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Applying the triple correlation functions to characterizing high-frequency repetition trains of picosecond optical pulses

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1. Introduction.

The triple correlation of an ordinary function on the real time is the integral of the product of that function with two independently shifted copies of itself. Triple correlation methods are frequently used in signal processing for treating signals that are corrupted by additive Gaussian noise; in particular, triple correlation techniques perform well when multiple observations of the signal are available and the signal may be translating in between the observations, e.g. a sequence of images of an object translating on a noisy background.

2.- The triple auto-correlation function for a one-dimensional signal

The triple auto-correlation function of the signal $\mathbf{F(t)}$ is determined by the following integral

$$\mathbf{F_3(t_1,t_2)} = \int_{-\infty}^{\infty} \mathbf{F(t)F(t+t_1)F(t+t_2)\,dt} \ . \tag{1}$$

The Fourier transformation of Eq.(1) with the kernel $\mathbf{\exp[-2\pi i\ (f_1 t_1 + f_2 t_2)]}$ gives the bispectrum

$$\text{a)}\ \ \mathbf{F_3(f_1,f_2) = F(f_1)F(f_2)F(-f_1-f_2)}\ , \qquad \text{b)}\ \ \mathbf{F(f)} = \int_{-\infty}^{\infty} \mathbf{F(t)\exp[(-2\pi i\ f\ t)\,dt}\ , \tag{2}$$

where $\mathbf{F(f)}$ is the spectrum of signal. The bispectrum $\mathbf{F_3(f_1,f_2)}$ has two following symmetries

$$\mathbf{F_3(f_1,f_2) = F_3(f_2,f_1) = F_3(-f_1-f_2,f_1)}\ , \tag{3}$$

which lead to the redundancy of three fourths of the frequency plane $\mathbf{(f_1,f_2)}$ with determining the bispectrum $\mathbf{F_3(f_1,f_2)}$. For a real-valued temporal signal $\mathbf{F(t)}$, both the spectrum $\mathbf{F(f)}$ and bispectrum $\mathbf{F_3(f_1,f_2)}$ are Hermitian conjugate functions, i.e.

$$\text{a)}\ \ \mathbf{F(f) = F^*(-f)}\ , \qquad \text{b)}\ \ \mathbf{F_3(f_1,f_2) = F_3^*(-f_1,-f_2)}\ , \tag{4}$$

so that just in this particular case even one eight of the frequency plane $\mathbf{(f_1,f_2)}$ is quite enough for determining the bispectrum $\mathbf{F_3(f_1,f_2)}$. If a system is linear in behavior and temporally invariant relative to the signal $\mathbf{F(t)}$, this system is linear in behavior and temporally invariant relative to the triple auto-correlation function as well. It follows from the constraint equation coupling the input and output temporal signals, $\mathbf{F_{in}(t)}$ and $\mathbf{F_{out}(t)}$, through the response function $\mathbf{P(t)}$, namely,

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$$\text{a)}\quad F_{out}(t) = \int_{-\infty}^{\infty} F_{in}(t-\tau)\, P(\tau)\, d\tau\ , \qquad\qquad \text{b)}\quad F_{out}(f) = F_{in}(f)\, P(f)\ . \tag{5}$$

It is seen from Eqs.(2) and (5b) that $F_{3,\,out}(f_1, f_2) = F_{3,\,in}(f_1, f_2)\ P_3(f_1, f_2)$, so that

$$F_{3,\,out}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{3,\,in}(t_1-\tau_1, t_2-\tau_2)\, P_3(t_1-\tau_1, t_2-\tau_2)\, d\tau_1\, d\tau_2\ , \tag{6}$$

where $P_3(t_1, t_2)$ and $P_3(f_1, f_2)$ are the corresponding response functions. Then, Eq.(2) shows that the bispectrum becomes to be not varied when an arbitrary exponential factor is included into the spectrum of the temporal signal. Let us take $F^S(f) = F(f)\exp(\gamma f)$, where γ is an arbitrary complex-valued constant. In this case, one can calculate

$$F_3^S(f_1,f_2) = F^S(f_1)\,F^S(f_2)\,F^S(-f_1-f_2) = F(f_1)\,F(f_2)\,F(-f_1-f_2)\exp[\gamma(f_1+f_2-f_1-f_2)] = F_3(f_1,f_2) \tag{7}$$

This example demonstrates that the process of recovering the signal from the triple auto-correlation function or the bispectrum can be not always unambiguous and conclusive. Now, we can illustrate these considerations by a few particular graphical cases.

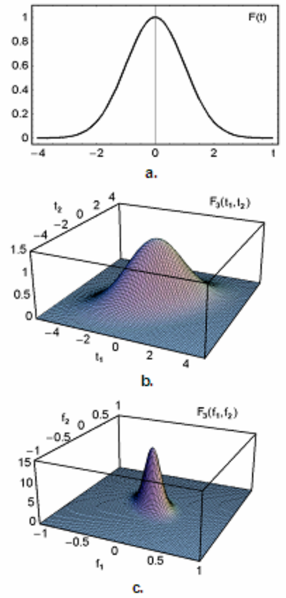

Figure 1. The envelope, triple auto-correlation, and bispectrum for real Gaussian pulse.

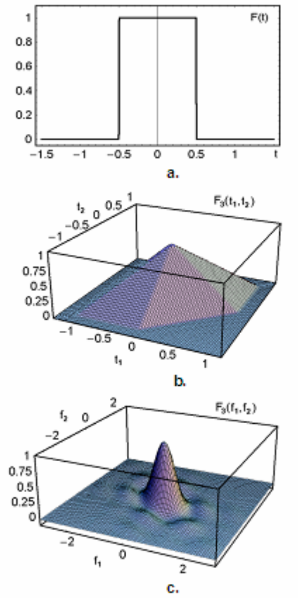

Figure 2. The envelope, triple auto-correlation, and bispectrum for real rectangular pulse.

Figures 1 and 2 demonstrate the envelopes, triple auto-correlations, and bispectra for typical Gaussian and rectangular pulses of unit width given by

$$\text{a)}\quad F^G(t) = \exp\left(-t^2/2\right), \qquad\qquad \text{b)}\quad F^R(t) = \theta\left(x+0.5\right) - \theta\left(x-0.5\right), \tag{8}$$

The corresponding analytical expressions describing Figs. 1b, 1c, 2b, and 2c are described by

$$\text{a)}\quad F_3^G(t_1,t_2) = \sqrt{\frac{2\pi}{3}}\exp\left[-\frac{1}{3}\left(t_1^2+t_2^2-t_1 t_2\right)\right],$$

$$\text{b)}\quad F_3^G(f_1,f_2) = (2\pi)^{3/2}\exp\left[-4\pi^2\left(f_1^2+f_2^2+f_1 f_2\right)\right],$$

c) $F_3^R(t_1,t_2) = 0.25\left[\,\text{sign}\,(t_1 - t_2 - 1)\,(|1-t_1| - |1+t_2| + |t_1| - |t_2|) + \text{sign}\,(t_1 - t_2 + 1)\cdot\right.$

$$\left.(|1+t_1| + |1-t_2| - |t_1| - |t_2|) + \text{sign}\,(t_1 - t_2)\,(|1-t_1| - |1+t_1| + |1+t_2| - |1-t_2|)\,\right],$$

$$\text{d) } F_3^R(f_1,f_2) = \frac{\sin(\pi f_1)}{\pi f_1}\,\frac{\sin(\pi f_2)}{\pi f_2}\,\frac{\sin[\pi(f_1+f_2)]}{\pi(f_1+f_2)}\ . \qquad (9)$$

3. The algorithm of recovering the temporal signal from its triple auto-correlation function

If the temporal signal $F(t)$ is, for example, real-valued as well as is of a finite extent, it can be retrieved from its triple auto-correlation function $F_3(t_1,t_2)$ almost uniquely apart from a shift. For a real signal $F(t)$ of finite extent, its spectrum $F(f)$ can be analytically continued by extending the frequency f to the complex variable $z = z' + i z''$. The analytic continuation $F(z)$ is determined by its complex zeros z_n and can be written as a Hadamard product

$$F(z) = \exp(\alpha + \beta z)\prod_n (z - z_n)\exp\left(\frac{z}{z_n}\right)\ . \qquad (10)$$

where α and β are some constants. This fundamental equation from the theory of complex functions cannot be applied directly to any arbitrary function of two variables. However, in the case of triple correlations, it is known how the two-dimensional function $F_3(z_1,z_2)$ is related to the one-dimensional function $F(z)$, because one can exploit Eq.(2), so that one can write

$$F_3(z_1,z_2) = F(z_1)F(z_2)F(-z_1 - z_2)\ . \qquad (11)$$

Consequently, one can insert Eq.(10) into Eq.(11) and obtain

$$F_3(z_1,z_2) = \exp(3\alpha)\prod_n (z_1 - z_n)(z_2 - z_n)(-z_1 - z_2 - z_n)\ . \qquad (12)$$

Using Eq.(12), one can derive the particular complex zeros of $F(z)$ from the complex zero subspaces of $F_3(z_1,z_2) = 0$. Ones the zeros z_n are known, one can compute $F(z)$, hence $F(f)$, and then $F(t)$. The detailed consideration of this proof shows that for the general case of the complex-valued function $F(t)$, its spectrum $F(f)$ can be reconstructed up to the exponential factor $\exp(\alpha + \beta f)$, where the factor β is an arbitrary, broadly speaking complex-valued, constant, while $\alpha = \{0, 2\pi i - (1/3), 2\pi i - (2/3)\}$.

One can consider a retrieval algorithm. For the sake of simplicity let us assume reality of the signal $F(t)$ and hence Hermitian symmetry of its spectrum $F(f)$, see Eq.(4a). We begin by setting $f_2 = 0$ in a bispectrum $F_3(f_1,f_2)$, so that

$$F_3(f_1,0) = F(f_1)F(0)F(-f_1) = |F(f_1)|^2 F(0) \qquad (13)$$

and therefore the Fourier amplitude $|F(f)|$ is available directly on the f_1-axis of $F_3(f_1,f_2)$. Then, one can retrieve the Fourier phase $\varphi(f)$, defined by $F(f) = |F(f)|\exp[\varphi(f)]$. To that end we concentrate on a straight line, being parallel to the f_1-axis, but above it by one sampling step δf.

$$F_3(f_1,\delta f) = F(f_1)F(\delta f)F(-f_1 - \delta f) = F(\delta f)\cdot|F(f_1)\,F(-f_1 - \delta f)|\exp[i\varphi(f_1) - i\varphi(f_1 + \delta f)] \qquad (14)$$

Defining $\varphi_3(f_1,f_2)$ as the phase of the bispectrum $F_3(f_1,f_2)$, one can derive from Eq.(14) the following phase equation

$$\varphi_3(f_1, f_2) = \varphi(\delta f) + [\varphi(f_1) - \varphi(f_1 + \delta f)] \quad . \tag{15}$$

Finally, one can extract $\varphi(f_1)$ itself, apart from an additive constant, and a term liner in f_1, that reflects the lack of knowledge about t_0 in $F(t - t_0)$.

4. Application to the characterization of picosecond optical pulses

Since a new field of researches had been developed, and now the pulse durations reported by experiments are approaching the theoretical limits. Nevertheless, up to now there are no detectors being fast enough to measure such ultrashort pulses directly. That is why a lot of the elaborated methods of measuring are based on the analysis of various auto-correlation functions. Unfortunately, the auto-correlation functions of the second order is symmetric in behavior, so that they cannot give us any information about asymmetry if optical pulses under investigation. At this point, one can benefit from a triple auto-correlation, which can provide the true pulse shape. A triple-intensity correlation interferometer is shown in Fig.3. This interferometer can be exploited to record the raw data of the experiments with a sequence of ultrashort optical pulses. The triplet of arms of this interferometer provides mutually delayed pulse trains with the intensities $I(t + t_1)$ and $I(t + t_2)$ together with the non-delayed intensity $I(t)$. Mixing of these three pulse trains on nonlinear crystal with the resulting third-harmonic generation, one can obtain the intensity triple auto-correlation function

$$I_3(t_1, t_2) = \int_{-\infty}^{\infty} I(t) I(t + t_1) I(t + t_2)\, dt \quad . \tag{16}$$

Once the intensity triple auto-correlation function of the optical pulses is known, the pulse shape $I(t)$ can be reconstructed using the algorithm described the previous section. Figure 4 illustrates the corresponding steps of such a reconstruction. Thus, this technique makes possible measuring asymmetric envelopes of ultrashort optical pulses and recovering signals almost unambiguously.

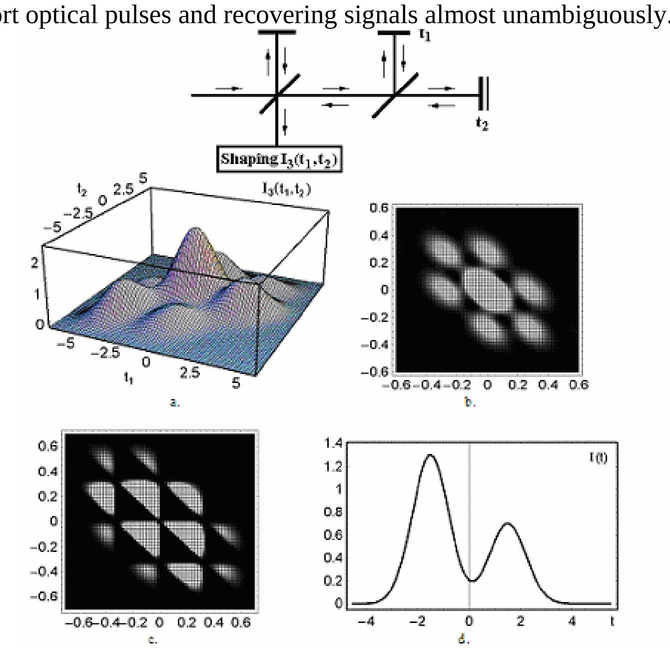

Figure 4. The steps of reconstructing an asymmetric optical pulse: (a) - a triple auto-correlation function; (b) and (c) - real and imaginary parts of the bispectrum; (d) – the reconstructed pulse.