Geometric form of volcanoes with a limited based

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Abstract

Many volcanic constructs have geometric different shapes depending on different phenomena as parasitic cones, erosion or coral growth. In Lacey, Ockendon and Turcotte [11] the authors proposed a nonlinear model proving that the shape of volcanoes is determined by the hydraulic resistance to the flow of magma, from a line source, through the porous edifice. This model was later extended in Angevine, Turcotte and Ockendon [2] to include the shape of aseismic, submarine ridges. In this communication we propose a modification of the above mentioned models in order to simulate the more realistic case of volcanoes with a limited base.

We start by proving that the free boundary (the volcano base) associated to the models described in the above mentioned references is not bounded as \( t \to +\infty \) (even if it is assumed that the flux generated by the magma supply \( Q_0(t) \) in the line source is a bounded function). As said before, this unrealistic fact (specially in the case of volcanoes located in islands) is the main reason to propose a modification of the involved nonlinear equations in order to obtain a new model giving rise to a bounded free boundary (even if \( t \to +\infty \)). By using some suitable variations of the modelling arguments of Angevine, Turcotte and Ockendon [2] and Lacey, Ockendon and Turcotte [11] we propose the new model,

\[
\begin{align*}
\frac{\partial H}{\partial t} &= K \frac{\partial^2 H^2}{\partial x^2} + \frac{\mu x}{|x|} \frac{\partial H}{\partial x}, \quad x \in \mathbb{R} - \{0\}, \quad t > 0 \\
-K \frac{\partial H}{\partial x}(0, t) &= Q_0(t), \quad t > 0, \\
H(0, x) &= H_0(x), \quad x \in \mathbb{R} - \{0\}.
\end{align*}
\]

Here we assume known the constants \( K, \mu, \lambda > 0 \) (which depend on the constitutive porous material) and that \( Q_0(t) \geq 0, \ H_0(x) \geq 0 \) and \( H_0 \) has compact support in \( \mathbb{R} - \{0\} \). The models proposed in Angevine, Turcotte and Ockendon [2] and Lacey,
Ockendon and Turcotte [11] correspond to the case of $\mu = 0$. We prove that when $\lambda \in (0, 2)$ and $Q_0(t)$ is a bounded function (as it corresponds to the more important examples) then, if we denote by $\xi_{\pm}(t)$ the free boundary (formed by two curves) given by support of $H(t, \cdot)$, i.e. $\text{supp } H(t, \cdot) = [\xi_-(t), 0] \cup [0, \xi_+(t)]$, necessarily $|\xi_{\pm}(t)| < \xi_\infty$ for any $t > 0$, for some $\xi_\infty < +\infty$. This conclusion leads to a better comparison between the bathymetric and theoretical profiles of many volcanoes.

1 Introduction

Let the governing equations for two-dimensional flow of uniform incompressible fluid through a rigid, isotropic porous medium were used in [2] to derive the geometrical form of aseismic volcanoes. They started from the basic equations

$$
\begin{cases}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \\
u = -\frac{k}{\mu \phi} \frac{\partial p}{\partial x}, \\
w = -\frac{k}{\mu \phi} \left(\frac{\partial p}{\partial z} + \rho_m g\right),
\end{cases}
$$

where $u$ and $w$ are the velocities in the $x$ and $z$ directions of the flow, $k$ is permeability, $\mu$ is dynamic viscosity, $\phi$ is porosity, $p$ is pressure, $\rho_m$ is magma density, and $g$ is the gravitational acceleration. These equations are combined to get

$$
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0.
$$

The boundary conditions considered in [2] let the following:

$$
\begin{cases}
z = h, \quad p = \rho_m g(d - h), \quad \text{pressure due to the overlying seawater}, \\
z = h, \quad w = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}, \quad \text{kinematic constraint on the upper surface}, \\
z = 0, \quad \frac{\partial p}{\partial z} = -\rho_m g, \quad \text{on the base of the ridges},
\end{cases}
$$

where $d$ is the depth of the ocean floor and $z = h(x, t)$. We also recall that the magma supply requires the additional condition

$$
u h \rightarrow \frac{Q_0}{2\phi} \text{ as } x \rightarrow 0.
$$

By introducing the small aspect ratio, a rescale is introduced originating the new terms; $Z = z/\epsilon, H = h/\epsilon, D = d/\epsilon$ and $T = t\epsilon$, with $\epsilon << 1$. Equations (3) and (4) become:

$$
\begin{cases}
\epsilon^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial Z^2} = 0, \\
Z = H, \quad p = \epsilon \rho_m g(D - H), \\
Z = H, \quad w = \epsilon^2 \frac{\partial H}{\partial T} + \epsilon u \frac{\partial H}{\partial x}, \\
Z = 0, \quad \frac{\partial p}{\partial Z} = -\epsilon \rho_m g.
\end{cases}
$$
Finally, after rescaling (2) and by using an expansion in the form

\[ p = \epsilon p_0 + \epsilon^3 p_1 + \ldots \]

it was proved in [2] that the velocities at \( Z = H \) must be given by

\[
\begin{aligned}
    u &= -\epsilon k(\rho_m - \rho_w) g \frac{\partial H}{\partial x}, \\
    w &= \epsilon^2 k(\rho_m - \rho_w) g H \frac{\partial^2 H}{\partial x^2}.
\end{aligned}
\]

Substituting these velocities into third equation of (6) they arrive to the degenerate quasi-linear equation of Boussinesq type

\[
\frac{\partial H}{\partial t} = K \frac{\partial^2 H^2}{\partial x^2} + \mu \frac{\partial H^\lambda}{\partial x}.
\]

The solution of this equation, satisfying the associated boundary conditions to (4) and (5) were studied in [2] by using their self-similar structure. Here we shall see (Theorem 1) that if we call as \( \xi(t) \) to the free boundary given by support of \( H(t, \cdot) = \left[-\xi(t), 0\right] \cup \left(0, \xi(t)\right) \), for any \( t > 0 \), then necessarily \( \xi(t) \to +\infty \) as \( t \to +\infty \), which does not seems to be very realistic. So, if we assume symmetry conditions on the initial data \( H_0(x) \), to improve the model, avowing such conclusion, we consider the system

\[
P(\mu, Q_0) \equiv \begin{cases} 
    \frac{\partial H}{\partial t} = K \frac{\partial^2 H^2}{\partial x^2} + \mu \frac{\partial H^\lambda}{\partial x} & x \in (0, +\infty), t > 0, \\
    -KH \frac{\partial H}{\partial x}(0, t) = Q_0(t) & t > 0, \\
    H(0, x) = H_0(x) & x \in (0, +\infty).
\end{cases}
\]

Notice that \( P(0, Q_0) \) corresponds to the Boussinesq type equation (8). Here we assume a renormalization of the constants \( K, \mu > 0 \) and that \( H_0(x) \geq 0 \) has a compact support. We point out that a more general framework is possible (we can detail it in a subsequent draft of the paper which would contain as well the exact definition of weak solution, and other details). The main result for the new model is given in Theorem 2 and shows that if

\[ 0 < \lambda < 2, \]

and

\[ 0 \leq Q_0(t) \leq Q_{0,\infty} \text{ for any } t > 0. \]

then the support of \( H(t, \cdot) = \left[-\xi(t), 0\right] \cup \left(0, \xi(t)\right) \), for any \( t > 0 \), but it has a limited penetration in the sense that

\[ |\xi(t)| \leq \xi_\infty \text{ for any } t \geq 0, \]

for some finite \( \xi_\infty < \infty \) depending on \( \lambda, K, \mu, Q_{0,\infty} \) and \( H_0(x) \). We mention that without the symmetry assumption on \( H_0(x) \) we must work on the spatial domain \( \mathbb{R} - \{0\} \) and by replacing the nonlinear pde by

\[
\frac{\partial H}{\partial t} = K \frac{\partial^2 H^2}{\partial x^2} + \frac{\mu x}{|x|} \frac{\partial H^\lambda}{\partial x}.
\]
The new modelling argument consists in introducing the new velocities:

$$u = \epsilon \frac{k(\rho_m - \rho_w)g_2 H}{\mu \phi} \frac{\partial H}{\partial x} - \nu(\epsilon)\left(\frac{\rho_m - \rho_w}{\mu \phi}\right) g_1 H^{\lambda - 1},$$

$$w = \epsilon^2 \frac{k(\rho_m - \rho_w)g_2 H}{\mu \phi} \frac{\partial^2 H}{\partial x^2},$$

for some $0 < \lambda < 2$ (the justification of this new exponent $\lambda$ may come from some other terms in the asymptotic expansion or from other form of the boundary conditions). Notice also that if $\lambda \in (1, 2)$ the new term is small for $H \in (0, H_0)$, if $\lambda \in (0, 1)$ the new term is very big if $H \in (0, H_0)$ and, which is more useful, when $\lambda = 1$ the new term is a constant. We also point out that $\epsilon^2 \frac{\partial H}{\partial t} = w - \nu(\epsilon)\left(\frac{\rho_m - \rho_w}{\mu \phi}\right) g_1 H^{\lambda - 1} \frac{\partial H}{\partial x}.$

So,

$$\epsilon^2 \frac{\partial H}{\partial t} = \epsilon^2 \frac{k(\rho_m - \rho_w)g_2 H}{\mu \phi} \frac{\partial H}{\partial x} \left(\frac{H \frac{\partial H}{\partial x}}{\lambda \frac{\partial x}}\right) + \nu(\epsilon)\left(\frac{\rho_m - \rho_w}{\mu \phi}\right) g_1 H^{\lambda - 1} \frac{\partial H}{\partial x},$$

and then, we must assume that $\nu(\epsilon) = \epsilon.$

## 2 Unlimited volcanoes base according the previous model ($\mu = 0$).

We are going to prove that the free boundary is not bounded as $t \to +\infty$, for this we are going to prove next theorem.

**Theorem 1** Let $\zeta(t)$ the free boundary of the problem $P(0, Q_0)$, then $\zeta(t) \to +\infty$ if $t \to +\infty$.

We shall built the proof in two different steps. In a first one, we shall prove that if $U(t, x)$ and $H(t, x)$ are solutions of the respective problems $P(0, Q_0)$ and $P(0, 0)$ with the same initial data then, we have $U \leq H$.

In a second step, we shall prove that if $\zeta(t)$ and $\xi(t)$ are the free boundaries of the problems $P(0, Q_0)$ and $P(0, 0)$, with the same initial data, then $\zeta(t) \to +\infty$ as $t \to +\infty$. This will conclude the proof since, the first step proves that $0 < \zeta(t) \leq \xi(t)$, and thus, necessarily, $\xi(t) \to +\infty$ if $t \to +\infty$.

The first step is a special conclusion of a more general statement:

**Proposition 1** Let $H_1$ and $H_2$ be the solutions of $P(\mu, Q_0)$ corresponding to $\mu \geq 0, Q_{1,0}, Q_{2,0} \text{ and } H_{1,0}, H_{2,0}$ respectively. Then, for any $t > 0$ we have

$$\int_{\Omega} (H_1(t, x) - H_2(t, x))_+ dx \leq \int_{\Omega} (H_{1,0}(x) - H_{2,0}(x))_+ dx + \int_0^t (Q_{1,0}(\tau) - Q_{2,0}(\tau))_+ d\tau$$

where, we used the notation, $a_+(x) = \max(0, a(x))$, for any general function defined on $\Omega$. 
Notice that as a direct consequence of the Proposition 1, and of the fact that \( a_+(x) = 0 \) implies that \( a(x) \leq 0 \), we have

**Corollary 1** Let \( H_1 \) and \( H_2 \) be the solutions of \( P(\mu,Q_0) \) corresponding to \( \mu \geq 0, Q_{1,0}, Q_{2,0} \) and \( H_{1,0}, H_{2,0} \) respectively such that \( Q_{1,0}(t) \leq Q_{2,0}(t) \) for any \( t > 0 \) and \( H_{1,0}(x) \leq H_{2,0}(x) \) for \( x \in \Omega \). Then \( H_1(t,x) \leq H_2(t,x) \) for any \( t > 0 \) and for \( x \in \Omega \).

We also get from Proposition 1 a quantitative expression of the continuous dependence of solution \( H \) of \( P(\mu,Q_0) \) with respect to the data \( Q_0 \) and \( H_0 \).

**Corollary 2** Let \( H_1 \) and \( H_2 \) be the solutions of \( P(\mu,Q_0) \) corresponding to \( \mu \geq 0, Q_{1,0}, Q_{2,0} \) and \( H_{1,0}, H_{2,0} \) respectively. Then for any \( t > 0 \)

\[
\int_{\Omega} |H_1(t,x) - H_2(t,x)| \, dx \leq \int_{\Omega} |H_{1,0}(x) - H_{2,0}(x)| \, dx + \int_0^t |(Q_{1,0}(\tau) - Q_{2,0}(\tau))| \, d\tau.
\]

**Proof of Corollary 2.** It is enough to observe that, for any general function \( a(x) \) defined on \( \Omega \) we have that \( |a(x)| = a_+(x) + a_-(x) \) and that, if for fixed \( t > 0 \) we define \( a(x) = H_1(t,x) - H_2(t,x) \) then \( a_-(x) = - \min(0,a(x)) = (H_2(t,x) - H_1(t,x))_+ \). Since, the order of \( H_1 \) and \( H_2 \), taken in Proposition 1, is arbitrary, by reversing the roles of \( H_1 \) and \( H_2 \), we get that

\[
\int_{\Omega} (H_1(t,x) - H_2(t,x))_+ \, dx \leq \int_{\Omega} (H_{1,0}(x) - H_{2,0}(x))_+ \, dx + \int_0^t (Q_{1,0}(\tau) - Q_{2,0}(\tau))_+ \, d\tau,
\]

which concludes the proof.

**Proof of the Proposition 1.** The main idea is to multiply the difference of the two equations by a regular approximation \( p_n(r) \), \( n \in \mathbb{N} \), of the Heaviside function

\[
\text{sign}_{+0}(r) = 0 \text{ if } r \leq 0 \text{ and } \text{sign}_{+0}(r) = 1 \text{ if } r > 0,
\]

taking as \( r = (H_1^2(t,x) - H_2^2(t,x)) \). For instance, we can take \( p_n \)

\[
p_n(r) = \begin{cases} 
0 & \text{if } r \leq -\frac{1}{n}, \\
nr & \text{if } r \in \left[-\frac{1}{n},\frac{1}{n}\right], \\
1 & \text{if } r > \frac{1}{n}.
\end{cases}
\]

Then,

\[
\int_{\Omega} \left( \frac{\partial H_1(t,x)}{\partial t} - \frac{\partial H_2(t,x)}{\partial t} \right) p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx = K \int_{\Omega} \frac{\partial}{\partial x}((\frac{\partial}{\partial x}H_1^2(t,x)) - \\
\frac{\partial}{\partial x}H_2^2(t,x))p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx + \mu \int_{\Omega} \frac{\partial}{\partial x}(H_1^\lambda(t,x) - H_2^\lambda(t,x))p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx.
\]

By the definition of weak solution (i.e., by integrating by parts) we get

\[
\int_{\Omega} \left( \frac{\partial H_1(t,x)}{\partial t} - \frac{\partial H_2(t,x)}{\partial t} \right) p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx + K \int_{\Omega} \frac{\partial}{\partial x}H_1^2(t,x) - \\
\frac{\partial}{\partial x}H_2^2(t,x))^2p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx = \mu \int_{\Omega} \frac{\partial}{\partial x}(H_1^\lambda(t,x) - H_2^\lambda(t,x))p_n(H_1^2(t,x) - H_2^2(t,x)) \, dx + \\
K(\frac{\partial}{\partial x}H_1^2(t,0) - \frac{\partial}{\partial x}H_2^2(t,0))p_n(H_1^2(t,0) - H_2^2(t,0))
\]
where we used the facts that support of $H^2_1(t, .) - H^2_2(t, .)$ is a compact set for any $t \geq 0$. Then, since $0 \leq p_n(r) \leq 1$ for any $r$, and passing to the limit, as $n \to +\infty$ we have that
\[
\text{sign}_+(H^2_1(t, x) - H^2_2(t, x)) = \text{sign}_+(H_1(t, x) - H_2(t, x)) = \text{sign}_+(H^\lambda_1(t, x) - H^\lambda_2(t, x)).
\]

Finally, it is enough to remember that
\[
\frac{\partial H(t, x)}{\partial t} \text{sign}_+(H(t, x)) = \frac{\partial [H(t, x)]_+}{\partial t} \quad \text{and} \quad \frac{\partial H(t, x)}{\partial x} \text{sign}_+(H(t, x)) = \frac{\partial [H(t, x)]_+}{\partial x}
\]
for any general function $H(t, x)$ and so the result follows by interating in $t$ and using that support of $H^\lambda_1(t, .) - H^\lambda_2(t, .)$ is a compact set for any $t \geq 0$ and that $[H^\lambda_1(t, 0) - H^\lambda_2(t, 0)]_+ \geq 0$.

3 Limited volcanoes base for $\mu > 0$.

Concerning the theory of existence and uniqueness of weak solutions we send the reader to the works \cite{1}, \cite{10}, \cite{6}, \cite{9}, \cite{8}, \cite{4} and their references.

**Theorem 2** Assume $H_0(x)$ bounded and with compact support,

\[ 0 < \lambda < 2, \]

and let

\[ 0 \leq Q_0(t) \leq Q_{0,\infty}, \text{ for any } t > 0 \]

for a suitable $Q_{0,\infty}$. Then the support $H(t, .) = [-\xi(t), 0) \cup (0, \xi(t)]$, for any $t > 0$,

\[ |\xi(t)| \leq \xi_\infty \text{ for any } t \geq 0, \]

for some finite $\xi_\infty < \infty$ depending on $\lambda, K, \mu, Q_{0,\infty}$ and $H_0(x)$.

**Proof.** Thanks to Corollary 1 it is enough to construct a supersolution $H_2(t, x)$ with a bounded support for any $t \geq 0$. In fact, we can construct such a function as $H_2(t, x) = U(x)$ solution of the ordinary differential equation

\[
\begin{cases}
K(U^2)_x + C_1 U^\lambda = 0 & x \in (0, +\infty), \\
U(0) = C_2.
\end{cases}
\]

Using that $\lambda < 2$ the support of $U$ is compact and since $H(t, x)$ is bounded we can choose $C_1, C_2 > 0$ suitably as to have

\[ Q_{1,0}(t) \leq C_1 C_2^\lambda \text{ for any } t > 0 \text{ and } H_{1,0}(x) \leq U(x) \text{ for } x \in \Omega, \]

and the proof is complete.

**Remark.** Other supersolutions leading to other qualitative properties of the free boundary can be found in the works \cite{10}, \cite{6}, \cite{7}, \cite{8}, \cite{5} and \cite{3}.
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