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A theorem for rigid motions in Post-Newtonian celestial mechanics

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Abstract. The velocity field distribution for rigid motions in the Born's sense applied to Post-Newtonian Relativistic Celestial Mechanics is examined together with its compatibility with the Newtonian distribution.

Un teorema para movimientos rígidos en mecánica celeste Post-Newtoniana

Resumen. Se examinan la distribución del campo de velocidades para movimientos rígidos en el sentido de Born aplicado a la mecánica celeste Post-Newtoniana junto con su compatibilidad con la distribución Newtoniana.

1. Introduction

As is known, it is of common use the introduction of some concept of rigidity as an hypothesis to simplify the equations and, then, to obtain solutions to the general problem of motion of extended bodies in Post-Newtonian Relativistic Celestial Mechanics (PNRCM), both when global or local reference systems are used to describe the motions, and particularly when these last systems are, at the same time, rigid reference frames in an analogous way as they are in Newtonian Celestial Mechanics. The hypothesis most used at the Post-Newtonian level (see, for example, [1], [4], [5], [7], [8], [10], [11], [12], [14], [16], [17]) is the Newtonian hypothesis, which assumes in its most general form that the velocity field $\tilde{v}^i$ of a body B in rigid motion can be decomposed in the form

$$\tilde{v}^i(\mathbf{x}, t) = \tilde{v}^i_B(t) + \tilde{\Omega}_{ij}(t) r^j(\mathbf{x}, t) \tag{1}$$

for some $\tilde{v}^i_B$ and $\tilde{\Omega}_{ij}$, with $\tilde{\Omega}_{ij}$ antisymmetric, and where r^i is the position vector, in the reference system chosen, for an arbitrary point $\mathbf{x}$ of the body B measured with respect to a comoving origin $x_B(t)$, so that

$$r^i(\mathbf{x}, t) = x^i - x^i_B(t).$$

(In this paper the notations are as in Brumberg [1]; in particular, latin indices run from 1 to 3 and greek from 0 to 3; comma denotes usual derivative, and semicolon covariant derivative; repeated indices imply an Einstein summation; round brackets surrounding indices denote symmetrization, and square brackets anti-symmetrization).

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But, since Eq.(1) preserves the fact that the Euclidean distance between any two points in B is constant, and the concept of Euclidean distance is not covariant in General Relativity (GR), its use in PNRCM needs to be justified. The aim of the present paper is to give a new velocity field distribution (Eq.(24) below) which, because is derived from a covariant definition of rigidity in GR, and also because it reduces to Eq. (1) in the sense we will see below, lets us justify, and also give the limitations for the use of Eq.(1) in the two problems mentioned above. For this purpose we use the definition of rigidity given by Born (see, for example, Synge [18]) because, although it allows only for three degrees of freedom in an arbitrary space-time instead of the six allowed by Eq. (1) ([9], [18]) it is, besides covariant, the most natural extension to GR of the rigidity definition for Classical Mechanics. This is so because this definition states, when it is applied to a body B, that the relativistic distance between any of its neighbouring particles remains constant ([18]). In fact, if it is taken into account that any velocity gradient in GR can be decomposed in the form ([15])

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + A_\alpha u_\beta, \tag{2}$$

with

$$\begin{aligned} \omega_{\alpha\beta} &= P^\rho_\alpha P^\sigma_\beta u_{[\rho;\sigma]} & (3)\\ \sigma_{\alpha\beta} &= P^\rho_\alpha P^\sigma_\beta u_{(\rho;\sigma)} & (4)\\ A_\alpha &= u_{\alpha;\beta} u^\beta & (5)\\ P^\alpha_\beta &= \delta^\alpha_\beta - u^\alpha u_\beta, & (6) \end{aligned}$$

mathematically Born's definition can also be established on B by assuming in it that the rate-of-strain tensor $\sigma_{\alpha\beta}$ satisfies (see [18])

$$\sigma_{\alpha\beta} = 0. \tag{7}$$

(Here u^α is the 4-velocity satisfying $u^\alpha u_\alpha = 1$ and A^α denotes the 4-acceleration of the particles in B).

The question, therefore, is in the search of the structure of the rigid motions of a body B when the rigidity condition on it is given by the Post-Newtonian approximation of Eq.(7), i.e., for weak fields and small velocities. According to Synge, although the Born criterion involves no difficulty when applied to a one-dimensional body, difficulties accumulate with increase of dimensionality, and it must be stated emphatically that the three-dimensional concept of rigidity does not pass from Newtonian physics into relativity. The difficulties inherent in relativistic rigidity are, however, connected with non-integrability, and are avoided if we work in an infinitesimal domain (Synge,[18], p.115) or with weak fields and small velocities. In fact, we will see next how, under these hypothesis of the PNRCM, the decomposition of a rigid motion at the Post-Newtonian level is given by a formula similar to Eq.(1) although, in general, this formula does not reduce exactly to Eq.(1) at the Newtonian level. For general results, attempts and alternatives to overcome the difficulties associated to the limited number of degrees of freedom given by Eq (7), both in Special and General Relativity, see [9], [6], [18], [3] and references therein. These alternatives seem less plausible because no one preserves the constancy of the relativistic distance between particles.

2. Post-Newtonian rigid motions

The kinematics of a material system in an arbitrary space-time, which as is known deals with the relative behaviour of neighbouring stream-lines, may be discussed in various ways. For our purposes the expressions (2)-(6) are enough. It is easy to see that, according to them, the rate-of-strain tensor is given by:

$$\sigma_{\alpha\beta} = \frac{1}{2}[u_{\alpha;\beta} + u_{\beta;\alpha} - A_\alpha u_\beta - A_\beta u_\alpha], \tag{8}$$

with

$$\sigma_{\alpha\beta} u^\beta = 0. \tag{9}$$

Since, on the other hand, and accordingly to PNRCM, we shall restrict our attention only to N-body systems which are in slow motion and have weak gravitational fields (everywhere including the interiors of the bodies) the two fundamental factors relevant to our problem that characterize this situation (i.e; $v \ll c$, v being the characteristic velocity of the bodies; $U \ll c^2$, U being the Newtonian potential) enable the introduction of the small parameter $\varepsilon \sim \dfrac{v}{c} \sim \left(\dfrac{U}{c^2}\right)^{1/2} \ll 1$ so that, since in most practical problems of PNRCM it is required to know the metric $g_{\alpha\beta}$ at the first post-Newtonian level, we may assume that ([1])

$$ds^2 = g_{\alpha\beta}(t, x^i) dx^\alpha dx^\beta, \tag{10}$$

where

$$\begin{aligned}
g_{00}(t,x^i) &= 1 + h^{(2)}_{00}(t,x^i) + h^{(4)}_{00}(t,x^i) + O(\varepsilon^6), &(11)\\
g_{0i}(t,x^i) &= h^{(3)}_{0i}(t,x^i) + O(\varepsilon^5), &(12)\\
g_{ij}(t,x^i) &= -\delta_{ij} + h^{(2)}_{ij}(t,x^i) + O(\varepsilon^4), &(13)
\end{aligned}$$

with

$$\begin{aligned}
h^{(2)}_{00}(t,x^i) &= -2c^{-2}U(t,x^i), &(14)\\
h^{(4)}_{00}(t,x^i) &= 2c^{-4}((U(t,x^i))^2 - W(t,x^i)), &(15)\\
h^{(3)}_{0i}(t,x^i) &= 4c^{-3}U^i(t,x^i), &(16)\\
h^{(2)}_{ij}(t,x^i) &= -2c^{-2}\delta_{ij}U(t,x^i), &(17)
\end{aligned}$$

and where $U^i(t,x^i)$ and $W(t,x^i)$ are the vector and complementary potentials respectively, and the symbols $h^{(k)}_{\alpha\beta}$ $(k = 2, 3, 4)$ on the left-hand sides of Eqs.(14)-(17) stand for terms of the order ε^k in the respective expansions of $g_{\alpha\beta}$.

For our purposes we need first to expand the components u_0 and u_i of the 4-velocity up to $O(\varepsilon^6)$ and $O(\varepsilon^7)$ respectively, and then the components A_0 and A_i of the 4-acceleration to $O(\varepsilon^7)$ and $O(\varepsilon^6)$ respectively; next, the appropriated expansions for the Christoffel symbols and, finally, the corresponding expansions for the covariant derivatives appearing in (8). Then, from these expansions, and according to Eqs.(10)-(17) we find that ($c = 1$):

$$\sigma_{\alpha\beta} = \begin{pmatrix} \sigma_{00} & \sigma_{i0} \\ \sigma_{0i} & \sigma_{ij} \end{pmatrix} = \begin{pmatrix} \sigma^{(3)}_{00} + \sigma^{(5)}_{00} & \sigma^{(2)}_{i0} + \sigma^{(4)}_{i0} \\ \sigma^{(2)}_{0i} + \sigma^{(4)}_{0i} & \sigma^{(1)}_{ij} + \sigma^{(3)}_{ij} \end{pmatrix} + \begin{pmatrix} O(\varepsilon^7) & O(\varepsilon^6) \\ O(\varepsilon^6) & O(\varepsilon^5) \end{pmatrix}, \tag{18}$$

where

$$\begin{aligned}
\sigma^{(1)}_{ij} &= -\frac{1}{2}(\tilde{v}^i_{,j} + \tilde{v}^j_{,i}), \qquad \sigma^{(2)}_{i0} = \frac{1}{2}(\tilde{v}^i_{,k} + \tilde{v}^k_{,i})\tilde{v}^k, \qquad \sigma^{(3)}_{00} = -\frac{1}{2}(\tilde{v}^2)_{,k}\tilde{v}^k, &(19)\\
\sigma^{(3)}_{ij} &= -\frac{1}{2}\left[(\bar{v}^i_{,j} + \bar{v}^j_{,i}) - (6U + \tilde{v}^2)\sigma^{(1)}_{ij} + \tilde{v}^i(\tilde{v}^j_{,0} + 2\sigma^{(2)}_{j0}) + \tilde{v}^j(\tilde{v}^i_{,0} + 2\sigma^{(2)}_{i0}) + 2\delta_{ij}\dot{U}\right], &(20)\\
\sigma^{(4)}_{i0} &= \frac{1}{2}\left[\tilde{v}^2\tilde{v}^i_{,0} + (6U + 3\tilde{v}^2)\sigma^{(2)}_{i0} + 2\sigma^{(2)}_{i0}\bar{v}^k + (\bar{v}^i_{,k} + \bar{v}^k_{,i})\tilde{v}^k + \tilde{v}^i(\dot{U} - \sigma^{(3)}_{00})\right], &(21)\\
\sigma^{(5)}_{00} &= -\frac{1}{2}\left[\tilde{v}^2(2\dot{U} + \tilde{v}^2_{,0}) - \sigma^{(3)}_{00}(6U + 5\tilde{v}^2) + 2(\tilde{v}^i_{,k}\bar{v}^i + \tilde{v}^i\bar{v}^i_{,k})\tilde{v}^k + \tilde{v}^2_{,k}\bar{v}^k\right], &(22)
\end{aligned}$$

with $v^\mu = \frac{dx^\mu}{dt} = (1, v^1, v^2, v^3)$, $v^i = \tilde{v}^i + \bar{v}^i + O(\varepsilon^5)$, $\tilde{v}^i = O(\varepsilon)$, $\bar{v}^i = O(\varepsilon^3)$ and $\dot{U} = U_{,0} + U_{,k}v^k$.

Notice here that, since Eq.(9) is an identity, the integration of $\sigma_{\alpha\beta} = 0$ may greatly be simplified since this integration reduces to the integration of $\sigma_{ij} = 0$. Notice also that since the principal part of σ_{ij} corresponds to the Newtonian rate-of-strain tensor then, when we integrate the equations $\sigma_{ij} = 0$ for the velocity field distribution of an extended body B in rigid motion at the Newtonian level, that is to say, the equations $\tilde{v}^i_{,j} + \tilde{v}^j_{,i} = 0$, we shall have Eq.(1) as solutions of them in the Newtonian approximation.

Now, in order to prepare the integration at the Post-Newtonian level and according to these last observations, the Post-Newtonian definition of rigid motion can easily be achieved by means of the the following definition:

Definition 1 *A motion is rigid in the Post-Newtonian Born's sense when $\sigma_{ij} = O(\varepsilon^5)$.*

Now, taking into account that from the vanishing of σ_{ij} we have that the expansion $\theta \equiv u^\alpha_{;\alpha}$ also vanishes, and, since at the Post-Newtonian level it results that the expansion is given by $\theta = \theta^{(1)} + \theta^{(3)} + O(\varepsilon^5)$, with $\theta^{(1)} = \tilde{v}^k_{,k}, \theta^{(3)} = \left(\frac{1}{2}(\tilde{v})^2 + U\right)\theta^{(1)} + \bar{v}^k_{,k} + \frac{d}{dt}\left(\frac{1}{2}(\tilde{v})^2 + 3U\right)$, then, from Def. 1 and the condition $\theta = O(\varepsilon^5)$ we necessarily have that $\frac{d}{dt}\left(\frac{1}{2}(\tilde{v})^2 + 3U\right) = 0$.

Therefore, the equations to integrate at the Post-Newtonian level for a rigid motion at this level are

$$\bar{v}^i_{,j} + \bar{v}^j_{,i} = -(\tilde{v}^i\tilde{v}^j_{,0} + \tilde{v}^j\tilde{v}^i_{,0} - \frac{2}{3}\delta_{ij}\tilde{v}^k\tilde{v}^k_{,0}), \tag{23}$$

where $\tilde{v}^i$ are the solution of a rigid motion at the Newtonian level, $\sigma^{(1)}_{ij} = 0$, so that they are given by Eq. (1), that is to say, in vector notation by $\tilde{\boldsymbol{v}}(x^i,t) = \tilde{\boldsymbol{v}}_B(t) + \tilde{\boldsymbol{\Omega}}(t) \times \mathbf{r}$, where $\tilde{\boldsymbol{\Omega}} = (\tilde{\Omega}^1, \tilde{\Omega}^2, \tilde{\Omega}^3)^T = (\tilde{\Omega}_{32}, \tilde{\Omega}_{13}, \tilde{\Omega}_{21})^T$.

3. The velocity field

We may state the following theorem:

Theorem 1 *The velocity field distribution (1) is compatible with the system (23) if, and only if, $\tilde{\Omega}_{ij}(t) = Cnt$. Moreover, in this case, the velocity field distribution for a general rigid motion in the Post-Newtonian Born's sense is given by:*

$$\boldsymbol{v}(x^i,t) = \left(1 - \frac{2}{3}(\mathbf{r}\cdot\tilde{\boldsymbol{v}}_{,0})\right)\tilde{\boldsymbol{v}}_B + \left(1 - \frac{2}{3}(\mathbf{r}\cdot\tilde{\boldsymbol{v}}_{,0})\right)(\tilde{\boldsymbol{\Omega}}\times\mathbf{r}) + \frac{1}{3}(\boldsymbol{\Lambda}\times\tilde{\boldsymbol{v}}_{,0}) + \bar{\boldsymbol{v}}_B + \bar{\boldsymbol{\Omega}}\times\mathbf{r} + O(\varepsilon^5), \tag{24}$$

where

$$\boldsymbol{\Lambda}(x^i,t) = (\tilde{\boldsymbol{v}}\times\mathbf{r}) + \frac{1}{2}(\mathbf{r}\cdot\mathbf{r})\tilde{\boldsymbol{\Omega}} = O(\varepsilon). \tag{25}$$

PROOF. The first part is a direct consequence of the conditions of compatibility of Saint-Venant (see, for example, Love [13]). In fact, from these conditions we have:

$$\tilde{\Omega}^1\dot{\tilde{\Omega}}^1 + \tilde{\Omega}^2\dot{\tilde{\Omega}}^2 = 7\tilde{\Omega}^3\dot{\tilde{\Omega}}^3, \quad \tilde{\Omega}^2\dot{\tilde{\Omega}}^2 + \tilde{\Omega}^3\dot{\tilde{\Omega}}^3 = 7\tilde{\Omega}^1\dot{\tilde{\Omega}}^1, \quad \tilde{\Omega}^1\dot{\tilde{\Omega}}^1 + \tilde{\Omega}^3\dot{\tilde{\Omega}}^3 = 7\tilde{\Omega}^2\dot{\tilde{\Omega}}^2,$$

from which we have $\tilde{\boldsymbol{\Omega}} = Cnt$. This fact simplifies the system (23) since now we have $\tilde{\boldsymbol{v}}_{,0}(t) = \dot{\tilde{\boldsymbol{v}}}_B(t) - \tilde{\boldsymbol{\Omega}} \times \tilde{\boldsymbol{v}}_B(t)$.

Then, integrating Eq.(23) for $i = j$ we have for $v^1(x^i,t)$

$$\begin{aligned} v^1(x^i,t) &= \frac{1}{3}\left(\tilde{v}^2_{,0}\left[\tilde{v}^2_B r^1 + \frac{1}{2}\tilde{\Omega}^3(r^1)^2 - \tilde{\Omega}^1 r^1 r^3\right] + \tilde{v}^3_{,0}\left[\tilde{v}^3_B r^1 - \frac{1}{2}\tilde{\Omega}^2(r^1)^2 + \tilde{\Omega}^1 r^1 r^2\right]\right. \\ &- \left. 2\tilde{v}^1_{,0} r^1\left[\tilde{v}^1_B + \tilde{\Omega}^2 r^3 - \tilde{\Omega}^3 r^2\right]\right) + \alpha^1(x^2,x^3,t), \end{aligned} \tag{26}$$

and similar expressions for $v^2(x^i,t)$ and $v^3(x^i,t)$, where α^i are functions (smooth enough) to be determined. Now, from these last equations and using Eq.(23) for $i \neq j$, it can be seen that α^1 is given by

$$\begin{aligned} \alpha^1 = & - \tilde{v}^2_{,0}\left[\tilde{v}^1_B r^2 - \frac{1}{2}\tilde{\Omega}^3((r^2)^2+(r^3)^2) + r^3(\tilde{\Omega}^2 r^2 + \tilde{\Omega}^3 r^3)\right] + \frac{1}{6}\left(\tilde{v}^3_{,0}\Omega^2 + 2\tilde{v}^2_{,0}\Omega^3\right)(r^2)^2 \\ & - \tilde{v}^3_{,0}\left[\tilde{v}^1_B r^3 + \frac{1}{2}\tilde{\Omega}^2((r^2)^2+(r^3)^2) - r^2(\tilde{\Omega}^2 r^2 + \tilde{\Omega}^3 r^3)\right] - \frac{1}{6}\left(\tilde{v}^2_{,0}\Omega^3 + 2\tilde{v}^3_{,0}\Omega^2\right)(r^3)^2 \end{aligned} \quad (27)$$

having similar expressions for α^2 and α^3.

Now, from (26) and (27) another set of functions may be determined so that, finally, we have:

$$\begin{aligned} v^1(x^i,t) &= \frac{1}{3}\left(\tilde{v}^2_{,0}\left[\tilde{v}^2_B r^1 - \tilde{v}^1_B r^2 - (\tilde{\Omega}^k r^k) r^3 + \frac{1}{2}(r^k r^k)\tilde{\Omega}^3\right] - 2(\tilde{v}^k_{,0} r^k)\left[\tilde{v}^1_B + \tilde{\Omega}^2 r^3 - \tilde{\Omega}^3 r^2\right]\right. \\ &+ \left.\tilde{v}^3_{,0}\left[\tilde{v}^3_B r^1 - \tilde{v}^1_B r^3 + (\tilde{\Omega}^k r^k) r^2 - \frac{1}{2}(r^k r^k)\tilde{\Omega}^2\right]\right), \\ v^2(x^i,t) &= \frac{1}{3}\left(\tilde{v}^1_{,0}\left[\tilde{v}^1_B r^2 - \tilde{v}^2_B r^1 + (\tilde{\Omega}^k r^k) r^3 - \frac{1}{2}(r^k r^k)\tilde{\Omega}^3\right] - 2(\tilde{v}^k_{,0} r^k)\left[\tilde{v}^2_B + \tilde{\Omega}^3 r^1 - \tilde{\Omega}^1 r^3\right]\right. \\ &+ \left.\tilde{v}^3_{,0}\left[\tilde{v}^3_B r^2 - \tilde{v}^2_B r^3 - (\tilde{\Omega}^k r^k) r^1 + \frac{1}{2}(r^k r^k)\tilde{\Omega}^1\right]\right), \\ v^3(x^i,t) &= \frac{1}{3}\left(\tilde{v}^1_{,0}\left[\tilde{v}^1_B r^3 - \tilde{v}^3_B r^1 - (\tilde{\Omega}^k r^k) r^2 + \frac{1}{2}(r^k r^k)\tilde{\Omega}^2\right] - 2(\tilde{v}^k_{,0} r^k)\left[\tilde{v}^3_B + \tilde{\Omega}^1 r^2 - \tilde{\Omega}^2 r^1\right]\right. \\ &+ \left.\tilde{v}^2_{,0}\left[\tilde{v}^2_B r^3 - \tilde{v}^3_B r^2 + (\tilde{\Omega}^k r^k) r^1 - \frac{1}{2}(r^k r^k)\tilde{\Omega}^1\right]\right), \end{aligned}$$

which, written in vector notation, give the result (24) when the solution of the homogeneous system $\alpha^i_{,j} + \alpha^j_{,i} = 0$ is added to them. ■

4. Conclusions

Since, as is known, the relativistic distances involved in the Born's criterion between the particles of a body B have a chronometric measure, i.e., half ($c = 1$) the trip-time from emission and return of photons from any particle to every other of B which, for rigidity, are required to be constants, the validity of the two alternatives Eq.(1) and Eq.(24) may, ideally, be tested by means of an interferometer, which instrument is essentially a device for comparing trip-times. This will either give a model with euclidean or relativistic distances, provided that B is accepted close enough to rigid by interferometry. In effect, as a consequence of this theorem it must be concluded that, the Newtonian velocity distributions (1) applied to PNRCM is compatible with the Post-Newtonian approximation of the Born's rigidity condition (7), that is to say, with (23), only when $\tilde{\Omega}_{ij}(t)$ in (1) is constant. On the other hand, taking into account precisely that Eq.(24) reduces to Eq.(1) for a body B if this is in stationary motion, it should be desirable to use Eq.(1) in the manner made, e.g., by Brumberg in dealing with the N-body problem ([1]) since, under this hypothesis, the general covariance principle of GR is respected and, according to theorem 1, the total number of degrees of freedom generally allowed by Born's definition does not restrict the solution to this problem as is stated in the present paper.

Finally, in the type of solutions like those of [1], [12] and [2], there still remains the possibility of apply Eq.(24) to study motions more general than the stationary motions. This possibility may simplify some of the calculations in the N-body problem, both in the construction of local rigid reference frames at the Post-Newtonian level from their standard covariant definitions in GR (to describe the motions) as well as in simplifying the own equations of motion (being, these last, referred to some local rigid reference frame or to any other local or global reference system) since, as far as we know, the only similar available simplifications so far have been Eq.(1).

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