}{(z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta))^k}\, \mathrm{d}\beta, \tag{4.7}$$

is uniformly bounded on $c(\alpha) > 0$, $0 < t \le T$ and $\lambda, \mu \in [-1, 1]$ with $\lambda \neq \mu$.

Proof First of all, by adding and subtracting $\partial_\alpha z'_\mu / \partial_\alpha z_\mu$ (recall Sect. 1 Notation) we split

$$C^k_{\lambda,\mu}(t,\alpha) := \int_{\mathbb{T}} \frac{\beta^{k-1}}{(z_\lambda - z'_\mu)^k} \left(1 - \frac{\partial_\alpha z'_\mu}{\partial_\alpha z_\mu}\right) \mathrm{d}\beta + \frac{1}{\partial_\alpha z_\mu} \int_{\mathbb{T}} \beta^{k-1} \frac{\partial_\alpha z'_\mu}{(z_\lambda - z'_\mu)^k}\, \mathrm{d}\beta. \tag{4.8}$$

The first term is controlled by $\mathcal{C}(c, z)$ and $\|\partial_\alpha z_\mu\|_{C_t C^{0,\delta}}$ (recall (2.24)). The identity (4.8) allows to prove the result by induction on k. For $k = 1$, the second integral in (4.8) is explicit (cf. (2.28))

$$\int_{\mathbb{T}} \frac{\partial_\alpha z'_\mu}{z_\lambda - z'_\mu} \, d\beta = (1 + \operatorname{sgn}(\lambda - \mu))\pi i,$$

where we have applied the Cauchy's argument principle for $\lambda \neq \mu$ and $tc(\alpha) > 0$. For $k \geq 2$, an integration by parts yields

$$\begin{aligned} \int_{\mathbb{T}} \beta^{k-1} \frac{\partial_\alpha z'_\mu}{(z_\lambda - z'_\mu)^k} \, d\beta &= -\frac{1}{k-1} \left(\frac{\beta}{z_\lambda - z'_\mu} \right)^{k-1} \Big|_{\beta=-\ell_\circ/2}^{\beta=+\ell_\circ/2} + \int_{\mathbb{T}} \frac{\beta^{k-2}}{(z_\lambda - z'_\mu)^{k-1}} \, d\beta \\ &= S^{k-1}_{\lambda,\mu} + C^{k-1}_{\lambda,\mu}, \end{aligned}$$

where

$$S^j_{\lambda,\mu}(t, \alpha) := \frac{(-1)^j - 1}{j} \left(\frac{\ell_\circ/2}{z_\lambda(t, \alpha) - z_\mu(t, \alpha + \ell_\circ/2)} \right)^j. \tag{4.9}$$

This $S^{k-1}_{\lambda,\mu}$ is controlled by $\mathcal{C}(c, z)$ while $C^{k-1}_{\lambda,\mu}$ is bounded by induction hypothesis. Furthermore, this recursive formula for $C^k_{\lambda,\mu}$ yields ($S^0_{\lambda,\mu} = 0$)

$$C^k_{\lambda,\mu} = \sum_{j=0}^{k-1} \frac{1}{(\partial_\alpha z_\mu)^{k-j}} \left(\int_{\mathbb{T}} \beta^j \frac{\partial_\alpha z_\mu - \partial_\alpha z'_\mu}{(z_\lambda - z'_\mu)^{j+1}} \, d\beta + S^j_{\lambda,\mu} \right) + \frac{(1 + \operatorname{sgn}(\lambda - \mu))\pi i}{(\partial_\alpha z_\mu)^k}, \tag{4.10}$$

which is bounded by $\mathcal{C}(c, z)$ and $\|\partial_\alpha z_\mu\|_{C_t C^{0,\delta}}$. □

Lemma 4.2 *For* $0 \leq t \leq T$ *it holds that*

$$E - B_a = O(t(c + |\partial_\alpha c|)).$$

Proof Notice that $E = B = B_+ = B_-$ for $tc(\alpha) = 0$. Consider now $tc(\alpha) \neq 0$. Recall from (2.31) that $B_{a,b}$ is split into

$$B_{a,b} = A_{a,b} - i(a - b)t\partial_\alpha(c\tau)\frac{1}{2\pi}(C^1_{a,b})_1,$$

where $C^1_{a,b}$ is given in (4.7) and

$$A_{a,b} := \frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{1}{z_a - z'_b} \right)_1 (\partial_\alpha z_b - \partial_\alpha z'_b) \, d\beta.$$

Then, for $b=-a$, we have

$$E-B_a=B_{b,b}-B_{a,b}=B_{b,b}-A_{a,b}+iat\partial_\alpha(c\tau)\frac{1}{\pi}(C^1_{a,b})_1. \tag{4.11}$$

The last term is $O(t|\partial_\alpha(c\tau)|)$ by Lemma 4.1. For the first term, the fundamental theorem of calculus yields

$$\begin{aligned} B_{b,b}-A_{a,b}&=\frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{z_b-z_b'}-\frac{1}{z_a-z_b'}\right)_1(\partial_\alpha z_b-\partial_\alpha z_b')\,\mathrm{d}\beta\\ &=-\frac{a}{2\pi}tc\int_{\mathbb{T}}\int_{-1}^1\left(\frac{\tau^\perp}{(z_\lambda-z_b')^2}\right)_1(\partial_\alpha z_b-\partial_\alpha z_b')\,\mathrm{d}\lambda\,\mathrm{d}\beta. \end{aligned} \tag{4.12}$$

Let us check that we can apply the Fubini's theorem. By using the regularity conditions (2.21)–(2.24), then (4.12) can be bounded by

$$tc\int_0^{\ell_\circ/2}\int_0^1\frac{\beta}{\beta^2+((1-\lambda)tc)^2}\,\mathrm{d}\lambda\,\mathrm{d}\beta=tc\int_0^1\int_0^{\frac{\ell_\circ/2}{(1-\lambda)tc}}\frac{\beta}{1+\beta^2}\,\mathrm{d}\beta\,\mathrm{d}\lambda<\infty. \tag{4.13}$$

Hence, the Fubini's theorem allows to interchange the order of integration of λ and β in (4.12). Then, by adding and subtracting $\beta\partial_\alpha^2 z_b$ in (4.12), we get

$$\begin{aligned} B_{b,b}-A_{a,b}=&-\frac{a}{2\pi}tc\int_{-1}^1\int_{\mathbb{T}}\left(\frac{\tau^\perp}{(z_\lambda-z_b')^2}\right)_1(\partial_\alpha z_b-\partial_\alpha z_b'-\beta\partial_\alpha^2 z_b)\,\mathrm{d}\beta\,\mathrm{d}\lambda\\ &-\frac{a}{2\pi}tc\partial_\alpha^2 z_b\left(\tau^\perp\int_{-1}^1 C^2_{\lambda,b}\,\mathrm{d}\lambda\right)_1, \end{aligned} \tag{4.14}$$

where, by applying the Taylor's theorem on the first term and Lemma 4.1 on the second one, we see that (4.14) is $O(tc)$ in terms of $\mathcal{C}(c,z)$ and $\|\partial_\alpha^2 z_b\|_{C_tC^{0,\delta}}$. $\square$

The next lemma shows that indeed $\kappa=\partial_t(E-B)|_{t=0}$.

Lemma 4.3 *For $0\le t\le T$ it holds that*

$$E-B-t\kappa=O(t^2(c+|\partial_\alpha c|)).$$

Proof Notice that

$$E-B=\tfrac{1}{2}(E-B_-)+\tfrac{1}{2}(E-B_+),$$

with $E-B_a$ given in (4.11). We start with some auxiliary computations. By combining (4.10) and (4.14) we get, for $b=-a$,

$$
\begin{aligned}
B_{b,b} - A_{a,b} = &-\frac{a}{2\pi} tc \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau^{\perp}}{(z_\lambda - z_b')^2} \right)_1 (\partial_\alpha z_b - \partial_\alpha z_b' - \beta \partial_\alpha^2 z_b) \, \mathrm{d}\beta \, \mathrm{d}\lambda \\
&-\frac{a}{2\pi} tc \partial_\alpha^2 z_b \sum_{j=0,1} \left(\frac{\tau^{\perp}}{(\partial_\alpha z_b)^{2-j}} \int_{-1}^{1} \left(\int_{\mathbb{T}} \beta^j \frac{\partial_\alpha z_b - \partial_\alpha z_b'}{(z_\lambda - z_b')^{j+1}} \, \mathrm{d}\beta + S_{\lambda,b}^j \right) \mathrm{d}\lambda \right)_1 \\
&+\frac{a}{2} tc \partial_\alpha^2 z_b \left(\frac{\tau}{(\partial_\alpha z_b)^2} \int_{-1}^{1} (1 + \operatorname{sgn}(\lambda - b)) \, \mathrm{d}\lambda \right)_1 ,
\end{aligned}
$$

where $S_{\lambda,b}^j$ is given in (4.9). Notice that

$$
\int_{-1}^{1} (1 + \operatorname{sgn}(\lambda - b)) \, \mathrm{d}\lambda = 2(1 + a).
$$

Furthermore, it is easy to see that $S_{\lambda,b}^j - S_{\lambda,0}^j = O(t)$ by the regularity conditions (2.21)–(2.24). Then, by writing $\partial_\alpha z_b = \partial_\alpha z - itb\partial_\alpha(c\tau)$, it follows that

$$
\begin{aligned}
B_{b,b} - A_{a,b} = &-\frac{a}{2\pi} tc \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau^{\perp}}{(z_\lambda - z_b')^2} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta \partial_\alpha^2 z) \, \mathrm{d}\beta \, \mathrm{d}\lambda \\
&- \frac{a}{2\pi} tc \partial_\alpha^2 z \sum_{j=0,1} \left(\frac{\tau^{\perp}}{(\partial_\alpha z)^{2-j}} \int_{-1}^{1} \left(\int_{\mathbb{T}} \beta^j \frac{\partial_\alpha z - \partial_\alpha z'}{(z_\lambda - z_b')^{j+1}} \, \mathrm{d}\beta + S_{\lambda,0}^j \right) \mathrm{d}\lambda \right)_1 \\
&+ (1 + a) tc \partial_\alpha^2 z \left(\frac{\tau}{(\partial_\alpha z)^2} \right)_1 \\
&+ O(t^2 c).
\end{aligned}
$$

Therefore, analogously to (4.12) (4.13), the fundamental theorem of calculus jointly with the Fubini's theorem yield (recall (4.11))

$$
\begin{aligned}
&E - B - t\kappa \\
&= \tfrac{1}{2}(E - B_-) + \tfrac{1}{2}(E - B_+) - t\kappa \\
&= \frac{1}{\pi} t^2 c \int_{-1}^{1} \int_{-1}^{1} \int_{\mathbb{T}} \left(\frac{\tau c' \tau'}{(z_\lambda - z_\mu')^3} \right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta \partial_\alpha^2 z) \, \mathrm{d}\beta \, \mathrm{d}\lambda \, \mathrm{d}\mu && =: I_1 \\
&\quad - \frac{1}{2\pi} t^2 c \partial_\alpha^2 z \sum_{j=0,1} \left(\frac{(j+1)\tau}{(\partial_\alpha z)^{2-j}} \int_{-1}^{1} \int_{-1}^{1} \int_{\mathbb{T}} c' \tau' \beta^j \frac{\partial_\alpha z - \partial_\alpha z'}{(z_\lambda - z_\mu')^{j+2}} \, \mathrm{d}\beta \, \mathrm{d}\lambda \, \mathrm{d}\mu \right)_1 && =: I_2 \\
&\quad - it\partial_\alpha(c\tau) \frac{1}{\pi} (C_{-,+}^1 - C_{+,-}^1)_1 - 2it\partial_\alpha(c\tau) \left(\frac{1}{\partial_\alpha z^\circ} \right)_2 && =: I_3 \\
&\quad + 2tc\partial_\alpha^2 z \left(\frac{\tau}{(\partial_\alpha z)^2} \right)_1 - 2tc\partial_\alpha^2 z^\circ \left(\frac{\tau}{(\partial_\alpha z^\circ)^2} \right)_1 && =: I_4 \\
&\quad + O(t^2 c).
\end{aligned}
$$

For I_1, by adding and subtracting $c\tau$ and $\frac{1}{2}\beta^2 \partial_\alpha^3 z$, we split it into

$$
I_1 = -\frac{1}{2\pi} (tc)^2 \partial_\alpha^3 z \left(\tau^2 \int_{-1}^{1} \int_{-1}^{1} C_{\lambda,\mu}^3 \, \mathrm{d}\lambda \, \mathrm{d}\mu \right)_1 + \text{commutators},
$$

where

$$\begin{aligned}\text{commutators} &= \frac{1}{\pi}t^2 c \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} \left(\frac{\tau((c\tau)' - c\tau)}{(z_\lambda - z'_\mu)^3}\right)_1 (\partial_\alpha z - \partial_\alpha z' - \beta\partial_\alpha^2 z)\,\mathrm{d}\beta\,\mathrm{d}\lambda\,\mathrm{d}\mu \\ &\quad + \frac{1}{\pi}(tc)^2 \int_{-1}^{1}\int_{-1}^{1}\int_{\mathbb{T}} \left(\frac{\tau^2}{(z_\lambda - z'_\mu)^3}\right)_1 \\ &\quad \times (\partial_\alpha z - \partial_\alpha z' - \beta\partial_\alpha^2 z + \tfrac{1}{2}\beta^2\partial_\alpha^3 z)\,\mathrm{d}\beta\,\mathrm{d}\lambda\,\mathrm{d}\mu.\end{aligned}$$

The first term of I_1 is $O((tc)^2)$ by Lemma 4.1 and the commutators are $O(t^2c)$ in terms of $\|z\|_{C_tC^{3,\delta}} \lesssim \|z\|_{C_tH^4}$. Similarly, by adding and subtracting $c\tau$ and $\beta\partial_\alpha^2 z$ for I_2, we gain commutators of order $O(t^2c)$, while the remaining term reads as

$$-\frac{1}{2\pi}(tc)^2\partial_\alpha^2 z \sum_{j=0,1}\left(\frac{(j+1)\tau^2\partial_\alpha^2 z}{(\partial_\alpha z)^{2-j}}\int_{-1}^{1}\int_{-1}^{1} C_{\lambda,\mu}^{j+2}\,\mathrm{d}\lambda\,\mathrm{d}\mu\right)_1 = O((tc)^2),$$

where we have applied Lemma 4.1. For I_3, (4.10) yields

$$C_{\lambda,\mu}^1 = \frac{1}{\partial_\alpha z_\mu}\left(\int_{\mathbb{T}} \frac{\partial_\alpha z_\mu - \partial_\alpha z'_\mu}{z_\lambda - z'_\mu}\,\mathrm{d}\beta + (1 + \operatorname{sgn}(\lambda - \mu))\pi i\right),$$

and thus

$$\frac{1}{\pi}(C_{-,+}^1 - C_{+,-}^1)_1 = O(t) - 2\left(\frac{1}{\partial_\alpha z_\mu}\right)_2.$$

Finally, a direct computation shows that

$$\left(\frac{1}{\partial_\alpha z_\mu}\right)_2 - \left(\frac{1}{\partial_\alpha z^\circ}\right)_2 = O(t),$$

and also for I_4

$$\partial_\alpha^2 z\left(\frac{\tau}{(\partial_\alpha z)^2}\right)_1 - \partial_\alpha^2 z^\circ\left(\frac{\tau}{(\partial_\alpha z^\circ)^2}\right)_1 = O(t),$$

in terms of $\|\partial_t z\|_{C_tC^{2,0}} \lesssim \|\partial_t z\|_{C_tH^3}$. □

The next lemmas deal with D, h and $E^{1)}$.

Lemma 4.4 *For $0 \le t \le T$ it holds that*

$$D - D^{(0)}\partial_\alpha z \cdot \partial_\alpha z^\circ = O(t).$$

Proof Recall that $D = -\frac{1}{2}(B_+ - B_-) + D^{(0)}$. Then, the statement follows from Lemma 4.2 since

$$B_+ - B_- = (B_+ - E) + (E - B_-) = O(t),$$

and using that $\partial_\alpha z \cdot \partial_\alpha z^\circ = 1 + O(t)$. □

Lemma 4.5 *For $0 \le t \le T$ it holds that*

$$h, E^{1)} - E = O(t^2).$$

Proof Recall the definition of $E^{(0)}$ and $E^{(1)}$ in (4.2). First of all we observe that, in analogy with the Hilbert transform, it follows that $E^{(0)} \in H^{k_\circ - 1}$ and also $E^{(1)} \in H^{k_\circ - 2}$ in terms of the chord-arc constant $\mathcal{C}(z^\circ)$ and $\|z^\circ\|_{H^{k_\circ}}$. Similarly, by differentiating E in time

$$\begin{aligned}\partial_t E = \frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{z_b(t,\alpha) - z_b(t,\beta)}\right)_1 (\partial_\alpha \partial_t z_b(t,\alpha) - \partial_\alpha \partial_t z_b(t,\beta))\,\mathrm{d}\beta \\ - \frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{\partial_t z_b(t,\alpha) - \partial_t z_b(t,\beta)}{(z_b(t,\alpha) - z_b(t,\beta))^2}\right)_1 (\partial_\alpha z_b(t,\alpha) - \partial_\alpha z_b(t,\beta))\,\mathrm{d}\beta,\end{aligned} \tag{4.15}$$

Theorem 4.1 provides enough regularity (recall $k_\circ \ge 6$) and validity of chord-arc condition to obtain that $\partial_t E \in C_t H^{k_\circ - 4}$ as long as $0 \le t \le T$. In particular, since $E^{(0)} = E|_{t=0}$, the mean value theorem yields

$$\|E - E^{(0)}\|_{C_t H^{k_\circ - 4}} = O(t). \tag{4.16}$$

Hence, recalling the definition of h (4.4) together with Lemmas 4.3 and 4.4, it follows that $h = O(t)$ as well. Then, by writing

$$F - E^{(0)} = \psi_0(E - E^{(0)}) + t\psi_1 E^{(1)} - (t\kappa + i(tD^{(0)} \cdot \partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

it follows from (4.16) and the regularity of the remaining terms that $\|F - E^{(0)}\|_{C_t H^{k_\circ - 4}} = O(t)$. As a result from (4.1), (4.2) and (4.15), we have $\partial_t E - E^{(1)} = O(t)$. Notice this implies $E^{(1)} = \partial_t E|_{t=0}$. Therefore, by applying the fundamental theorem of calculus

$$E - E^{1)} = \int_0^t (\partial_t E(s) - E^{(1)})\,\mathrm{d}s,$$

we get $E - E^{1)} = O(t^2)$. Finally, this is enough to update the estimate for h to $O(t^2)$ as well. □

Proof of Theorem 4.1 *Proof of* (2.33a). On $tc(\alpha) = 0$ we directly have that $\partial_t z = E = B$.

Proof of (2.36). Consider now $tc(\alpha) > 0$. For (2.36a), the expression (4.5) and a direct use of Lemmas 4.2 and 4.5 yield that

$$\partial_t z - B_a = O(t).$$

For (2.36b), we use the expression (4.6). Then, Lemmas 4.3–4.5 imply that

$$\begin{aligned}
\left|\int_{\alpha_1}^{\alpha}(E - B - t\kappa)\cdot\partial_\alpha z^\perp \,\mathrm{d}\alpha'\right| &\lesssim t^2\int_{\alpha_1}^{\alpha}(c + |\partial_\alpha c|)\,\mathrm{d}\alpha', \\
\left|\int_{\alpha_1}^{\alpha}\psi_1(E^{1)} - E)\cdot\partial_\alpha z^\perp \,\mathrm{d}\alpha'\right| &\lesssim t^2\int_{\alpha_1}^{\alpha}\psi_1\,\mathrm{d}\alpha', \\
\left|\int_{\alpha_1}^{\alpha} t(D - D^{(0)}\partial_\alpha z\cdot\partial_\alpha z^\circ)\cdot\partial_\alpha(c\tau)\,\mathrm{d}\alpha'\right| &\lesssim t^2\int_{\alpha_1}^{\alpha}(c + |\partial_\alpha c|)\,\mathrm{d}\alpha', \\
\left|\int_{\alpha_1}^{\alpha} h\psi_1\partial_\alpha z\cdot\partial_\alpha z^\circ\,\mathrm{d}\alpha'\right| &\lesssim t^2\int_{\alpha_1}^{\alpha}\psi_1\,\mathrm{d}\alpha',
\end{aligned}$$

uniformly on $c(\alpha) > 0$. Firstly, recall that $\psi_1 \lesssim c$ by Lemma 3.1. Secondly, if α is closer to α_1, Corollary 3.1 controls the integrals $\int_{\alpha_1}^{\alpha} c\,\mathrm{d}\alpha'$ and $\int_{\alpha_1}^{\alpha}|\partial_\alpha c|\,\mathrm{d}\alpha'$ in terms of $c(\alpha)$. Hence, the four terms above are $O(t^2 c(\alpha))$.

If α is closer to α_2, Corollary 3.1 yields control on $\int_{\alpha}^{\alpha_2} c\,\mathrm{d}\alpha'$ and $\int_{\alpha}^{\alpha_2}|\partial_\alpha c|\,\mathrm{d}\alpha'$ in terms of $c(\alpha)$. However, by (4.4), it holds that

$$\int_{\alpha_1}^{\alpha}(H - h\psi_1\partial_\alpha z\cdot\partial_\alpha z^\circ)\,\mathrm{d}\alpha' = -\int_{\alpha}^{\alpha_2}(H - h\psi_1\partial_\alpha z\cdot\partial_\alpha z^\circ)\,\mathrm{d}\alpha',$$

and thus we can integrate on (α, α_2). Therefore, for all $\alpha \in (\alpha_1, \alpha_2)$ the full expression is $O(t^2 c(\alpha))$. Finally, by dividing by $tc(\alpha)$ we have proven that (2.36b) holds. □

5 Existence of *z*

Theorem 5.1 *Let $z^\circ \in H^{k_\circ}(\mathbb{T};\mathbb{R}^2)$ be a closed chord-arc curve with $k_\circ \geq 6$. Then, there exists $z \in C_t H^{k_\circ - 2}$ with $\partial_t z \in C_t H^{k_\circ - 3}$ solving* (4.1) *and satisfying* (2.22)*–*(2.24) *for some $0 < T \ll 1$ depending on the chord-arc constant $\mathcal{C}(z^\circ)$ and the norm $\|z^\circ\|_{H^{k_\circ}}$.*

Remark 5.1 The initial regularity required $k_\circ = 6$ is due to the fact that the energy estimates are easier in H^4, some estimates in the proof of Lemmas 4.3–4.5 and that the pseudo-interface loses two derivatives on the mixing region as in [43, 64].

We split the proof of this theorem into two parts. Firstly, we obtain a priori energy estimates for the Eq. (4.1). Secondly, we explain briefly how (4.1) is regularized in order to use the a priori estimates to show the existence of the desired solution z.

5.1 A Priori Energy Estimates

We will take our energy as

$$\mathcal{E}(z) := \|z\|^2_{H^{k_\circ-2}} + \mathcal{A}(z)^{-1} + \mathcal{C}(z) + \mathcal{S}(z)^{-1}, \tag{5.1}$$

where $\mathcal{A}(z)$, $\mathcal{C}(z)$ are the angle and the chord-arc constants given in (2.19), (2.20) respectively, and

$$\mathcal{S}(z) := \inf\{\sigma(\alpha) \,:\, \alpha \in \operatorname{supp}\psi_0\}$$

measures the RT-stability of z on supp ψ_0 (recall $\sigma = (\rho_+ - \rho_-)\partial_\alpha z_1$ with $\rho_\pm = \pm 1$). Notice that $\mathcal{C}(z^\circ) < \infty$ by hypothesis and that $\mathcal{A}(z^\circ) = 1$, $\mathcal{S}(z^\circ) \geq 2\eta(1-2s) > 0$ by construction (recall Lemma 3.3). It turns out that $\frac{d}{dt}(\mathcal{A}(z)^{-1} + \mathcal{C}(z) + \mathcal{S}(z)^{-1})$ is a lower order term w.r.t. $\mathcal{E}(z)$. Analogous terms to $\mathcal{C}$ and $\mathcal{S}$ are rigorously analyzed in [17, 28]. The term $\mathcal{A}$ can be treated with a similar technique.

Next, we analyze the Sobolev norm of (5.1). Given $0 \leq k \leq k_\circ - 2$, we split

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_\alpha^k z\|^2_{L^2} &= \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k F \,\mathrm{d}\alpha \\
&= \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k(\psi_0 E)\,\mathrm{d}\alpha && =: I \\
&\quad + \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k(\psi_1 E^{1)} - (t\kappa + itD^{(0)} \cdot \partial_\alpha(c\tau)\partial_\alpha z^\circ))\,\mathrm{d}\alpha && =: I_\circ \\
&\quad - \int_{\mathbb{T}} \partial_\alpha^k z \cdot \partial_\alpha^k(i\psi_1\partial_\alpha z^\circ) h\,\mathrm{d}\alpha && =: I_h
\end{aligned}$$

We claim that the terms I, $I_\circ$ and I_h are controlled from above in terms of $\|z^\circ\|_{H^{k_\circ}}$, $\|c\|_{H^{k_\circ-1}}$, $\|\psi_j\|_{H^{k_\circ-2}}$ for $j = 0, 1$, $\|z\|_{H^{k_\circ-2}}$, $\mathcal{A}(z)^{-1}$, $\mathcal{C}(z)$ and $\mathcal{S}(z)^{-1}$.

The term $I_\circ$ is controlled because ψ_1, $E^{1)}$, κ, $D^{(0)}$, c and τ only depends on z°. Indeed, it is clear that ψ_1 and c are smooth by definition (3.3) (3.4), while τ and $D^{(0)}$ lose one derivative and κ loses two (recall (1.12) (4.3)). In analogy with the Hilbert transform (recall (4.2)) it follows that $E^{(0)}$ loses one derivative and similarly $E^{(1)}$ loses two, namely

$$\|E^{1)}\|_{H^{k_\circ-2}} \lesssim \|z^\circ\|_{H^{k_\circ}} + \|z^\circ\|^q_{H^{k_\circ}},$$

in terms of $\mathcal{C}(z^\circ)$ and $\|c\|_{H^{k_\circ-1}}$, for some $q \in \mathbb{N}$.

The term I_h is controlled because $h(t)$ does not depend on α. As we saw in Lemmas 4.2–4.5, it follows that $\|h\|_{L^\infty}$ is controlled in terms of $\|z\|_{H^3}$, $\|z^\circ\|_{H^4}$, $\mathcal{A}(c, z)^{-1}$ and $\mathcal{C}(c, z)$. These quantities (2.23) and (2.24) are controlled by $\mathcal{A}(z)^{-1}$ and $\mathcal{C}(z)$ for small times (cf. Lemma B.1).

The term I is expected to be controlled because at the linear level

$$\psi_0 E \sim -\sum_{b=\pm} \psi_0\sigma_b\Lambda z_b,$$

where $\Lambda := (-\Delta)^{1/2}$ and $\sigma_b = (\rho_+ - \rho_-)(\partial_\alpha z_b)_1$, which satisfies $\psi_0\sigma_b \geq 0$ for small times as our energy controls $\mathcal{S}(z)^{-1}$ (see the next subsection for a detailed explanation).

5.1.1 Analysis of *I*

As we mentioned in the introduction, the analysis of I is classical for curves in the fully stable regime. In our case, all the terms are treated classically but the most singular one which needs further analysis. Let us present here the estimate for the main term and discuss the rest in the appendix. We will assume w.l.o.g. that $\mathbb{T} = [-\pi, \pi]$ ($\ell_\circ = 2\pi$). The most singular term in I, for each $b = \pm$, is (recall Sect. 1 Notation)

$$J := \frac{1}{2\pi}\int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{1}{\delta_\beta z_b}\right)_1 \partial_\alpha^{k+1}\delta_\beta z_b \,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

Since $z_b = z - ibtc\tau$, the term with $\partial_\alpha^{k+1}\delta_\beta(c\tau)$ is controlled by $\mathcal{C}(z)$ and $\|c\tau\|_{H^{k+1}}$. For $\partial_\alpha^{k+1}\delta_\beta z$, by adding and subtracting a suitable term, we split it into

$$J_\sigma + J_\Phi := -\frac{1}{2}\int_{\mathbb{T}} \left(\frac{\psi_0}{\partial_\alpha z_b}\right)_1 \partial_\alpha^k z \cdot \Lambda(\partial_\alpha^k z)\,\mathrm{d}\alpha + \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \Phi_b \partial_\alpha^{k+1}\delta_\beta z \,\mathrm{d}\beta\,\mathrm{d}\alpha,$$

where $\Lambda = (-\Delta)^{1/2} : H^1 \to L^2$ is the operator

$$\Lambda f(\alpha) := \frac{1}{2\pi}\mathrm{pv}\int_{\mathbb{T}} \frac{\partial_\beta f(\beta)}{\tan(\frac{\alpha-\beta}{2})}\,\mathrm{d}\beta = \frac{1}{4\pi}\mathrm{pv}\int_{\mathbb{T}} \frac{f(\alpha) - f(\beta)}{\sin^2(\frac{\alpha-\beta}{2})}\,\mathrm{d}\beta,$$

and Φ_b is the bounded kernel

$$\Phi_b(t, \alpha, \beta) := \frac{1}{2\pi}\left(\frac{1}{\delta_\beta z_b} - \frac{1}{\partial_\alpha z_b(2\tan(\beta/2))}\right)_1. \tag{5.2}$$

For J_σ we proceed as follows. Recall that $\partial_\alpha z_1^\circ > 0$ uniformly on $\operatorname{supp}\psi_0$. Indeed, this is why we have opened the mixing zone slightly inside the stable regime. Our energy (5.1) allows to assume that $(\partial_\alpha z_b)_1 > 0$ on $\operatorname{supp}\psi_0$ for later times. Hence, using the Córdoba-Córdoba pointwise inequality $2f \cdot \Lambda f \geq \Lambda(|f|^2)$ (see [27]) and the fact that Λ is self-adjoint, we deduce that

$$\begin{aligned} 4J_\sigma &\leq -\int_{\mathbb{T}} \left(\frac{\psi_0}{\partial_\alpha z_b}\right)_1 \Lambda(|\partial_\alpha^k z|^2)\,\mathrm{d}\alpha \\ &= -\int_{\mathbb{T}} \Lambda\left(\frac{\psi_0}{\partial_\alpha z_b}\right)_1 |\partial_\alpha^k z|^2\,\mathrm{d}\alpha \leq \left\| \Lambda\left(\frac{\psi_0}{\partial_\alpha z_b}\right)_1 \right\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2}^2, \end{aligned}$$

with the first term controlled by $\mathcal{C}(z)$ and $\|z_b\|_{C^{2,\delta}}$. Notice that, since the evolution of $(\partial_\alpha z_b)_1$ is controlled in terms of our energy $\mathcal{E}$, the time of positiveness of $(\partial_\alpha z_b)_1$ on $\operatorname{supp}\psi_0$ depends just on the initial data z°.

The estimate of J_Φ is classical. We present it here for completeness. By writing $\partial_\alpha^{k+1}\delta_\beta z_b = \partial_\alpha\partial_\alpha^k z_b + \partial_\beta\partial_\alpha^k z_b'$, we split

$$\begin{aligned} J_\Phi = & -\frac{1}{2}\int_{\mathbb{T}} |\partial_\alpha^k z|^2 \partial_\alpha\left(\psi_0\int_{\mathbb{T}}\Phi_b \,\mathrm{d}\beta\right)\mathrm{d}\alpha && =: L_1 \\ & -\int_{\mathbb{T}}\psi_0\partial_\alpha^k z\cdot\left(\int_{\mathbb{T}}\partial_\alpha^k z_b'\partial_\beta\Phi_b\,\mathrm{d}\beta\right)\mathrm{d}\alpha && =: L_2 \end{aligned}$$

where we have integrated by parts w.r.t. α and β for L_1 and L_2 respectively. On the one hand, L_1 is controlled because we can write

$$\int_{\mathbb{T}}\Phi_b\,\mathrm{d}\beta = \frac{1}{2\pi}\left(\mathrm{pv}\int_{\mathbb{T}}\frac{\mathrm{d}\beta}{\delta_\beta z_b}\right)_1 = \frac{1}{2\pi}\left(\frac{1}{\partial_\alpha z_b}\left(\int_{\mathbb{T}}\frac{\delta_\beta\partial_\alpha z_b}{\delta_\beta z_b}\,\mathrm{d}\beta + \underbrace{\mathrm{pv}\int_{\mathbb{T}}\frac{\partial_\alpha z_b'}{\delta_b z_b}\,\mathrm{d}\beta}_{=\pi i}\right)\right)_1,$$

which is bounded in C^1 by $\mathcal{C}(z)$ and $\|z_b\|_{C^{2,\delta}}$. On the other hand, L_2 is controlled because $\partial_\beta\Phi_b$ is bounded in terms of $\mathcal{C}(z)$ and $\|z_b\|_{C^3}$.
The analysis of the remaining terms is standard (see e.g. [28]). For completeness, we have presented a compact version in Lemma B.2.

5.2 Regularization

In order to be able to apply the Picard's theorem we regularize the Eq. (4.1) via

$$\begin{aligned} \partial_t z &= \phi_\varepsilon * F_\varepsilon(t, z^\circ, z), \\ z|_{t=0} &= z^\circ, \end{aligned} \tag{5.3}$$

in terms of the parameter $\varepsilon > 0$, where

$$F_\varepsilon := \psi_0 E_\varepsilon + \psi_1 E^{1)} - (t\kappa + i(tD^{(0)}\cdot\partial_\alpha(c\tau) + h\psi_1)\partial_\alpha z^\circ),$$

which agrees with F except that E has been replaced by

$$E_\varepsilon(t,\alpha) := \frac{1}{2\pi}\sum_{b=\pm}\int_{\mathbb{T}}\left(\frac{1}{\delta_\beta z_b}\right)_1\partial_\alpha\delta_\beta(\phi_\varepsilon * z_b)\,\mathrm{d}\beta.$$

Let us fix the open set where the Picard's theorem is applied. Firstly, let O_k be the open subset of H^k formed by chord-arc curves

$$O_k := \{z \in H^k(\mathbb{T};\mathbb{R}^2) \,:\, \mathcal{C}(z) < \infty\}.$$

Secondly, given $z^\circ \in O_{k_\circ}$ and $k \le k_\circ$, we define the open neighborhood $O_k(z^\circ)$ of z° in H^k as the set of curves $z \in O_k$ satisfying, for some fixed parameters $0 <$

$A, C, R, S < \infty$,

$$\mathcal{A}(z) > A, \qquad \mathcal{C}(z) < C, \qquad \|z\|_{H^k} < R, \tag{5.4}$$

and also

$$\mathcal{S}(z) > S. \tag{5.5}$$

From left to right, the conditions in (5.4) establish that the angle between $\partial_\alpha z$ and τ is uniformly non-perpendicular, which is necessary for our construction of the mixing zone (cf. Rem. 2.3), and that the chord-arc constant and the H^k-norm of z are uniformly bounded respectively. Since we want $z^\circ \in O_k(z^\circ)$, necessarily $A < \mathcal{A}(z^\circ) = 1$, $C > \mathcal{C}(z^\circ)$ and $R > \|z^\circ\|_{H^k}$. The condition (5.5) establishes that z remains uniformly stable on supp ψ_0. By Sect. 3 (cf. Lemma 3.3) we consider $S < 2\eta(1-2s)$.

Lemma 5.1 *Assume that there exists $z^\varepsilon \in C([0,T_\varepsilon]; O_{k_\circ-2}(z^\circ))$ solving (5.3) for some $0 < T_\varepsilon \ll 1$. Then, there exists $q \in \mathbb{N}$ satisfying*

$$\frac{d}{dt}\mathcal{E}(z^\varepsilon) \lesssim \mathcal{E}(z^\varepsilon) + \mathcal{E}(z^\varepsilon)^q, \tag{5.6}$$

in terms of A, C, R, S and $\|z^\circ\|_{H^{k_\circ}}$, but independently of ε.

Proof This is totally analogous to the a priori energy estimates of the previous section (see [28]). □

Proof of Theorem 5.1 *Step 1 Approximation sequence z^ε.* For all $\varepsilon > 0$, a standard Picard iteration yields a time-dependent curve $z^\varepsilon \in C([0,T_\varepsilon]; O_{k_\circ-2}(z^\circ))$ satisfying

$$z^\varepsilon(t) = z^\circ + \int_0^t \phi_\varepsilon * F_\varepsilon(s, z^\circ, z^\varepsilon(s))\,\mathrm{d}s.$$

This T_ε is taken in terms of the parameters defining $O_{k_\circ-2}(z^\circ)$ in such a way that the conditions (B.1) (B.2) hold. Thus, the operators $B_{a,b}$ (and so h) are well-defined.

As usual, the Gronwall's inequality applied to (5.6) implies that $\|z^\varepsilon\|_{C([0,T_\varepsilon];H^{k_\circ-2})}$ is uniformly bounded in ε. Furthermore, z^ε satisfies (5.3) with $\|\partial_t z\|_{C([0,T_\varepsilon];H^{k_\circ-3})}$ uniformly bounded in ε. We notice that our system is not autonomous but smooth in time. All these facts guarantee that the times of existence T_ε do not vanish as $\varepsilon \to 0$, namely $T_\varepsilon \geq T > 0$.

Step 2 Convergence to z. By the Rellich–Kondrachov and the Banach–Alaoglu theorems, we may assume (taking a subsequence if necessary) that there exists $z \in C([0,T]; O_{k_\circ-2}(z^\circ))$ such that $z^\varepsilon \to z$ in $C_t H^{k_\circ-3}$ and also $\partial_\alpha^{k_\circ-2} z^\varepsilon \rightharpoonup \partial_\alpha^{k_\circ-2} z$ as $\varepsilon \to 0$. Furthermore, $\partial_t z \in C([0,T]; H^{k_\circ-3})$. Finally, it follow that z solves (4.1) and satisfies (2.22)–(2.24). □

6 Proof of the Main Results and Generalizations

In this section we glue the several proofs of the previous sections to gain clarity of how the Theorems 1.1 and 1.2 are proved. In addition, we recall how this construction is generalized for piecewise constant coarse-grained densities.

6.1 Proof of Theorems 1.1 and 1.2

First of all we construct the growth-rate c and the partition of the unity $\{\psi_0, \psi_1\}$ as in (3.4) and (3.3) respectively, in terms of z° and some small parameters η, s (e.g. $\eta = s = \frac{1}{4}, \delta = 1$). By Lemma 3.2, this c satisfies the inequality (2.35) and the regularity condition (2.21) (indeed $c \in C^\infty$).

Once these functions are fixed, the Theorem 5.1 implies the existence of a time-dependent pseudo-interface z satisfying the Eq. (4.1) and the regularity conditions (2.22)–(2.24) for some $T \ll 1$. By Theorem 4.1, this z satisfies the growth conditions (2.33a) (2.36).

Next, we construct the mixing zone Ω_{mix} and the non-mixing zones $\Omega_\pm$ by (1.10) (2.17) respectively. Then, we define the triplet $(\bar{\rho}, \bar{v}, \bar{m})$ by (2.26) (2.27) (2.32). Hence, by Proposition 2.3 and Lemma 2.3, $(\bar{\rho}, \bar{v}, \bar{m})$ is a subsolution to IPM for some $0 < T' \le T$.

Finally, the h-principle in IPM (Theorem 2.1) yields infinitely many mixing solutions to IPM starting from (1.1) (1.6).

The proof of Theorem 1.2 is analogous to the one of Theorem 1.1. The main difference for the asymptotically flat case is that, since the domain of integration is $\mathbb{R}$ instead of $\mathbb{T}$, most of the integrals are taken with the Cauchy's principal value at infinity. In this case, (2.28) reads as

$$\begin{aligned} \operatorname{Ind}_{z_+(t)}(x) &= \tfrac{1}{2}\mathbb{1}_{\Omega_+(t)}(x), \\ \operatorname{Ind}_{z_-(t)}(x) &= \tfrac{1}{2}(1 - \mathbb{1}_{\Omega_-(t)}(x)), \end{aligned} \tag{6.1}$$

which changes the limits (2.30) but not the result. The proof of Theorem 1.2 in the x_1-periodic case is even closer to the one of Theorem 1.1. In this case (6.1) holds as well, but we do not have to deal with the infinity since the domain is $\mathbb{T}$.

6.2 Piecewise Constant Coarse-Grained Densities

Following [43, 64] we split the mixing zone into several levels $L = \{\lambda_j : 1 \le |j| \le N\}$ for $N \ge 1$ with

$$\lambda_j = \operatorname{sgn} j \, \tfrac{2|j|-1}{2N-1},$$

namely we consider

$$\Omega^j_{\text{mix}}(t) := \{z_\lambda(t, \alpha) : c(\alpha) > 0, \ \lambda \in (-\lambda_j, \lambda_j)\},$$

with z_λ defined as in (1.11), which satisfies $\Omega_{\mathrm{mix}}^1 \subset \cdots \subset \Omega_{\mathrm{mix}}^N =: \Omega_{\mathrm{mix}}$. In addition we define $\Omega_\pm$ as in (2.17) (or (2.18)).

Analogously to [43, 64], we define the piecewise constant (coarse-grained) density as ($|L| = 2N$)

$$\bar{\rho}(t,x) := \frac{2}{|L|}\sum_{b\in L}\mathrm{Ind}_{z_b(t)}(x) - 1, \tag{6.2}$$

for the closed case (1.6), while for asymptotically flat curves (1.7) the definition (6.2) needs to remove the last -1. Observe that $\bar{\rho} = \pm 1$ on $\Omega_\pm$ while $\bar{\rho}$ approaches the linear profile in [16, 19, 71] inside the mixing zone.

Analogously to (2.27), the Biot-Savart law yields

$$\begin{aligned}\bar{v}(t,x) &= -\left(\frac{1}{\pi i|L|}\sum_{b\in L}\int\frac{(\partial_\alpha z_b(t,\beta))_2}{x - z_b(t,\beta)}\,\mathrm{d}\beta\right)^* \\ &= -\frac{1}{\pi|L|}\sum_{b\in L}\int\left(\frac{1}{x - z_b(t,\beta)}\right)_1\partial_\alpha z_b(t,\beta)\,\mathrm{d}\beta, \qquad x \neq z_b(t,\beta).\end{aligned} \tag{6.3}$$

Analogously to (2.32), we write the relaxed momentum as

$$\bar{m} := \bar{\rho}\bar{v} - (1 - \bar{\rho}^2)(\gamma + \tfrac{1}{2}i),$$

in terms of some

$$\gamma := \sum_{j=1}^N \nabla^\perp g_j \mathbb{1}_{\Omega_{\mathrm{mix}}^j},$$

with $g_j(t,x)$ to be determined. Hence, analogously to (2.41), we define g_j in (α,λ)-coordinates as

$$G_j(t,\alpha,\lambda) := \int_{\alpha_1}^{\alpha}\left(\sum_{a=\pm\lambda_j}\frac{\lambda + a}{2}(\partial_t z - B_a)\cdot\partial_\alpha z_a^\perp - \frac{1}{N}(\lambda_j c\tau + \tfrac{1}{2})\cdot\partial_\alpha z_\lambda\right)\mathrm{d}\alpha',$$

where

$$B_a := \sum_{b\in L} B_{a,b}.$$

Finally, using that

$$\frac{1}{N}\sum_{j=1}^N \lambda_j = \frac{N}{2N-1},$$

the condition $|\gamma| < \frac{1}{2}$ yields the more general regime for c given in (1.8) as $N \to \infty$ (cf. (2.40)) . The rest follows analogously to the case $N = 1$ (see [43, 64]).

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A The Pressure

Lemma A.1 *Let (ρ, v) be a mixing solution from Theorem 1.1 or 1.2. Then, there exists a pressure p satisfying the Darcy's law*

$$\int_0^t \int_{\mathbb{R}^2} ((v + \rho i) \cdot \Phi - p\nabla \cdot \Phi)\, \mathrm{d}x\, \mathrm{d}s = 0, \tag{A.1}$$

for every test function $\Phi \in C_c^2(\mathbb{R}^3; \mathbb{R}^2)$. Observe that (A.1) agrees with (2.11c) for $\Phi = \nabla^{\perp}\phi$.
Moreover, $v = \nabla^{\perp}\psi$ with p and ψ the continuous functions given by

$$\begin{aligned}(p + i\psi)(t, x) = {} & \frac{1}{2\pi} \sum_{b=\pm} \int \log|x - z_b(t, \beta)| \partial_\alpha z_b(t, \beta)^* \, \mathrm{d}\beta \\ & + \frac{1}{2\pi i} \int_{\Omega_{\mathrm{mix}}(t)} \frac{1}{x - y} (\rho - \bar{\rho})(t, y)\, \mathrm{d}y.\end{aligned}$$

The first term corresponds to the macroscopic contribution of $\bar{\rho}$ (cf. (A.3)). *The second one is the fluctuation coming from $\rho - \bar{\rho}$ and vanishes outside Ω_{mix} (cf. (A.2)). Furthermore, for any fixed $\mathscr{E} \in C(\mathbb{R}_+; \mathbb{R}_+)$ with $\mathscr{E}(r) > 0$ for $r > 0$, we can select these (infinitely many) mixing solutions satisfying*

$$|((p + i\psi) - (\bar{p} + i\bar{\psi}))(t, x)| \leq \mathscr{E}(\mathrm{dist}((t, x), \Omega_+ \cup \Omega_-)), \tag{A.2}$$

where

$$(\bar{p} + i\bar{\psi})(t, x) := \frac{1}{2\pi} \sum_{b=\pm} \int \log|x - z_b(t, \beta)| \partial_\alpha z_b(t, \beta)^* \, \mathrm{d}\beta. \tag{A.3}$$

Proof Notice that $\bar{v} = \nabla^{\perp}\bar{\psi}$ by (2.27). In particular,

$$\nabla(\bar{p} + i\bar{\psi}) = -i\bar{\rho},$$

in the sense of distributions. Following [19, 57, 71] we consider the convex integration sequence $(\rho_k, v_k) \to (\rho, v)$ in $C_t L^{\infty}_{w^*}$. In fact, $\rho_k \to \rho$ in $C_t L^q_{\mathrm{loc}}$ for all $1 < q < \infty$ (see [57,p. 12]). Let us split $\rho_k = \bar{\rho} + \rho_k'$, $v_k = \bar{v} + v_k'$ and $p_k := \bar{p} + p_k'$ for some p_k' to be determined. By construction (see [57,Lemma 3.1]) $(\rho_k', v_k') = (\Delta\varphi_k', -\nabla^{\perp}\partial_1\varphi_k')$ for some real-valued function φ_k' which is smooth and compactly supported on Ω_{mix}. Notice that $v_k' = \nabla^{\perp}\psi_k'$ for $\psi_k' := -\partial_1\varphi_k'$. Hence, p_k satisfies $\nabla(p_k + i\psi_k) = -i\rho_k$ if and only if p_k' satisfies

$$\nabla(p_k' + i\psi_k') = -i\rho_k'.$$

Therefore $(\Delta = \nabla\nabla^*)$

$$p_k' + i\psi_k' = -i\nabla^*\varphi_k' + f_k,$$

for some (time-dependent) entire function f_k. Since φ_k' is compactly supported on Ω_{mix}, necessarily $(f_k(t,x))_2 \to 0$ as $|x| \to \infty$. Therefore, the Liouville's theorem ($e^{if_k(t)}$ is entire and bounded) implies that f_k equals to a (time-dependent) real constant. Hence, as we are choosing p_k', we may assume that $f_k = 0$. Finally, the Cauchy-Pompeiu's formula yields

$$\nabla^*\varphi_k'(t,x) = \frac{1}{2\pi}\int_{\Omega_{\mathrm{mix}}(t)} \frac{1}{x-y}\rho_k'(t,y)\,\mathrm{d}y.$$

This concludes the proof by taking the limit $k \to \infty$. The inequality (A.2) can be guaranteed by following the proof of the quantitative h-principle in [19]. □

B Auxiliary Lemmas

Lemma B.1 *Let $z \in C([0,T]; C^{1,\delta}(\mathbb{T};\mathbb{R}^2))$. Assume that, for some parameters $0 < A, C, R, S < \infty$,*

$$\mathcal{A}(z(t)) > A, \qquad \mathcal{C}(z(t)) < C, \qquad |\partial_\alpha z(t)|_{C^\delta} < R, \qquad \mathcal{S}(z(t)) > S,$$

for all $0 \le t \le T$. Then, there exists $0 < T'(A, C, R, S, \delta, \|c\tau\|_{C^{1,\delta}}) \le T$ such that the equi-chord-arc condition holds:

$$|z_\lambda(t,\alpha) - z_\mu(t,\alpha-\beta)|^2 \ge D\left(\frac{\beta^2}{(2C)^2} + ((\lambda-\mu)tc(\alpha))^2\right), \tag{B.1}$$

for all $\alpha, \beta \in \mathbb{T}$, $\lambda, \mu \in [-1, 1]$ *and* $0 \le t \le T'$, *where* $D \equiv 1 - \sqrt{1 - (A/2)^2}$. *In addition,*

$$\sup_{\lambda \in [-1,1]} \mathcal{A}(z_\lambda(t)) > A/2, \qquad \sup_{\lambda \in [-1,1]} \mathcal{S}(z_\lambda(t)) > S/2. \tag{B.2}$$

Proof First of all notice that $A < 1$, and thus $D < 1$ as well. In particular, (B.1) holds for $tc(\alpha) = 0$. Henceforth, let $c(\alpha) > 0$ and $0 < t \le T'$ for some $0 < T' \le T$ to be determined. Notice that we can take T' satisfying (recall Sect. 1 Notation)

$$|\delta_\beta z_\mu| \ge |\delta_\beta z| - t|\delta_\beta(c\tau)| \ge \left(\frac{1}{C} - t|c\tau|_{C^1}\right)|\beta| \ge \frac{|\beta|}{2C}. \tag{B.3}$$

We split the proof into two cases, depending on the following parameter

$$r \equiv \left(\frac{A}{2^3 C R}\right)^{1/\delta}.$$

Case $|\beta| \le r$. By writing,

$$z_\lambda - z'_\mu = \delta_\beta z_\mu - (\lambda - \mu)tc\tau^\perp, \tag{B.4}$$

we split the l.h.s. of (B.1) into

$$|z_\lambda - z'_\mu|^2 = |\delta_\beta z_\mu|^2 + ((\lambda-\mu)tc)^2 - 2(\lambda-\mu)tc\delta_\beta z_\mu \cdot \tau^\perp. \tag{B.5}$$

Let us analyze its third term. By our choice of r and using $|\partial_\alpha z| \ge 1/C$, we can take T' satisfying

$$\left|\frac{\delta_\beta z_\mu}{|\delta_\beta z_\mu|} - \frac{\partial_\alpha z}{|\partial_\alpha z|}\right| \le 2\frac{|\triangle_\beta z_\mu - \partial_\alpha z|}{|\partial_\alpha z|} \le 2C(|\partial_\alpha z|_{C^\delta}|\beta|^\delta + t|c\tau|_{C^1}) \le A/2.$$

Then, by adding and subtracting $\partial_\alpha z/|\partial_\alpha z|$, we deduce that (recall (1.11) (1.12))

$$\frac{|\delta_\beta z_\mu \cdot \tau|}{|\delta_\beta z_\mu|} \ge \frac{\partial_\alpha z}{|\partial_\alpha z|}\cdot\tau - \left|\frac{\delta_\beta z_\mu}{|\delta_\beta z_\mu|} - \frac{\partial_\alpha z}{|\partial_\alpha z|}\right| \ge A/2,$$

which implies that

$$(\delta_\beta z_\mu \cdot \tau^\perp)^2 = |\delta_\beta z_\mu|^2 - (\delta_\beta z_\mu \cdot \tau)^2 \le (1 - (A/2)^2)|\delta_\beta z_\mu|^2. \tag{B.6}$$

Finally, by applying (B.3) and (B.6) into (B.5), we deduce that

$$\begin{aligned} |z_\lambda - z'_\mu|^2 &\ge (1 - \sqrt{1-(A/2)^2})(|\delta_\beta z_\mu|^2 + ((\lambda-\mu)tc)^2) \\ &\ge (1 - \sqrt{1-(A/2)^2})\left(\frac{\beta^2}{(2C)^2} + ((\lambda-\mu)tc)^2\right). \end{aligned}$$

Case $|\beta| > r$. On the one hand, by applying (B.3) (B.4), the l.h.s. of (B.1) can be bounded from below as

$$|z_\lambda - z'_\mu| \geq |\delta_\beta z_\mu| - 2t\|c\|_{C^0} \geq \frac{|\beta|}{2C} - 2t\|c\|_{C^0}.$$

On the other hand, the r.h.s. of (B.1) can be bounded from above as

$$\frac{\beta^2}{(2C)^2} + ((\lambda - \mu)tc)^2 \leq \frac{\beta^2}{(2C)^2} + (2t\|c\|_{C^0})^2.$$

Thus, it is enough to guarantee that

$$\left(\frac{|\beta|}{2C} - 2t\|c\|_{C^0}\right)^2 \geq D\left(\frac{\beta^2}{(2C)^2} + (2t\|c\|_{C^0})^2\right),$$

or equivalently

$$\frac{(1-s)^2}{1+s^2} \geq D \quad \text{with} \quad s \equiv \frac{4C\|c\|_{C^0}}{|\beta|}t.$$

Since $D \ll 1$, this holds for all $|\beta| > r$ by taking T' small enough.

Finally, it is clear that (B.2) holds for small times. □

Lemma B.2 *The remaining terms of I from Sect. 5.1.1 are lower order terms.*

Proof By combining the general Leibniz rule applied to $(\psi_0, K_b, \partial_\alpha \delta_\beta z_b)$ where

$$K_b(t, \alpha, \beta) := \left(\frac{1}{\delta_\beta z_b(t, \alpha)}\right)_1,$$

with the Faà di Bruno's formula applied to the kernel K_b, we split ($j = (j_0, j_1, j_2)$)

$$I = \frac{1}{2\pi}\sum_{b=\pm}\sum_{|j|=k}\sum_{n\in\pi_{j_1}}\binom{k}{j}(-1)^{|n|}C_n I_b(j, n),$$

where

$$I_b(j, n) := \int_{\mathbb{T}} (\partial_\alpha^{j_0}\psi_0)\partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\prod_{i=1}^{j_1}(\partial_\alpha^i \delta_\beta z_b)^{n_i}}{(\delta_\beta z_b)^{|n|+1}}\right)_1 (\partial_\alpha^{j_2+1}\delta_\beta z_b)\, \mathrm{d}\beta\, \mathrm{d}\alpha,$$

with $\pi_{j_1} := \{n \in \mathbb{N}_0^{j_1} \,:\, n_1 + 2n_2 + \cdots + j_1 n_{j_1} = j_1\}$ and

$$C_n := \frac{|n|!\, j_1!}{n_1! 1!^{n_1} \cdots n_{j_1}! j_1!^{n_{j_1}}} > 0.$$

The most singular term $j = (0, 0, k)$ ($\Rightarrow n = 0$) has been analyzed in Sect. 5.1.1.

Second singular term. Let us consider $j = (0, k, 0)$. If $n_k = 1$ then

$$I_b(j, n) = \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\partial_\alpha^k \delta_\beta z_b}{(\delta_\beta z_b)^2} \right)_1 \partial_\alpha \delta_\beta z_b \, \mathrm{d}\beta \, \mathrm{d}\alpha.$$

By splitting $\partial_\alpha \delta_\beta z_b$ into its real and imaginary part and comparing

$$\frac{(\partial_\alpha \delta_\beta z_b)_l}{(\delta_\beta z_b)^2} \sim \frac{(\partial_\alpha^2 z_b)_l}{(\partial_\alpha z_b)^2 (2 \tan(\beta/2))}, \qquad l = 1, 2,$$

we obtain a Hilbert transform acting on $\partial_\alpha^k z_b$ while the commutator is a bounded kernel as in (5.2). If $n_k = 0$, notice that for any $k \geq 3$ we have $n_{k-1} \leq \frac{k}{k-1} < 2$, that is $n_{k-1} = 0$ or 1 (the case $k < 3 < k_\circ - 2$ is easier). If $n_{k-1} = 1$ ($\Rightarrow n_1 = 1$) then simply

$$|I_b(j, n)| \lesssim \mathcal{C}(z)^3 \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} |\partial_\alpha z_b|_{C^1}^2 |\partial_\alpha^{k-1} z_b|_{C^\delta},$$

and for $n_{k-1} = 0$

$$|I_b(j, n)| \lesssim \mathcal{C}(z)^{|n|+1} \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} \|z_b\|_{C^{k-2,1}}^{|n|+1}. \tag{B.7}$$

Third singular term. Let us consider $j = (0, 1, k - 1)$ ($\Rightarrow n = 1$):

$$I_b(j, n) = \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\partial_\alpha \delta_\beta z_b}{(\delta_\beta z_b)^2} \right)_1 (\partial_\alpha^k \delta_\beta z_b) \, \mathrm{d}\beta \, \mathrm{d}\alpha.$$

This is analogous to the case $(0, k, 0)$. The case $j = (1, 0, k - 1)$ is analogous too. Let us consider now $j = (0, k - 1, 1)$. If $n_{k-1} = 1$ then

$$I_b(j, n) = \int_{\mathbb{T}} \psi_0 \partial_\alpha^k z \cdot \int_{\mathbb{T}} \left(\frac{\partial_\alpha^{k-1} \delta_\beta z_b}{(\delta_\beta z_b)^2} \right)_1 (\partial_\alpha^2 \delta_\beta z_b) \, \mathrm{d}\beta \, \mathrm{d}\alpha,$$

and so

$$|I_b(j, n)| \lesssim \mathcal{C}(z)^2 \|\psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} |\partial_\alpha^{k-1} z_b|_{C^\delta} |\partial_\alpha^2 z_b|_{C^1}.$$

If $n_{k-1} = 0$ then (B.7) holds. The case $j = (1, k - 1, 0)$ is analogous.
Harmless terms. For $0 \leq j_1, j_2 \leq k - 2$, simply

$$|I_b(j, n)| \lesssim \mathcal{C}(z)^{|n|+1} \|\partial_\alpha^{j_0} \psi_0\|_{L^\infty} \|\partial_\alpha^k z\|_{L^2} \|z_b\|_{H^k}^{|n|+1}.$$

This concludes the proof. □

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