Unitary evolution and uniqueness of the Fock quantization in flat cosmologies

G A Mena Marugán[1], D Martín-de Blas[1] and L Castelló Gomar[2]

[1] Instituto de Estructura de la Materia, IEM-CSIC, Serrano 121, 28006 Madrid, Spain.
[2] Facultad de C.C. Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain.

E-mail: `mena@iem.cfmac.csic.es, daniel.martin@iem.cfmac.csic.es`

Abstract. We study the Fock quantization of scalar fields with a time dependent mass in cosmological scenarios with flat compact spatial sections. This framework describes physically interesting situations like, e.g., cosmological perturbations in flat Friedmann-Robertson-Walker spacetimes, generally including a suitable scaling of them by a background function. We prove that the requirements of vacuum invariance under the spatial isometries and of a unitary quantum dynamics select (a) a unique canonical pair of field variables among all those related by time dependent canonical transformations which scale the field configurations, and (b) a unique Fock representation for the canonical commutation relations of this pair of variables. The proof is generalizable to any compact spatial topology in three or less dimensions, though we focus on the case of the three-torus owing to the especially relevant implications.

1. Introduction

The Fock quantization of linear fields in curved spacetimes (either physical backgrounds or auxiliary spacetimes in effective descriptions) is subject to ambiguities which affect the physical predictions. The canonical commutation relations (CCR's) of the fieldlike system admit an infinite number of nonequivalent Fock representations [1]. In simple cases, like e.g. in Minkowski spacetime, one can remove this ambiguity and select a preferred representation by imposing that the vacuum be invariant under the spacetime symmetries. But for less symmetric backgrounds (or when the curved spacetime is only an auxiliary structure) one cannot appeal to this criterion, and other vacuum properties must be required in order to pick out a unique Fock representation.

In this work we will consider the case of scalar fields in cosmological spacetimes which have spatial sections with the topology of a three-torus. We focus our study on the case of three-torus topology owing to its physical relevance, for instance in the study of cosmological perturbations around flat Friedmann-Robertson-Walker (FRW) spacetime with compact sections [2]. The analysis is nonetheless generalizable to other compact spatial sections [3] (see e.g. [2, 4] for three-spheres). We will show that one can select a unique class of unitarily equivalent Fock quantizations by means of a requirement which, furthermore, is itself stable under unitary transformations. This requirement includes the invariance under the spatial isometries. However, this invariance does not suffice to fix the Fock representation, essentially because, in cosmological scenarios, no timelike symmetry is available. The requirement is then completed by demanding instead that the dynamics be unitary in the quantum theory.

On the other hand, in cosmological settings like those that we want to study, it is natural to consider time dependent linear canonical transformations in which the field is scaled [4, 5]. In

these transformations, the dynamics is modified (owing to the time dependent changes) and, in general, a new fieldlike system is reached. This introduces a new kind of ambiguity when facing the Fock quantization. Nonetheless, we will show that our requirement of spatial symmetry invariance and unitary evolution fixes this freedom completely as well.

Before proving these results, let us emphasize the importance of determining a unique Fock quantum description for fields in cosmological spacetimes, since this uniqueness provides robustness to the quantum predictions and makes them significant.

2. Fock quantization with unitary evolution

Let us begin by considering the quantization of a real scalar field φ with a time dependent mass $s(t)$ propagating in a globally hyperbolic static spacetime, $\mathbb{I} \times \Sigma$, where $\mathbb{I}$ is a closed time interval and the spatial sections Σ have the topology of a three-torus. A system of this type describes, for instance, the gauge-invariant energy density perturbations around an FRW spacetime after a suitable field scaling (in conformal time and in isotropic and adiabatic conditions) [4], as well as the ultraviolet behavior of the scalar perturbations after gauge fixing and rescaling [6]. The phase space of the system is given by $(\varphi, P_\varphi) = (\varphi_{|t_0}, \sqrt{h}\dot{\varphi}_{|t_0})$. The dot stands for the time derivative, t_0 is an initial time, and h is the determinant of the spatial metric, h_{ab}. The Hamilton field equations are $\dot{\varphi} = h^{-1/2}P_\varphi$ and $\dot{P}_\varphi = h^{1/2}[\Delta\varphi - s(t)\varphi]$, where Δ is the standard Laplace-Beltami (LB) operator for the three-torus metric. These field equations are invariant under the spatial isometries resulting from the composition of rigid rotations in each of the three spatial directions $i = 1, 2, 3$, $T_{\vec{\alpha}} = T_{\alpha_1} \circ T_{\alpha_2} \circ T_{\alpha_3}$, where $T_{\alpha_i} : \theta_i \to \theta_i + \alpha_i$ and $\alpha_i \in S^1$.

We make a Fourier decomposition of the scalar field using a basis of real eigenfunctions of the LB operator:

$$\varphi(t, \vec{\theta}) = \frac{1}{\pi^{3/2}} \sum_{j=0}^{3} \sum_{\vec{n}_j} \left[q_{\vec{n}_j}(t) \cos(\vec{n}_j \cdot \vec{\theta}) + x_{\vec{n}_j}(t) \sin(\vec{n}_j \cdot \vec{\theta}) \right]. \tag{1}$$

Here, $\vec{n}_0 = (n_1, n_2, n_3)$ with $n_i \in \mathbb{N}$, and $\vec{n}_i$ is obtained from $\vec{n}_0$ by flipping the sing of n_i. A similar expansion of the field momentum leads to modes determined by the coefficients $p_{\vec{n}_j} = \dot{q}_{\vec{n}_j}$ and $y_{\vec{n}_j} = \dot{x}_{\vec{n}_j}$. Then, the equations of motion for the field modes coincide in each eigenspace of the LB operator, with eigenvalue $-\omega_n^2 = -|\vec{n}_j|^2$ (the subindex $n \in \mathbb{N}$ designates here the ordering in the sequence of different eigenvalues):

$$\ddot{q}_{\vec{n}_j} + \left[\omega_n^2 + s(t) \right] q_{\vec{n}_j} = 0, \qquad \ddot{x}_{\vec{n}_j} + \left[\omega_n^2 + s(t) \right] x_{\vec{n}_j} = 0. \tag{2}$$

The possible Fock representations of the CCR's are totally characterized by the choice of a complex structure (CS) J [1], namely, of a real, canonical linear map on phase space whose square is minus the identity and which is compatible with the symplectic structure, Ω (containing the information about the canonical Poisson brackets), in the sense that the bilinear map $\Omega(J\cdot, \cdot)$ is positive definite.

As starting point, we choose the CS which would be naturally associated with a free massless field. This CS, J_0, is determined by the spatial metric, and hence its invariance under the three-torus isometries is warranted. The corresponding annihilationlike variables are

$$a_{\vec{n}_j} = \frac{1}{\sqrt{2\omega_n}}(\omega_n q_{\vec{n}_j} + ip_{\vec{n}_j}), \qquad \tilde{a}_{\vec{n}_j} = \frac{1}{\sqrt{2\omega_n}}(\omega_n x_{\vec{n}_j} + iy_{\vec{n}_j}). \tag{3}$$

For simplicity, we ignore the zero modes, which can be quantized separately (possibly by nonstandard methods). The creationlike variables $\{(a_{\vec{n}_j}^*, \tilde{a}_{\vec{n}_j}^*)\}$ are the complex conjugate of (3). The action of J_0 is diagonal in these variables: $J_0(a_{\vec{n}_j}) = ia_{\vec{n}_j}$, $J_0(a_{\vec{n}_j}^*) = -ia_{\vec{n}_j}^*$, and

likewise for $\tilde{a}_{\vec{n}_j}$ and $\tilde{a}_{\vec{n}_j}^*$. Note that these variables are precisely those promoted to annihilation and creation operators in the Fock representation defined by J_0.

The time evolution of the annihilationlike variables is given by the Bogoliubov transformation

$$a_{\vec{n}_j}(t) = \alpha_n(t,t_0)a_{\vec{n}_j}(t_0) + \beta_n(t,t_0)a_{\vec{n}_j}^*(t_0), \tag{4}$$

where the alpha and beta functions coincide in each eigenspace of the LB operator, and $|\alpha_n(t,t_0)|^2 - |\beta_n(t,t_0)|^2 = 1$. This transformation is unitary in the Fock representation defined by J_0 if and only if the set of beta functions, $\{\beta_{\vec{n}_j}(t,t_0)\}$, is square summable. Taking into account the eigenspace degeneracy, g_n, the considered condition amounts to the square summability of $\{g_n|\beta_n(t,t_0)|^2\}$. On the other hand, the asymptotic behavior when $n \to \infty$ (provided that the timelike function $s(t)$ has an integrable first derivative in each compact time subinterval [2]) is

$$\alpha_n(t,t_0) = e^{-i\omega_n(t-t_0)} + \mathcal{O}\left(\omega_n^{-1}\right), \qquad \beta_n(t,t_0) = \mathcal{O}\left(\omega_n^{-2}\right). \tag{5}$$

The variation of g_n with n in the three-torus case is quite involved, because of accidental degeneration. Nonetheless, one can prove that the sequence $\{g_n/\omega_n^4\}$ is summable if so is $\{D_n/n^4\}$, where D_n is the number of eigenstates with eigenvalue in the interval $(n-1,n]$. It is easy to realize [7] that $D_n = \mathcal{O}(n^2)$ for large n and, therefore, the evolution is indeed unitarily implementable with respect to the CS J_0.

3. Uniqueness results
Let us briefly sketch the proof that the requirement of invariance under the spatial symmetries and of a unitary quantum dynamics indeed selects a unique family of (unitarily equivalent) Fock representations, as well as a preferred canonical pair of field variables.

3.1. Fock representation of the CCR's
First, we characterize the complex structures, J, that are invariant under the group of transformations $T_{\vec{\alpha}}$. These transformations only mix modes with the same labels $\vec{n}_j$ (and the mixing is independent for each value of the label). As a consequence, any invariant CS must be block diagonal, with 4×4 blocks, in the considered basis on phase space. Furthermore, imposing the invariance and compatibility of the symplectic structure with the CS, we conclude that every invariant CS J can be obtained from J_0 by means of a transformation of the type $J = KJ_0K^{-1}$, where K is block diagonal, with 4×4 blocks of the form (see [7] for details):

$$K_{\vec{n}_j} = \begin{pmatrix} \mathcal{K}_{\vec{n}_j} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{\vec{n}_j} \end{pmatrix}; \qquad \mathcal{K}_{\vec{n}_j} = \begin{pmatrix} \kappa_{\vec{n}_j} & \lambda_{\vec{n}_j} \\ \lambda_{\vec{n}_j}^* & \kappa_{\vec{n}_j}^* \end{pmatrix}. \tag{6}$$

Here, $\kappa_{\vec{n}_j}$ and $\lambda_{\vec{n}_j}$ are complex numbers satisfying $|\kappa_{\vec{n}_j}|^2 - |\lambda_{\vec{n}_j}|^2 = 1$, so that K is a canonical transformation. The evolution $\mathcal{U}(t,t_0)$ is unitarily implementable in the representation given by J if and only if $K^{-1}\mathcal{U}(t,t_0)K$ is unitarily implementable in the representation given by J_0, for all instants of time. As before, unitarity amounts to the square summability of the corresponding, new beta functions, which are

$$\beta_{\vec{n}_j}^J(t,t_0) = (\kappa_{\vec{n}_j}^*)^2\beta_n(t,t_0) - \lambda_{\vec{n}_j}^2\beta_n^*(t,t_0) + 2i\kappa_{\vec{n}_j}^*\lambda_{\vec{n}_j}\mathfrak{I}[\alpha_n(t,t_0)]. \tag{7}$$

The symbol $\mathfrak{I}[\cdot]$ denotes the imaginary part. It is easy to realize that, if $\{\beta_{\vec{n}_j}^J(t,t_0)\}$ is square summable, then the same must happen with $\{\lambda_{\vec{n}_j}\mathfrak{I}[\alpha_n(t,t_0)]\}$ [2]. Owing to the asymptotic (ultraviolet) behavior of the functions $\alpha_n(t,t_0)$ (and making an average in time, together with a convenient use of Lusin's theorem [2]), one arrives then to the conclusion that $\{|\lambda_{\vec{n}_j}|^2\}$ must be square summable. But this immediately guarantees that the representations given by J and J_0 are unitarily equivalent (since the transformation K is then implemented unitarily).

3.2. Field description

As the final step in our analysis, we consider now the most general form of a time dependent, linear canonical transformation which scales the field configuration:

$$\phi = f(t)\varphi, \qquad P_\phi = f(t)^{-1}P_\varphi + g(t)\sqrt{h}\varphi. \tag{8}$$

The real functions $f(t)$ and $g(t)$ are assumed to be twice differentiable, and $f(t)$ nonvanishing. Besides, without loss of generality, we can set initially $f(t_0) = 1$ and $g(t_0) = 0$ [4].

One can show that a unitary evolution with respect to a CS which is invariant under the spatial symmetries is only possible when $f(t) = 1$ and $g(t) = 0 \ \forall t \in \mathbb{I}$ [7]. In order to prove this result, we consider the field dynamics of the new canonical pair and compute the alpha and beta functions for the CS J_0:

$$\bar{\alpha}_n = f_+(t)\alpha_n + f_-(t)\beta_n^* + i\frac{g(t)}{2\omega_n}[\alpha_n + \beta_n^*], \quad \bar{\beta}_n = f_+(t)\beta_n + f_-(t)\alpha_n^* + i\frac{g(t)}{2\omega_n}[\beta_n + \alpha_n^*], \tag{9}$$

where $2f_\pm(t) = f(t) \pm 1/f(t)$. Then, the beta functions $\bar{\beta}_{\vec{n}_j}^J$ for any other invariant CS, J, can be obtained as in expression (7). Once more, if the evolution is to be unitary, these beta functions have to be square summable at all times. Hence, in particular, the beta functions must tend to zero for large ω_n. Recalling the asymptotic behavior (5), one can then show that this vanishing limit is possible only if $f_-(t)$ is the zero function, a fact which implies that $f(t) = 1$ [7]. Moreover, by studying the behavior for large ω_n when $f(t)$ is the unity, one can prove that the square summability of the beta functions, needed for the unitarity of the dynamics, requires that $g(t)$ vanish identically [7]. In this way, the field description of the system turns out to be completely determined.

4. Conclusions

We have considered the Fock quantization of scalar fields in (possibly effective) cosmological spacetimes whose spatial sections have a three-torus topology. We have shown that the requirement of invariance under the spatial symmetries and of a unitary dynamics picks out a preferred canonical pair of variables for the field description –among all those related by time dependent canonical transformations which include a scaling of the field configuration. Besides, for such a canonical pair, the introduced requirement turns our to select a unique family of unitarily equivalent Fock representations of the CCR's. As a result, the ambiguity in the Fock quantization coming from the scaling of the field or from a choice of vacuum is removed. This allows one to reach robust physical predictions in the quantum theory for linear scalar fields with quadratic potentials, propagating in some types of nonstationary scenarios, which include the relevant case of compact flat FRW cosmologies.

References

[1] Wald R M, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Chicago University Press, Chicago, 1994).
[2] Cortez J, Mena Marugán G A, Olmedo J, and Velhinho J M, Phys. Rev. D **81**, 044037 (2010).
[3] Cortez J, Mena Marugán G A, Olmedo J, and Velhinho J M, Classical Quantum Gravity **28**, 172001 (2011).
[4] Cortez J, Mena Marugán G A, and Velhinho J M, Phys. Rev. D **83**, 025002 (2011).
[5] Mukhanov V, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
[6] Fernández-Méndez M, Mena Marugán G A, Olmedo J, and Velhinho J M, Phys. Rev. D **85**, 103525 (2012).
[7] Castelló Gomar L, Cortez J, Martín-de Blas D, and Mena Marugán G A, *in preparation*.