A representation theorem for finite Gödel algebras with operators

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Abstract. In this paper we introduce and study *finite Gödel algebras with operators* (GAOs for short) and their dual frames. Taking into account that the category of finite Gödel algebras with homomorphisms is dually equivalent to the category of finite forests with order-preserving open maps, the dual relational frames of GAOs are *forest frames*: finite forests endowed with two binary (crisp) relations satisfying suitable properties. Our main result is a Jónsson-Tarski like representation theorem for these structures. In particular we show that every finite Gödel algebra with operators determines a unique forest frame whose set of subforests, endowed with suitably defined algebraic and modal operators, is a GAO isomorphic to the original one.

Keywords: Finite Gödel algebras; modal operators; finite forests; representation theorem.

1 Introduction

Fuzzy modal logic is an active and relatively recent area of research aimed at generalizing classical modal logic to the many-valued or fuzzy framework. This is usually done by considering a Kripke-style relational semantics in which both accessibility relations and evaluations of modal formulas (in each world) are allowed to take values in the real unit interval $[0, 1]$, instead of the classical two-valued set $\{0, 1\}$ (see $[4, 5, 7]$ for instance).

In this contribution we put forward a new, algebraic-oriented perspective to the area of modal fuzzy logic, and in particular to Gödel modal logic by defining and studying the class of *finite Gödel algebras with operators* (*GAOs* for short). These structures are obtained by expanding the language of Gödel algebras (i.e. prelinear Heyting algebras) by means of two modal operators $\Diamond$ and $\Box$ equationally described by the same axioms used to define these operators in Boolean algebras with operators (BAOs), see [3].

Obviously, while in a BAO the operators $\Diamond$ and $\Box$ are inter-definable, this is not the general case in a GAO since the negation operator in a Gödel algebra is not involutive. Hence, the equation $\Diamond x = \neg\Box\neg x$ does not hold in general in a GAO.

In the same way the dual frames of BAOs are Kripke frames, the duality between finite Gödel algebras and finite forests (see [1]) leads us to introduce the dual structures of GAOs as triples $(\mathbf{F}, R_\Diamond, R_\Box)$, where $\mathbf{F} = (F, \leq)$ is a finite forest, while $R_\Diamond$ and $R_\Box$ are binary (crisp) relations on $\mathbf{F}$ satisfying suitable conditions of (anti-)monotonicity in their first argument.

The main result of this paper is a Jónsson-Tarski like representation theorem for GAOs. In particular we will show how, starting from a Gödel algebra with operators $(\mathbf{A}, \Diamond, \Box)$, one can uniquely define a forest frame $(\mathbf{F}, R_\Diamond, R_\Box)$ such that $(\mathbf{A}, \Diamond, \Box)$ is isomorphic to the GAO whose Gödel reduct is the algebra of subforests of $\mathbf{F}$ and whose modal operators are defined from the binary relations $R_\Diamond$ and $R_\Box$.

Finally, we will discuss the effect of a stronger axiomatization for $\Diamond$ and $\Box$ on the side of the corresponding forest frame. In particular we will see that the equations usually imposed on *positive modal algebras* [8, 6] allow for a simpler description of the forest frames needed in the representation theorem.

This paper is organized as follows. After this introduction, in the following Section 2 we will recall basic facts on finite Gödel algebras and finite forests. In Section 3 we will consider the case of Gödel algebras expanded by the modal operator $\Diamond$, while Gödel algebras with a $\Box$ operator will be studied in Section 4. Section 5 is dedicated to introduce Gödel algebras with both $\Diamond$ and $\Box$ and also to discuss the effect of the stronger axiomatization for these modalities obtained by adding the equations of positive modal algebras. We will end this paper in Section 6 where we present our future work.

2 Finite Gödel algebras and forests

Gödel algebras, the algebraic semantics of infinite-valued Gödel logic [9], are idempotent, bounded, integral, commutative residuated lattices of the form $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \bot, \top)$ satisfying the prelinearity equation: $(a \rightarrow b) \vee (b \rightarrow a) = \top$. In other words, Gödel algebras are *prelinear Heyting algebras*. If not unless specified, all algebras we will consider in this paper are finite.

Let $\mathbf{A}$ be a Gödel algebra and denote by $F_\mathbf{A}$ the set of its prime filters, i.e., filters principally generated by the join-irreducible elements of $\mathbf{A}$. Unlike the case of boolean algebras, prime and maximal filters are not the same for Gödel algebras and indeed $F_\mathbf{A}$ can be ordered in a nontrivial way. In particular, if for $f_1, f_2 \in F_\mathbf{A}$ we define $f_1 \leq f_2$ iff (as prime filters) $f_1 \supseteq f_2$, $\mathbf{F_A} = (F_\mathbf{A}, \leq)$ turns out to be a finite *forest*, i.e., a poset such that the downset of each element is totally ordered.

Finite forests play a crucial role in the theory of Gödel algebras. Indeed, let $\mathbf{F} = (F, \leq)$ be a finite forest, $S_\mathbf{F}$ be the set of all downward closed subsets of F (i.e., the *subforests* of $\mathbf{F}$) and consider the following operations on $S_\mathbf{F}$: for all $x, y \in F$,

1. $x \wedge y = x \cap y$ (the set-theoretic intersection);

2. $x \vee y = x \cup y$ (the set-theoretic union);
3. $x \to y = F \setminus \uparrow(x \setminus y)$ (where $\setminus$ denotes the set-theoretical difference and for every $z \in F$, $\uparrow z = \{k \in F \mid k \geq z\}$).[3]

The algebra $\mathbf{S_F} = (S_\mathbf{F}, \wedge, \vee, \to, \emptyset, F)$ is a Gödel algebra [1, §4.2] and the following is a Stone-like representation theorem for these structures.

Lemma 1 ([1, Theorem 4.2.1]). *Every Gödel algebra* $\mathbf{A}$ *is isomorphic to* $\mathbf{S_{F_A}}$ *through the map* $r : \mathbf{A} \to \mathbf{S_{F_A}}$

$$r : a \in A \mapsto \{f \in F_\mathbf{A} \mid a \in f\}.$$

Example 1. Let free_1 be the 1-generated free Gödel algebra (Fig. 1). Its prime filters, which are all principally generated as upsets of its join-irreducible elements, are $f_1 = \{y \in \mathsf{free}_1 \mid y \geq x\} = \{x, x \vee \neg x, \neg\neg x, \top\}$, $f_2 = \{y \in \mathsf{free}_1 \mid y \geq \neg x\} = \{\neg x, x \vee \neg x, \top\}$, and $f_3 = \{y \in \mathsf{free}_1 \mid y \geq \neg\neg x\} = \{\neg\neg x, \top\}$. The forest $\mathbf{F}_{\mathsf{free}_1}$ is obtained by ordering $\{f_1, f_2, f_3\}$ by reverse inclusion.

Let us consider the set $S_{\mathbf{F}_{\mathsf{free}_1}}$ of subforests of $\mathbf{F}_{\mathsf{free}_1}$:

$$S_{\mathbf{F}_{\mathsf{free}_1}} = \{\emptyset, F_{\mathsf{free}_1}, \{f_2\}, \{f_1\}, \{f_2, f_1\}, \{f_3, f_1\}\}$$

with operations $\wedge, \vee, \to$ as in (1-3) above. Lemma 1 shows that algebra $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ is a Gödel algebra which is isomorphic to free_1.

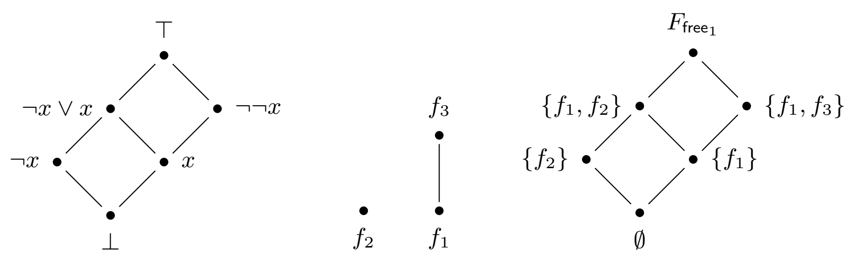

Fig. 1. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1, the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters, and the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$

3 Gödel algebras with $\Diamond$-operators

Definition 1. *A* $\Diamond$-Gödel algebra *is a pair* $(\mathbf{A}, \Diamond)$ *where* $\mathbf{A}$ *is a Gödel algebra and* $\Diamond : A \to A$ *satisfies the following equations:*

[3] Without danger of confusion, and thanks to the following result, we will not distinguish the symbols of a Gödel algebra $\mathbf{A}$ from those of $\mathbf{S_F}$

($\Diamond$1) $\Diamond(\bot) = \bot$;
($\Diamond$2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Definition 2. *A* $\Diamond$-forest frame *is a pair* $(\mathbf{F}, R)$ *where* $\mathbf{F} = (F, \leq)$ *is a finite forest and* $R \subseteq F \times F$ *satisfies the following condition:*

(A) *for all* $x, y, z \in F$*, if* $y \leq x$ *and* $R(x, z)$*, then* $R(y, z)$[4].

For every $\Diamond$-forest frame $(\mathbf{F}, R)$, let $\mathbf{S_F}$ be defined as in the previous section and consider the map $\delta_R : S_\mathbf{F} \to S_\mathbf{F}$ such that, for every $a \in S_\mathbf{F}$

$$\delta_R(a) = \{y \in F \mid \exists z \in a,\ R(y, z)\}. \quad (1)$$

Notice that, for all $a \in S_\mathbf{F}$, $\delta_R(a) \in S_\mathbf{F}$, i.e., $\delta_R(a)$ is a subforest of $\mathbf{F}$. Indeed if $x \in \delta_R(a)$ then there exists $z \in a$ such that $R(x, z)$. Let $y \leq x$ in $\mathbf{F}$. Then (A) of Definition 2 implies $R(y, z)$ as well, that is $y \in \delta_R(a)$ and hence $\delta_R(a)$ is downward closed. Further, the following properties hold.

Proposition 1. *For every* $\Diamond$*-forest frame* $(\mathbf{F}, R)$ *let* $\delta_R : S_\mathbf{F} \to S_\mathbf{F}$ *be defined as in (1). Then:*

1. $\delta_R(\bot) = \bot$;
2. *For all* $b \in S_\mathbf{F}$*,* $\delta_R(b) = \bigcup\{\delta_R(a) \mid a \leq b$ *and* a *is join-irreducible*$\}$.

Proof. (1) The bottom element of $\mathbf{S_F}$ is the empty forest, whence $\emptyset = \{y \in F \mid \exists z \in \emptyset,\ R(y, z)\} = \delta_R(\bot)$.
(2) The claim is trivial if b is join irreducible. Thus, let $b = a_1 \vee \ldots \vee a_m$ with the a_i's being join irreducible. Therefore, $\delta_R(b) = \{y \in F \mid \exists z \in b,\ R(y, z)\} = \{y \in F \mid \exists z \in a_1 \vee \ldots \vee a_m,\ R(y, z)\} = \{y \in F \mid \exists z \in a_1 \cup \ldots \cup a_m,\ R(y, z)\} = \bigcup_{i=1}^m(\{y \in F \mid \exists z \in a_i,\ R(y, z)\}) = \bigcup_{i=1}^m \delta_R(a_i)$. □

Lemma 2. *For each* $\Diamond$*-forest frame* $(\mathbf{F}, R)$*,* $(\mathbf{S_F}, \delta_R)$ *is a* $\Diamond$*-Gödel algebra.*

Proof. From Lemma 1, $\mathbf{S_F}$ is a Gödel algebra. Equation ($\Diamond$1) holds because of Proposition 1(1). Further, if $a, b \in S_\mathbf{F}$, by Proposition 1(2), $\delta_R(a \vee b) = \delta_R(a) \cup \delta_R(b) = \delta_R(a) \vee \delta_R(b)$ by definition of δ_R. Thus, δ_R satisfies ($\Diamond$2). □

Now, let $(\mathbf{A}, \Diamond)$ be a $\Diamond$-Gödel algebra, let $\mathbf{F_A}$ be as in Section 2 and define $Q_\Diamond \subseteq F_\mathbf{A} \times F_\mathbf{A}$ as follows: for all $f_1, f_2 \in F_\mathbf{A}$,

$$Q_\Diamond(f_1, f_2) \text{ iff } \Diamond(f_2) \subseteq f_1, \quad (2)$$

where, for every filter f, $\Diamond(f) = \{\Diamond x \mid x \in f\}$. Then the following holds.

Lemma 3. *For each* $\Diamond$*-Gödel algebra* $(\mathbf{A}, \Diamond)$*,* $(\mathbf{F_A}, Q_\Diamond)$ *is a* $\Diamond$*-forest frame.*

Proof. It is enough to prove that the condition (A) of Definition 2 holds. Let $f_1, f_2, f_3 \in F_\mathbf{A}$ and assume $Q_\Diamond(f_1, f_3)$ (i.e., $\Diamond(f_3) \subseteq f_1$) and $f_1 \geq f_2$, meaning that, as prime filters, $f_1 \subseteq f_2$. Then, $\Diamond(f_3) \subseteq f_1 \subseteq f_2$ and hence $Q_\Diamond(f_2, f_3)$.

[4] Along this paper we will adopt the notation $R(x, y)$ to denote that the pair (x, y) belongs to the relation R.

Now, our aim is to extend the isomorphism r of Lemma 1 to the case of $\Diamond$-Gödel algebras. Let hence $(\mathbf{A},\Diamond)$ be a $\Diamond$-Gödel algebra and define, for every $a\in A$,

$$r(\Diamond(a)) = \{f\in F_{\mathbf{A}} \mid \Diamond(a)\in f\}. \tag{3}$$

Theorem 1. *For every $\Diamond$-Gödel algebra $(\mathbf{A},\Diamond)$, the map $r:(\mathbf{A},\Diamond)\to(\mathbf{S}_{\mathbf{F_A}},\delta_{Q_\Diamond})$ is an isomorphism. In particular, for all $a\in A$,*

$$r(\Diamond(a)) = \delta_{Q_\Diamond}(r(a)). \tag{4}$$

Proof. We proved in Lemma 1 that $\mathbf{S}_{\mathbf{F_A}}$ is a Gödel algebra and the map $r:\mathbf{A}\to\mathbf{S}_{\mathbf{F_A}}$ is a Gödel isomorphism. Thus, it remains to show that (4) holds. First of all notice that it is sufficient to prove it for the case of a being a join-irreducible element of $\mathbf{A}$. Indeed, assume that (4) holds for join irreducible elements and let b be not join irreducible. Then b can be displayed as $b=a_1\vee\ldots\vee a_k$, where the a_i's are join irreducible. By ($\Diamond$2), $\Diamond(b)=\Diamond(a_1)\vee\ldots\vee\Diamond(a_k)$. Therefore, since r is a Gödel algebra isomorphism,

$$r(\Diamond(b)) = r(\Diamond(a_1))\vee\ldots\vee r(\Diamond(a_k)).$$

By assumption, $r(\Diamond a_i)=\delta_{Q_\Diamond}(r(a_i))$ for all $i=1,\ldots,k$. Thus, $r(\Diamond(b))=\delta_{Q_\Diamond}(a_1)\vee\ldots\vee\delta_{Q_\Diamond}(a_k)$ which equals $\delta_{Q_\Diamond}(b)$ by Proposition 1(2).

Let hence a be join irreducible. By Lemma 1, we have:

$$\begin{aligned}\delta_{Q_\Diamond}(r(a)) &= \{f\in F_{\mathbf{A}} \mid \exists g\in r(a),\ Q_\Diamond(f,g)\}\\ &= \{f\in F_{\mathbf{A}} \mid \exists g\in F_{\mathbf{A}},\ (a\in g\ \&\ Q_\Diamond(f,g)\}\\ &= \{f\in F_{\mathbf{A}} \mid \exists g\in F_{\mathbf{A}},\ (a\in g\ \&\ \Diamond(g)\subseteq f)\}\end{aligned}$$

Therefore, if $f\in\delta_{Q_\Diamond}(r(a))$, $\Diamond(a)\in f$ and hence $f\in r(\Diamond(a))$.

To prove the other inclusion we have to show that if $f'\in r(\Diamond(a))$, there exists an $f\in F_{\mathbf{A}}$ such that $a\in f$ and $\Diamond(f)\subseteq f'$. Since a is join irreducible, the filter $f_a=\{b\in A\mid b\geq a\}$ is prime. Let us prove that $\Diamond(f_a)\subseteq f'$.

Claim. $\Diamond(f_a)\subseteq f_{\Diamond(a)}=\{x\in A\mid x\geq\Diamond(a)\}$.

As a matter of fact, if $z\in\Diamond(f_a)$, then there exists $b\geq a$ such that $z=\Diamond(b)$. Since $\Diamond$ is monotone, $\Diamond(b)\geq\Diamond(a)$, whence $z=\Diamond(b)\in f_{\Diamond(a)}$.

Claim. For all $f'\in r(\Diamond(a))$, $f_{\Diamond(a)}\subseteq f'$.

Indeed, if $x\in f_{\Diamond(a)}$, then $x\geq\Diamond(a)$ and hence $x\in f'$ because $\Diamond(a)\in f'$ and f' is upward closed.

By the above claims, for all $f'\in r(\Diamond(a))$, $\Diamond(f_a)\subseteq f'$, whence

$$r(\Diamond(a))\subseteq\delta_{Q_\Diamond}(r(a)).$$

Thus, for all a, $r(\Diamond(a))=\delta_{Q_\Diamond}(r(a))$ which settles the claim. □

Example 2. Let free_1 be as in Example 1 and let $\Diamond:\mathsf{free}_1\to\mathsf{free}_1$ be the following map:

$$\Diamond(\bot) = \bot;\ \Diamond(x) = \neg x;\ \Diamond(\neg x) = \neg x \vee x;\ \Diamond(\neg x \vee x) = \neg x \vee x;$$
$$\Diamond(\neg\neg x) = \top;\ \Diamond(\top) = \top.$$

It is easy to check that $\Diamond$ satisfies ($\Diamond$1) and ($\Diamond$2) of Definition 1 and hence $(\mathsf{free}_1, \Diamond)$ is a $\Diamond$-Gödel algebra.

Let $\mathbf{F}_{\mathsf{free}_1}$ be the dual forest of free_1 as in Example 1 and let us compute $Q_\Diamond$ according to (2). First: $\Diamond(f_1) = \{\neg x, x \vee \neg x, \top\}$; $\Diamond(f_2) = \{x \vee \neg x, \top\}$ and $\Diamond(f_3) = \{\top\}$. Therefore, (see Figure 2)

$$Q_\Diamond = \{(f_1, f_2), (f_1, f_3), (f_2, f_2), (f_2, f_1), (f_2, f_3), (f_3, f_3)\}.$$

The relation $Q_\Diamond$ satisfies the property (A) of Definition 2. Indeed, $f_1 \leq f_3$, and for all $f \in F_{\mathsf{free}_1}$, if $Q_\Diamond(f_3, f)$ then $Q_\Diamond(f_1, f)$. Therefore $(F_{\mathsf{free}_1}, Q_\Diamond)$ is a $\Diamond$-forest frame.

Finally, let $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be the isomorphic copy of free_1 as in Example 1 and let $\delta_{Q_\Diamond} : \mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}} \to \mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be as in (1):

$\delta_{Q_\Diamond}(\emptyset) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \emptyset,\ Q_\Diamond(f, g)\} = \emptyset$;

$\delta_{Q_\Diamond}(\{f_1\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1\},\ Q_\Diamond(f, g)\} = \{f_2\}$;

$\delta_{Q_\Diamond}(\{f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_2\},\ Q_\Diamond(f, g)\} = \{f_1, f_2\}$;

$\delta_{Q_\Diamond}(\{f_1, f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1, f_2\},\ Q_\Diamond(f, g)\} = \{f_1, f_2\}$;

$\delta_{Q_\Diamond}(\{f_1, f_3\}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in \{f_1, f_3\},\ Q_\Diamond(f, g)\} = \{f_1, f_2, f_3\} = F_{\mathsf{free}_1}$;

$\delta_{Q_\Diamond}(F_{\mathsf{free}_1}) = \{f \in F_{\mathsf{free}_1} \mid \exists g \in F_{\mathsf{free}_1},\ Q_\Diamond(f, g)\} = F_{\mathsf{free}_1}$.

Therefore, $(\mathsf{free}_1, \Diamond)$ and $(\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}, \delta_{Q_\Diamond})$ are isomorphic $\Diamond$-Gödel algebras.

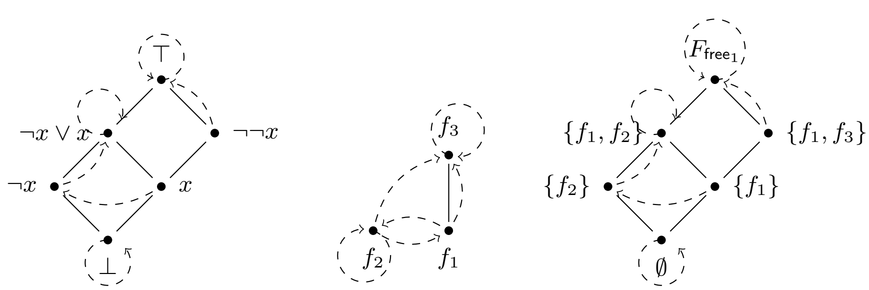

Fig. 2. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 and a $\Diamond$ operator (dotted arrows); the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters and the relation $Q_\Diamond$ (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ endowed with the operator $\delta_{Q_\Diamond}$ (dotted arrows).

4 Gödel algebras with $\Box$-operators

Definition 3. *A $\Box$-Gödel algebra is a pair $(\mathbf{A}, \Box)$ such that $\mathbf{A}$ is a Gödel algebra and $\Box : A \to A$ satisfies the following equalities:*

($\Box$1) $\Box(\top) = \top$;
($\Box$2) $\Box(a \wedge b) = \Box a \wedge \Box b$.

Definition 4. *A $\Box$-*forest frame *is a pair $(\mathbf{F}, R)$ where $\mathbf{F} = (F, \leq)$ is a finite forest and $R \subseteq F \times F$ satisfies the following condition:*

(M) *for all $x, y, z \in F$, if $x \leq y$ and $R(x, z)$, then $R(y, z)$.*

For every $\Box$-forest frame $(\mathbf{F}, R)$, let $\beta_R : S_{\mathbf{F}} \to S_{\mathbf{F}}$ be defined as follows: for all $a \in S_{\mathbf{F}}$,

$$\beta_R(a) = \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a)\}. \tag{5}$$

For all $a \in S_{\mathbf{F}}$, $\beta_R(a)$ is a subforest of $\mathbf{F}$. Indeed, if $x \in \beta_R(a)$ then $\forall z \in F$, $(R(x, z) \Rightarrow z \in a)$. Let $y \leq x$. Thus, for all $z \in F$ either $R(y, z)$ is false (and in this case the condition $R(y, z) \Rightarrow z \in a$ is trivially true), or $R(y, z)$ is true in which case $R(x, z)$ is true as well, because of (M), and hence $z \in a$. Thus $y \in \beta_R(a)$.

Proposition 2. *The following properties hold:*

1. $\beta_R(\top) = \top$;
2. *For all $b \in A_F$, $\beta_R(b) = \bigcup(\{\beta_R(a) \mid a \leq b$ and a is join irreducible$\})$.*

Proof. (1) Recall from Section 2 that the top element of $\mathbf{S_F}$ is F. Thus, $\beta_R(\top) = \beta_R(F) = \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in F)\}$. Obviously, the condition $(R(y, z) \Rightarrow z \in F)$ is true for all $z \in F$ and hence $\beta_R(F) = F$.

(2) Skipping the trivial case in which b is join irreducible, let $b = a_1 \vee \ldots \vee a_m$ with the a_i's join irreducible. Remember that in classical logic, for every finite k, $x \Rightarrow (\exists i \in \{1, \ldots, k\}(y_i)) = \exists i \in \{1, \ldots, k\}\ (x \Rightarrow y_i)$, hence

$$\begin{aligned}
\beta_R(b) &= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in b)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in \textstyle\bigvee_{i=1}^m a_i)\} \\
&= \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow (\exists i \in \{1, \ldots, m\}\ (z \in a_i)))\} \\
&= \{y \in F \mid \forall z \in F,\ \exists i \in \{1, \ldots, k\}\ (R(y, z) \Rightarrow z \in a_i)\} \\
&= \textstyle\bigcup_{i=1}^m \{y \in F \mid \forall z \in F,\ (R(y, z) \Rightarrow z \in a_i)\} \\
&= \textstyle\bigcup_{i=1}^m \beta_R(a_i).
\end{aligned}$$

The claim is hence settled. $\Box$

Lemma 4. *For every $\Box$-forest frame $(\mathbf{F}, R)$, $(\mathbf{S_F}, \beta_R)$ is a $\Box$-Gödel algebra.*

Proof. We already showed that $\beta_R(\top) = \top$. If $a, b \in S_{\mathbf{F}}$ and recalling that, as subforests of F, $a \wedge b = a \cap b$, one has

$$\begin{aligned}\beta_R(a \wedge b) &= \{y \in F \mid \forall z \in F,\ (R(y,z) \Rightarrow z \in a \wedge b)\}\\ &= \{y \in F \mid \forall z \in F,\ (R(y,z) \Rightarrow z \in a \cap b)\}\\ &= \{y \in F \mid \forall z \in F,\ (R(y,z) \Rightarrow z \in a)\} \cap\\ &\quad\ \{y \in F \mid \forall z \in F,\ (R(y,z) \Rightarrow z \in b)\}\\ &= \beta_R(a) \cap \beta_R(b)\\ &= \beta_R(a) \wedge \beta_R(b).\end{aligned}$$

□

Let $(\mathbf{A}, \Box)$ be a $\Box$-Gödel algebra and define $Q_\Box$ on $F_{\mathbf{A}} \times F_{\mathbf{A}}$ as follows: for each $f_1, f_2 \in F_{\mathbf{A}}$,

$$Q_\Box(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2, \tag{6}$$

where, for every filter f, $\Box^{-1}(f) = \{x \in A \mid \Box(x) \in f\}$.

Lemma 5. *For every $\Box$-Gödel algebra $(\mathbf{A}, \Box)$, $(\mathbf{F_A}, Q_\Box)$ is a $\Box$-forest frame.*

Proof. Let $f_1, f_2, f_3 \in F_{\mathbf{A}}$. If $f_1 \leq f_2$ in the order of $\mathbf{F_A}$, then $f_1 \supseteq f_2$ as prime filters, whence if $\Box^{-1}(f_1) \subseteq f_3$ then $\Box^{-1}(f_2) \subseteq f_3$. Therefore, if $Q_\Box(f_1, f_3)$, then $Q_\Box(f_2, f_3)$ which settles the claim.

The following result is the analogous of Theorem 1 in the case of $\Box$-Gödel algebras where r is the map of Lemma 1 which extends to all elements of a $\Box$-Gödel algebra $(\mathbf{A}, \Box)$ by the following stipulation:

$$r(\Box(a)) = \{f \in F_{\mathbf{A}} \mid \Box(a) \in f\}. \tag{7}$$

Theorem 2. *For every finite $G_\Box$-algebra $(\mathbf{A}, \Box)$, the map $r : (\mathbf{A}, \Box) \to (\mathbf{S}_{\mathbf{F_A}}, \beta_{Q_\Box})$ defined as above is an isomorphism. In particular, for all $a \in A$,*

$$r(\Box(a)) = \beta_{Q_\Box}(r(a)). \tag{8}$$

Proof. Let us start proving that for all $a \in A$, $\beta_{Q_\Box}(r(a)) \subseteq r(\Box(a))$. By definition,

$$\begin{aligned}\beta_{Q_\Box}(r(a)) &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}}\ (Q_\Box(f,g) \Rightarrow g \in r(a))\}\\ &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}}\ (\Box^{-1}(f) \subseteq g \Rightarrow a \in g)\}.\end{aligned}$$

Let $f \in \beta_{Q_\Box}(r(a))$ and assume, by way of contradiction, that $f \notin r(\Box(a))$, that is to say, $a \notin \Box^{-1}(f)$. Notice that this assumption forces $a \neq \top$.

Claim. $\Box^{-1}(f)$ is a filter of $\mathbf{A}$.

As a matter of facts, $\top \in \Box^{-1}(f)$ because $\top \in f$ and $\Box\top = \top$. Further, if $a, b \in \Box^{-1}(f)$, then $\Box a \in f$ and $\Box b \in f$, whence $\Box(a) \wedge \Box(b) \in f$ since f is a filter. Hence $\Box(a \wedge b) \in f$ by ($\Box$2) showing that $\Box^{-1}(f)$ is $\wedge$-closed. Finally, if $a \in \Box^{-1}(f)$ and $b \geq a$, then by the monotonicity of $\Box$, $\Box(b) \geq \Box(a)$, hence $\Box(b) \in f$ because f us upward closed. Therefore, $\Box^{-1}(f)$ is a filter of $\mathbf{A}$.

Going back to the proof of Theorem 2, if $a \in \Box^{-1}(f)$ and since $a \neq \top$, by [9, Lemma 2.3.15], there exists a prime filter p of $\mathbf{A}$ such that $p \supseteq \Box^{-1}(f)$ and $a \notin p$. On the other hand, $Q_{\Box}(f, p)$ because p extends $\Box^{-1}(f)$ and $a \notin p$. Thus, $f \notin \beta_{Q_{\Box}}(r(a))$ and a contradiction has been reached.

For the other inclusion, we have to prove that if $\Box(a) \in f$, then for all $g \in F_{\mathbf{A}}$, $Q_{\Box}(f, g) \Rightarrow a \in g$. If $\Box(a) \in f$, then $a \in \Box^{-1}(f)$. Therefore, for all $g \in F_{\mathbf{A}}$, if $Q_{\Box}(f, g)$, then $\Box^{-1}(f) \subseteq g$ and hence $a \in g$ which settles the claim. □

Example 3. As in the previous Examples 1 and 2 let free_1 the free, 1-generated Gödel algebra and consider the map $\Box : \mathsf{free}_1 \to \mathsf{free}_1$ defined as follows (dashed arrows in the leftmost picture of Figure 3):

$$\Box(\bot) = \bot;\ \Box(x) = x;\ \Box(\neg x) = \bot;\ \Box(\neg x \vee x) = x;\ \Box(\neg\neg x) = \neg\neg x;\ \Box(\top) = \top.$$

That operation makes $(\mathsf{free}_1, \Box)$ into a $\Box$-Gödel algebra.

For the reader convenience, let us compute $\Box^{-1}(f)$ (for $f \in F_{\mathsf{free}_1}$): Adopting the same notation of the previous examples,

$$\Box^{-1}(f_1) = \{\top\};\ \Box^{-1}(f_2) = f_2;\ \Box^{-1}(f_3) = f_3.$$

Therefore, by (6), $Q_{\Box} \subseteq F_{\mathsf{free}_1} \times F_{\mathsf{free}_1}$ is the following relation (check Figure 3, central picture):

$$Q_{\Box} = \{(f_1, f_1), (f_2, f_2), (f_3, f_3), (f_2, f_1), (f_2, f_3), (f_3, f_1)\}.$$

Notice that $(F_{\mathsf{free}_1}, Q_{\Box})$ is a $\Box$-forest frame. Indeed, $f_1 \leq f_3$ and for all $f \in F_{\mathsf{free}_1}$, $Q_{\Box}(f_1, f) \leq Q_{\Box}(f_3, f)$.

Finally, let $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ be the isomorphic copy of free_1 as in Example 1 and let us define $\beta_{Q_{\Box}}$ as above, i.e., for all $a \in S_{\mathbf{F}_{\mathsf{free}_1}}$,

$$\beta_{Q_{\Box}}(a) = \{f \in F_{\mathsf{free}_1} \mid \text{for all } g \in F_{\mathsf{free}_1},\ \text{if } Q_{\Box}(f, g) \text{ then } g \in a\}.$$

The computation is tedious and we will only show $\beta_{Q_{\Box}}(\{f_1, f_2\})$. The remaining cases are left to the reader.

$$\beta_{Q_{\Box}}(\{f_1, f_2\}) = \{f \in F_{\mathsf{free}_1} \mid \text{for all } g \in F_{\mathsf{free}_1},\ \text{if } Q_{\Box}(f, g) \text{ then } g \in \{f_1, f_2\}\}.$$

Let us enter a case distinction:

- $f_1 \in \beta_{Q_{\Box}}(\{f_1, f_2\})$. Let g be arbitrary in F_{free_1}. In particular, if $g = f_1$, then $Q_{\Box}(f_1, f_1)$ and $f_1 \in \{f_1, f_2\}$; if $g = f_2$, we have $Q_{\Box}(f_1, f_2)$ and again $f_2 \in \{f_1, f_2\}$; if $g = f_3$, $(f_1, f_3) \notin Q_{\Box}$ whence we conclude that $f_1 \in \beta_{Q_{\Box}}(\{f_1, f_2\})$.
- $f_2 \in \beta_{Q_{\Box}}(\{f_1, f_2\})$. Again we distinguish the following cases: for $g = f_1$ or $g = f_2$, $Q_{\Box}(f_2, g)$ and $g \in \{f_1, f_2\}$; if $g = f_3$, $Q_{\Box}(f_2, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_2 \notin \beta_{Q_{\Box}}(\{f_1, f_2\})$.
- $f_3 \in \beta_{Q_{\Box}}(\{f_1, f_2\})$. Notice immediately that for $g = f_3$ one has $Q_{\Box}(f_3, f_3)$ but $f_3 \notin \{f_1, f_2\}$, whence $f_3 \notin \beta_{Q_{\Box}}(\{f_1, f_2\})$.

Therefore, $\beta_{Q_{\Box}}(\{f_1, f_2\}) = \{f_1\}$ (see Figure 3, dashed arrows in the rightmost picture, for the remaining cases).

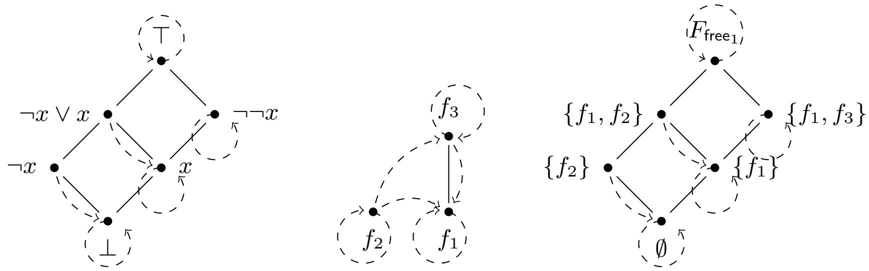

Fig. 3. From left to right: The Hasse diagram of the free Gödel algebra over one generator free_1 and a $\Box$ operator (dotted arrows); the forest $\mathbf{F}_{\mathsf{free}_1}$ of its prime filters and the relation $Q_\Box$ (dotted arrows); the Hasse diagram of its isomorphic copy $\mathbf{S}_{\mathbf{F}_{\mathsf{free}_1}}$ endowed with the operator $\beta_{Q_\Box}$ (dotted arrows).

5 Gödel algebras with $\Diamond$ and $\Box$ operators

The notions of results provided in the previous sections immediately give us the following

Definition 5. *A* Gödel algebra with operators *(GAO for short) is a triple* $(\mathbf{A}, \Diamond, \Box)$ *where* $\mathbf{A}$ *is a Gödel algebra,* $\Diamond$ *and* $\Box$ *are unary operators of* A *which satisfy the equations* $(\Diamond 1)$*-*$(\Diamond 2)$ *and* $(\Box 1)$*-*$(\Box 2)$ *of Definitions 1 and 3 respectively.*

Let us observe that the equations for $\Diamond$ and $\Box$ are *minimal* in the sense that $(\Diamond 1)$-$(\Diamond 2)$ and $(\Box 1)$-$(\Box 2)$ are the weakest requirements we may ask the modal operators to satisfy, taking into account that, since the negation operator in Gödel algebras is not involutive, $\Diamond$ and $\Box$ are not inter-definable as in the classical setting. A similar remark concerning the minimality of those equations, but in the more general setting of *Heyting algebras with operators*, can be found in [10].

This remark leads us to the following notion of *frame* for GAOs which, not surprisingly, includes both that of $\Diamond$- and $\Box$-forest frame.

Definition 6. *A* forest frame *is a triple* $(\mathbf{F}, R_\Diamond, R_\Box)$ *such that* $(\mathbf{F}, R_\Diamond)$ *is a* $\Diamond$*-forest frame and* $(\mathbf{F}, R_\Box)$ *is a* $\Box$*-forest frame.*

Given a GAO $(\mathbf{A}, \Diamond, \Box)$ and following exactly the same constructions and results described in the previous Sections 3 and 4, it is immediate to show that, indeed, $(\mathbf{F_A}, Q_\Diamond, Q_\Box)$ is a forest frame and, vice versa, given any forest frame $(\mathbf{F}, R_\Diamond, R_\Box)$, the algebra $(\mathbf{S_F}, \delta_{R_\Diamond}, \beta_{R_\Box})$ is a GAO. The following result, which is an immediate consequence of Theorem 1 and Theorem 2, is a Jónsson-Tarski like representation for GAOs.

Theorem 3. *Let* $(\mathbf{A}, \Diamond, \Box)$ *be a GAO. The map* $r : (\mathbf{A}, \Diamond, \Box) \to (\mathbf{A}_{F_\mathbf{A}}, \delta_{Q_\Diamond}, \beta_{Q_\Box})$*, where* $\delta_{Q_\Diamond}$ *and* $\beta_{Q_\Box}$ *are defined by equations (3) and (7), is an isomorphism. In particular, for all* $a \in A$*,*

$$r(\Diamond(a)) = \delta_{Q_\Diamond}(r(a)) \text{ and } r(\Box(a)) = \beta_{Q_\Box}(r(a)). \tag{9}$$

Following [8, 6], let us consider the following equations:

$(D1)$ $\Box(a \vee b) \leq \Box a \vee \Diamond b$;
$(D2)$ $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

For every Gödel algebra $\mathbf{A}$, let us denote by $\mathbf{A}^-$ its $\{\to, \neg\}$-free reduct. Then, if $(\mathbf{A}, \Diamond, \Box)$ satisfies $(D1)$ and $(D2)$, $(\mathbf{A}^-, \Diamond, \Box)$ is a *positive modal algebra* in the sense of [8, 6]. Since the set of prime filters of $\mathbf{A}$ and that of $\mathbf{A}^-$ coincide, $\mathbf{F}_{\mathbf{A}^-} = \mathbf{F}_{\mathbf{A}}$ and, following [6], let us define $R_{\mathbf{A}} \subseteq F_{\mathbf{A}^-} \times F_{\mathbf{A}^-}$ as follows: for all $f_1, f_2 \in F_{\mathbf{A}}$,

$$R_{\mathbf{A}}(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2 \subseteq \Diamond^{-1}(f_1).$$

Observing that $f_2 \subseteq \Diamond^{-1}(f_1)$ iff $\Diamond(f_2) \subseteq f_1$, by [6, Lemma 2.1(1)], we have that $R_{\mathbf{A}} = Q_\Diamond \cap Q_\Box$, where $Q_\Diamond$ and $Q_\Box$ are defined as in (2) and (6) respectively.

Now, let $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}}$ be the Gödel algebra of subforests of $\mathbf{F}_{\mathbf{A}^-}$ and define $\delta_{R_{\mathbf{A}}}$ and $\beta_{R_{\mathbf{A}}}$ on $S_{\mathbf{F}_{\mathbf{A}^-}}$ by (1) and (5) respectively. Then the following is an immediate consequence of [6, Theorem 2.2] (see also [8, Theorem 8.1]).

Proposition 3. *Let* $(\mathbf{A}, \Diamond, \Box)$ *be a GAO which satisfies* $(D1)$ *and* $(D2)$*. Then its* $\{\to, \neg\}$*-free reduct* $(\mathbf{A}^-, \Diamond, \Box)$ *and the positive algebra* $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ *are isomorphic (as positive modal algebras).*

Clearly, $S_{\mathbf{F}_{\mathbf{A}^-}} = S_{\mathbf{F}_{\mathbf{A}}}$. Now, it is not difficult to extend the above result to GAOs satisfying $(D1)$ and $(D2)$ by expanding the algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ by the operator $\to$ defined as in Section 2: for all $x, y \in S_{\mathbf{F}_{\mathbf{A}}}$,

$$x \to y = F_{\mathbf{A}} \setminus \uparrow(x \setminus y).$$

Then, $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-$ plus $\to$ and $\neg$ (defined as usual by $\neg x = x \to \emptyset$) is a Gödel algebra isomorphic to $\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}$. Thus, the following holds.

Theorem 4. *Every GAO* $(\mathbf{A}, \Diamond, \Box)$ *satisfying* $(D1)$ *and* $(D2)$ *is isomorphic to* $(\mathbf{S}_{\mathbf{F}_{\mathbf{A}}}, \delta_{R_{\mathbf{A}}}, \beta_{R_{\mathbf{A}}})$ *(as Gödel algebras with operators).*

6 Conclusion and future work

In the present paper we have introduced finite Gödel algebras with modal operators and their dual forest frames. Our main result is a Jónsson-Tarski like representation theorem for these structures. Further, we have introduced a proper subclass of Gödel algebras with operators, and we have shown for them a simplified representation which uses, on the dual side of forest frames, only one accessibility relation. It is important to notice that, in contrast with [5] where the authors consider Kripke frames for Gödel modal logic with a unique $[0, 1]$-valued accessibility relation, the dual frames of our Gödel algebras with operators have two crisp accessibility relations. This latter observation offers, in our opinion, a fresh new perspective on the semantic approach to fuzzy modal logics which deserves to be further investigated.

As for future work we plan the address the following questions:

(1) To extend the results of this paper to the whole class of Gödel algebras. In this direction we will investigate an extension of Theorem 3 for *general* Gödel algebras with operators. In order to achieve this goal we will take into account that the prime spectrum of a Gödel algebra forms an Esakia space whose underline poset is a forest (see [11, Theorem 2.4]).

(2) The whole class of Gödel algebras with operators forms a variety which determines the equivalent algebraic semantics of a Gödel modal logic. This logic, denoted by $\mathbf{G}_{\Box\Diamond}$, can be regarded as the axiomatic extension of intuitionistic modal logic $\mathbf{IntK}_{\Box\Diamond}$ [12] by the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$. Our main plans in this direction are to show that $\mathbf{G}_{\Box\Diamond}$ has the finite model property and to compare $\mathbf{G}_{\Box\Diamond}$ with the other approaches to Gödel modal logic existing in the literature, in particular with that of [5]. In this paper the authors introduce a logic with both $\Box$ and $\Diamond$ operators, stronger than $\mathbf{G}_{\Box\Diamond}$[5], that is shown to be complete with respect to the class of Kripke models over the standard Gödel algebra (on the unit real interval $[0,1]$) where both the accessibility relation and formulas are evaluated on $[0,1]$.

(3) Finite Nilpotent Minimum (NM) algebras with (or without) a negation fixpoint are dually equivalent to the category of finite forests (and hence categorically equivalent to finite Gödel algebras) [1, Proposition 4.5.4 and §4.5] and [2, Corollary 4.10]. In particular, the connected (disconnected, respectively) rotation of the $\{\bot\}$-free reduct of any finite, directly indecomposable Gödel algebra $\mathbf{A}$ is a finite, directly indecomposable, NM-algebra with (without) negation fixpoint and each directly indecomposable NM-algebra with (without) negation fixpoint arises in this way (see [1, §4.5] and references therein). Taking into account this structural description, we plan to extend the analysis reported in this paper to the classes of NM-algebras with, or without, negation fixpoint.

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[5] In particular it includes Fisher-Servi connecting axioms $\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$ and $(\Diamond\varphi \to \Box\varphi) \to \Box(\varphi \to \psi)$.

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