Geometric Characterization of the Homogeneity of Continua with Microstructure

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1. Introduction

A continuum with microstructure may geometrically be modelled as an associated bundle with a principal bundle. The homogeneity is characterized by using the theory of connections in principal bundles.

A continuum with microstructure is a simple body $B$ each point of which has attached a manifold of parameters $[2]$. In geometrical terms, we have a body manifold $B$ and a fiber bundle $\tilde{\pi} : \tilde{E} \rightarrow B$ over $B$. Some kind of homogeneity is needed for each fiber and the geometrical measure of this homogeneity is supplied by the action of a Lie group on the manifold of parameters. In geometrical words, the fiber bundle is associated with a principal bundle $\pi : E \rightarrow B$ with structure group $\mathcal{G}$.

In this framework, a configuration is an embedding of principal bundles of $\pi : E \rightarrow B$ into the trivial bundle $\pi_0 : \mathbb{R}^n \times \mathcal{G} \rightarrow B$. A change of configuration is a deformation. The material response is supposed to depend on the 1-jet of the deformation. We introduce the notion of uniformity and isotropy group in terms of jets. If the body $\pi : E \rightarrow B$ enjoys smooth uniformity we can characterize the homogeneity in terms of three connections: one linear connection $\Gamma$ on $B$ and two connections in the principal bundle $\pi : E \rightarrow B$: $\Lambda$ (which is defined from a global section $\mathcal{P} : B \rightarrow E$) and $\tilde{\Lambda}$. In fact, it is proved that $B$ is locally homogeneous if and only if the torsion tensor

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of $\Gamma$ identically vanishes and the global section $\mathcal{P} : \mathcal{B} \rightarrow \mathcal{E}$ is parallel with respect to $\mathcal{A}$.

These results recover the ones for second grade materials [3, 4, 12, 9] (see also [7, 8]), Cosserat continua [11, 6] and continua with vector microstructure [5].

Our approach may be considered as the natural generalization of the continuous theories of inhomogeneities of Noll [18] and Wang [20] (see also [19]). An alternative approach based in a defective crystalline lattice due to Kondo, Bilby and Kröner [14, 1, 15]) was recently updated by Kröner [16]. The use of principal bundles formalism in elastoplasticity theories may enjoy interesting features as the recent work by Epstein and Maugin shows [10] (see also [17, 16]).

2. CONTINUUMS WITH MICROSTRUCTURE. UNIFORMITY AND MATERIAL SYMMETRIES

An $n$-dimensional body $\mathcal{B}$ is said to be a continuum with microstructure if there exists a bundle $\tilde{\pi} : \tilde{\mathcal{E}} \rightarrow \mathcal{B}$ associated with some principal bundle $\pi : \mathcal{E} \rightarrow \mathcal{B}$ with structure group $\mathcal{G}$. The standard fibre $\mathcal{F}$ of $\tilde{\mathcal{E}}$ is the manifold of parameters. We assume that $\mathcal{F}$ has dimension $m$ and dim $\mathcal{B} = n \leq 3$.

Denote by $\pi_0 : \mathbb{R}^3 \times \mathcal{G} \rightarrow \mathbb{R}^3$ the trivial bundle. Therefore, a configuration of $\mathcal{E}$ is a principal bundle embedding $\Phi : \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathcal{G}$ which induces the identity between the structure groups. We denote by $\Phi : \mathcal{B} \rightarrow \mathbb{R}^3$ the induced embedding between the bases.

A deformation is a change of configuration, that is, given two configurations $\tilde{\Phi}_i : \mathcal{E} \rightarrow \mathbb{R}^3 \times \mathcal{G}$, $i = 1, 2$, $\tilde{\kappa} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1}$, which is a principal bundle isomorphism from $\tilde{\Phi}_1(\mathcal{E})$ into $\tilde{\Phi}_2(\mathcal{E})$ inducing the identity between the structure groups and covering the diffeomorphism $\kappa = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(\mathcal{B}) \rightarrow \Phi_2(\mathcal{B})$.

We assume that the material response is completely characterized by a scalar function which depends on the first derivative of the deformation. The constitutive equation is:

$$W = W(j^1_{\tilde{\kappa}(\tilde{\kappa})} \tilde{\kappa}) \quad (1)$$

We can consider equivalence classes of local principal bundle isomorphisms (as in the Appendix A) and then the constitutive equation more appropriately reads as follows:

$$W = W(j^1_{\kappa(\kappa)} \kappa) \quad (2)$$
where $j^1_{X,\kappa(X)} \tilde{\kappa}$ denotes the equivalence class of $j^1_{X,\kappa(\tilde{X})} \tilde{\kappa}$.

From now on, we fix a reference configuration $\Phi_0$, and make the obvious identifications: $B = \Phi_0(B)$, $E = \Phi_0(E)$.

**Definition 1.** A continuum with microstructure $B$ is said to be uniform if for every pair of points $X, Y \in B$ there exists a local isomorphism of principal bundles $\Phi$ (inducing the identity between the structure groups) such that $\Phi(X) = Y$ and

$$W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa} \circ j^1_{X,\Phi(X)} \Phi) = W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa}) \quad \forall \tilde{X} \in \pi^{-1}(X), \forall j^1_{\Phi(\tilde{X}),\kappa(\Phi(\tilde{X}))} \tilde{\kappa},$$

where $\circ$ denotes the composition of jets.

With the obvious notations, the uniformity condition may be equivalently written as

$$W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa} \circ j^1_{X,\Phi(X)} \Phi) = W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa}) \quad \forall j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa},$$

where $\circ$ denotes the composition of equivalence classes of jets.

Such a 1-jet (and its class) will be called a local uniformity from $X$ to $Y$.

A material symmetry at a point $X \in B$ is a 1-jet $j^1_{X,\Phi(X)} \Phi$ where $\Phi$ is a local isomorphism of principal bundles (inducing the identity between the structure groups) such that $\pi(\Phi(\tilde{X})) = \pi(\tilde{X}) = X$ and

$$W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa} \circ j^1_{X,\Phi(X)} \Phi) = W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa}) \quad \forall \tilde{X} \in \pi^{-1}(X), \forall j^1_{\Phi(\tilde{X}),\kappa(\Phi(\tilde{X}))} \tilde{\kappa}.$$

Again, by using equivalence classes of jets, we can write Equation (5) as follows

$$W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa} \circ j^1_{X,\Phi(X)} \Phi) = W(j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa}) \quad \forall j^1_{\Phi(X),\kappa(\Phi(X))} \tilde{\kappa}.$$

From (5) (or 6) we deduce that the collection $\bar{G}(X)$ of all material symmetries at $X$ forms a group which is called the isotropy group at $X$. Of course, the collection $G(X)$ of all the induced 1-jets on the base $B$ also forms a group.
3. Uniform continua with microstructure

Consider the family $\tilde{\Omega}(B)$ of all the local uniformities $j^1_{X,\Phi(X)} \tilde{\Phi}$. (Here we use the notations introduced in Appendix A). We have $\tilde{\Omega}(B) \subset \tilde{J}^1(\mathcal{E})$.

**Definition 2.** A continuum with microstructure $B$ is said to be smoothly uniform if $\Omega(B)$ is a Lie subgroupoid which admits a smooth global section. Such a section it is called a global smooth uniformity.

From now on, we suppose that $B$ enjoys global smooth uniformity and $\sigma : B \times B \to \tilde{\Omega}(B)$ is a global uniformity, that is, $\sigma$ is a smooth global section of $(\tilde{\alpha}, \tilde{\beta}) : \tilde{\Omega}(B) \to B \times B$.

Choose a point $X_0 \in B$ and define $S : B \to \tilde{\Omega}(B)$ by $S(X) = \sigma(X_0, X)$. Next, choose a non-holonomic frame $\tilde{Z}_0 = j^1_{e_1, \tilde{\Psi}(e_1)} \tilde{\Psi}$ at $X_0$ (see Appendix C) and put:

$$S(X) = \tilde{S}(X) \circ \tilde{Z}_0, \forall X \in B,$$

where $\tilde{S}(X)$ is the representative in $S(X)$ with source $\tilde{\Psi}(e_1)$. In other words, $S : B \to \tilde{P}\mathcal{E}$ is a non-holonomic parallelism on $B$. Sometimes we shall refer to $S$ as a field of uniformities.

**Definition 1.** A non-holonomic frame $\tilde{Z}_0$ at $X_0$ will be called a reference crystal at that point.

By using a reference crystal $\tilde{Z}_0$ we obtain a Lie subgroup $\tilde{G}$ of $\tilde{G}(n, \mathcal{G})$ as follows:

$$\tilde{G} = \{ \tilde{Z}_0^{-1} \circ \tilde{Z} \circ \tilde{Z}_0 \mid Z \in \tilde{G}(X_0) \},$$

(7)

where $\tilde{Z}$ denotes the representative of the class $Z$ with source $\tilde{\Psi}(e_1)$.

By applying the results of Appendix D, we know that $S$ induces:

- A global section $\mathcal{P}$ of the principal bundle $\pi : \mathcal{E} \to B$ (and hence, a connection $\Lambda$ in $\mathcal{E}$).
- A linear parallelism $\mathcal{Q}$ on $B$ (and hence a linear connection $\Gamma$ on $B$).
- A connection $\tilde{\Lambda}$ in $\mathcal{E}$.

Notice that there is a degree of freedom in the choice of $S$. In fact, if we change the reference crystal $\tilde{Z}_0$ to a new one $\tilde{Z}_0'$, then $\tilde{Z}_0' = \tilde{Z}_0(A, B, C)$, where $(A, B, C) \in \tilde{G}(n, \mathcal{G})$. Thus, $S' = S(A, B, C)$ is a new field of uniformities.

Furthermore, there is another degree of freedom. Suppose that $\tilde{G}$ is a continuous Lie subgroup of $\tilde{G}(n, \mathcal{G})$. Hence we prolongate a non-holonomic
parallelism $\mathcal{S}$ by $\tilde{G}$ and obtain a $\tilde{G}$-reduction of $\tilde{F}\mathcal{E}$. Therefore, all the sections $\mathcal{S}(A(X), B(X), C(X))$, where $(A, B, C) : B \to \tilde{G}$, are admissible non-holonomic parallelisms or, in other words, new fields of uniformities.

Remark 1. There is a more general class of continua with microstructure. Suppose that $\mathcal{B}$ only enjoys local smooth uniformity, that is, $\tilde{\Omega}(\mathcal{B})$ is a Lie subgroupoid which only admits local sections (in other words, local uniformities). As above, we fix a point $X_0$ at $\mathcal{B}$ and a non-holonomic frame at $X_0$. Proceeding in the same way, we obtain local sections of $\tilde{\pi} : \tilde{\mathcal{B}} \to \tilde{F}\mathcal{E}$ and, by prolongation, a $\tilde{G}$-reduction. We call such a reduction a $\tilde{G}$-structure.

4. HOMOGENEOUS CONTINUA WITH MICROSTRUCTURE

Definition 3. We say that $\mathcal{B}$ is homogeneous if there exists a global configuration $\tilde{\kappa}$ such that:

1. $\kappa : \mathcal{B} \to R^3$ is an embedding into $R^n$, i.e. $\kappa(\mathcal{B}) \subset R^n$; and
2. $\mathcal{S} = \tilde{\kappa}^{-1}$ is a uniformity field.

More precisely, for each $X \in \mathcal{B}$, let $\tilde{\mathcal{A}}_X : R^n \times \mathcal{G} \to \mathcal{E}$ be the bundle isomorphism defined by

$$\tilde{\mathcal{A}}_X(r, R) = \tilde{\kappa}^{-1}(r + \kappa(X), R).$$

(8)

Then $\mathcal{B}$ is homogeneous if $\mathcal{S}(X) = \tilde{\mathcal{A}}_{X(X)}^1 \tilde{\mathcal{A}}_X$ is a uniformity field. The continuum $\mathcal{B}$ is said to be locally homogeneous if every point of $\mathcal{B}$ has a neighborhood which is homogeneous.

In that case, there exist local coordinates $(x^i)$ in $R^n$ such that

$$\mathcal{S}(x^i) = \left(x^i, P^\alpha(x), 1, \frac{\partial P^\alpha}{\partial x^j}\right).$$

(9)

That is, $\mathcal{S}$ is an integrable prolongation.

Conversely, let $\mathcal{S}$ be a uniformity field for $\mathcal{B}$. If $\mathcal{S}$ is an integrable prolongation, then $\mathcal{B}$ is locally homogeneous.

Thus, we deduce the following result which characterizes geometrically the homogeneity of a medium with structure.

Theorem 1. A continuum with microstructure $\mathcal{B}$ is locally homogeneous if and only if it admits a field of uniformities which is an integrable prolongation.
In order to decide if a continuum $B$ is locally homogeneous, we proceed as follows. Suppose first that there are no material symmetries except the identity, i.e., $\tilde{G} = (e, 1, 0)$. Take a field of uniformities $S$, with associated connections $\Gamma$, $\Lambda$, and $\tilde{\Lambda}$. Compute the torsion tensor of the linear connection $\Gamma$. If it vanishes, we then check if the global section $\mathcal{P}$ is parallel with respect to $\tilde{\Lambda}$. If it is not, we change to another field of uniformities $S' = S(A, B, C)$ by means of a change of reference crystal and consider the new three connections $\Gamma'$, $\Lambda'$, and $\tilde{\Lambda}'$. Clearly, $\Gamma' = \Gamma$, and $\mathcal{P}A$ is parallel with respect to $\tilde{\Lambda}'$ if and only if $\mathcal{P}$ is so also. If we can choose $(A, B, C) \in \tilde{G}(n, \mathcal{G})$ such that $\mathcal{P}$ is parallel with respect to $\tilde{\Lambda}'$, we have finished, and $B$ is locally homogeneous.

Now, suppose that $\tilde{G}$ is not trivial. In this case, we have many choices for a uniformity field. The geometrical answer for a local homogeneity characterization needs to develop an appropriate study of the integrability problem for $\tilde{G}$-structures.

A. LIE GROUPOID ASSOCIATED WITH A PRINCIPAL BUNDLE

Let $\pi : \mathcal{E} \rightarrow B$ be a principal bundle with structure group $\mathcal{G}$. Denote by $J^1(\mathcal{E})$ the manifold of 1-jets $j^1_{X, \Phi(X)} \Phi$ of local automorphisms $\Phi$ of $\mathcal{E}$ such that $\Phi(\tilde{Y}A) = \tilde{\Phi}(\tilde{Y})A$, $\forall \tilde{Y} \in \mathcal{E}, \forall A \in \mathcal{G}$. Notice that $J^1(\mathcal{E}) \subset \Pi^1(\mathcal{E}, \mathcal{E})$, the Lie groupoid of the invertible 1-jets of the manifold $\mathcal{E}$. We define an equivalence relation on $J^1(\mathcal{E})$ as follows: $j^1_{X, \Phi(X)} \Phi \sim j^1_{Y, \Phi(Y)} \Phi$. The equivalence class of $j^1_{X, \Phi(X)} \Phi$ will be denoted by $j^1_{X, \Phi(X)} \tilde{\Phi}$, where $X = \pi(\tilde{X})$ and $\Phi$ is the induced diffeomorphism between the bases. Denote by $\tilde{J}^1(\mathcal{E})$ the quotient space $J^1(\mathcal{E})/\mathcal{G}$. If we define

$$\tilde{\alpha}([j^1_{X, \Phi(X)} \Phi]) = X, \tilde{\beta}([j^1_{X, \Phi(X)} \Phi]) = \Phi(X),$$

we can easily check that $\tilde{J}^1(\mathcal{E})$ is a Lie groupoid over $B$ with source and target maps $\tilde{\alpha}, \tilde{\beta} : \tilde{J}^1(\mathcal{E}) \rightarrow B$.

Furthermore, the set of induced 1-jets $j^1_{X, \Phi(X)} \Phi$ is just $\Pi^1(B, B)$.

B. BUNDLES ASSOCIATED WITH PRINCIPAL BUNDLES

Let $\pi : \mathcal{E} \rightarrow B$ be a principal bundle with structure group $\mathcal{G}$. Suppose that $\mathcal{G}$ acts on the left on a manifold $\mathcal{F}$, namely $\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}$. We define on
the product manifold $\mathcal{E} \times \mathcal{F}$ the following action of $\mathcal{G}$:

$$
(\mathcal{E} \times \mathcal{F}) \times \mathcal{G} \rightarrow \mathcal{E} \times \mathcal{F},
$$

$$(\tilde{X}, \xi)A \sim (\tilde{X}A, A^{-1}\xi).$$

Denote by $\tilde{\mathcal{E}} = \frac{\mathcal{E} \times \mathcal{F}}{\mathcal{G}}$ the quotient space and by $\tilde{\pi} : \tilde{\mathcal{E}} \rightarrow \mathcal{B}$ the canonical projection. We have that $\tilde{\pi} : \tilde{\mathcal{E}} \rightarrow \mathcal{B}$ is a fibre bundle with standard fibre $\mathcal{F}$ which is called an associated fibre bundle with $\mathcal{E}$.

C. Non-holonomic frames of a principal bundle

Let $\pi : \mathcal{E} \rightarrow \mathcal{B}$ be a principal bundle with projection $\pi$ and structure group $\mathcal{G}$. Consider the trivial principal bundle $R^n \times \mathcal{G} \rightarrow \mathbb{R}^n$, where $\dim \mathcal{B} = n$. Denote by $e_1$ the element $e_1 = (0, e)$, where $e$ is the neutral element of $\mathcal{G}$.

A non-holonomic frame of $\mathcal{E}$ at a point $X \in \mathcal{B}$ is a 1-jet $j^1_{e_1, \Phi(e_1)} \Phi$ of a local principal bundle isomorphism $\Phi : \mathbb{R}^n \times \mathcal{G} \rightarrow \mathcal{E}$, where $\Phi$ induces the identity between the structure groups, and $\pi(\Phi(e_1)) = X$. The collection of all non-holonomic frames at all the points of $\mathcal{B}$ is denoted by $\tilde{\mathcal{E}}$ and we have $\tilde{\mathcal{E}} \subset F(\mathcal{E})$, where $F(\mathcal{E})$ denotes the linear frame bundle of the manifold $\mathcal{E}$.

Take canonical coordinates $(r^i)$, $1 \leq i, j, k, \ldots \leq n$, on $\mathbb{R}^n$ and coordinates $(R^\alpha)$, $1 \leq \alpha, \beta, \gamma, \ldots \leq \dim \mathcal{G}$, on $\mathcal{G}$ (we can choose normal coordinates on $\mathcal{G}$, for instance). On $\mathcal{E}$ we have fibred coordinates $(x^i, X^\alpha)$. We have

$$
\tilde{\Phi}(r, R) = (\Phi(r), \varphi(r)R),
$$

where $\Phi : \mathbb{R}^n \rightarrow \mathcal{B}$ and $\varphi : \mathbb{R}^n \rightarrow \mathcal{G}$. We get

$$
\tilde{\Phi} = \left( \Phi^i(0), \varphi^\alpha(0), \frac{\partial \Phi^i}{\partial r^j}(0), 0, \frac{\partial \varphi^\alpha}{\partial r^j}(0), \varphi^\alpha(0) \right).
$$

We have used the following local coordinates:

$$
\mathcal{B} : (x^i),
$$

$$
\mathcal{E} : (x^i, X^\alpha),
$$

$$
F(\mathcal{E}) : (x^i, X^\alpha; x^i_j, x^i_{\beta}, X^\alpha_j, X^\alpha_{\beta}).
$$

With these notations the coordinates of $j^1_{e_1, \Phi(e_1)} \Phi$ are $(x^i, X^\alpha; x^i_j, X^\alpha_j)$. We deduce that $\tilde{\mathcal{E}}$ is a $(n + \dim \mathcal{G}) (n + 1)$-dimensional submanifold of $F(\mathcal{E})$. 

Furthermore, if we consider the elements \( j^1_{e_1, \Phi(e_1)} \Phi \) from \( \mathbb{R}^n \times \mathcal{G} \) into itself such that \( \Phi(0) = 0 \), we obtain a Lie group denoted by \( \mathcal{G}(n, \mathcal{G}) \) whose elements are of the form \( (A, B, C) \), where \( A \in \mathcal{G}, B \in GL(n, \mathbb{R}) \) and \( C \in \operatorname{Lin} (\mathbb{R}^n, \mathfrak{g}) \), \( \mathfrak{g} \) being the Lie algebra of \( \mathcal{G} \). Thus, \( \mathcal{G}(n, \mathcal{G}) \) may be identified with the product \( \mathcal{G} \times GL(n, \mathbb{R}) \times \operatorname{Lin} (\mathbb{R}^n, \mathfrak{g}) \), the multiplication law given by the following formula obtained by applying the chain rule:

\[
(A_1, B_1, C_1)(A_2, B_2, C_2) = (A = A_1 A_2, B = B_1 B_2, C = A_2 C_1 B_2 + A_1 C_2),
\]

with the following definitions:

- If \( A \in \mathcal{G} \) and \( C \in \operatorname{Lin} (\mathbb{R}^n, \mathfrak{g}) \), then \( AC \) is the composition \( \mathbb{R}^n \xrightarrow{C} \mathfrak{g} \xrightarrow{A} \mathfrak{g} \), the second mapping being the induced one from the right translation by \( A \).
- If \( B \in GL(n, \mathbb{R}) \) and \( C \in \operatorname{Lin} (\mathbb{R}^n, \mathfrak{g}) \), then \( CB \) is the composition \( \mathbb{R}^n \xrightarrow{C} \mathfrak{g} \xrightarrow{B} \mathbb{R}^n \).

A simple computation shows that \( \hat{\pi} : \tilde{F}E \rightarrow B \), where \( \hat{\pi} \) is the canonical projection, is a principal bundle over \( B \) and with structure group \( \mathcal{G}(n, \mathcal{G}) \). \( \tilde{F}E \) will be called the non-holonomic frame bundle of \( E \). We denote by \( \rho : \tilde{F}B \rightarrow FB \) and \( \theta : \tilde{F}B \rightarrow E \) the canonical projections.

D. NON-HOLONOMIC PARALLELISMS

**Definition 4.** A global section \( S : B \rightarrow \tilde{F}E \) is called a non-holonomic parallelism on \( E \).

By using the projections \( \rho \) and \( \theta \), \( S \) determines:

- A global section \( \mathcal{P} : B \rightarrow E \);
- A linear parallelism \( \mathcal{Q} \) on \( B \);
- A connection \( \hat{\Lambda} \) in \( \pi : E \rightarrow B \), by defining the horizontal subspaces as follows. Let \( S(X) = j^1_{e_1, \hat{\Phi}(e_1)} \hat{\Phi} \) be such that \( \hat{\Phi}(r, R) = \varphi(r) R \), where \( \varphi(r) = \hat{\Phi}(r, e) \). We define a horizontal subspace \( H_{\mathcal{P}(X)} = d\varphi(0)(T_0 \mathbb{R}^n) \) and, then we transport \( H_{\mathcal{P}(X)} \) by the action of \( \mathcal{G} \).

**Remark 2.** Roughly speaking, a nonholonomic frame at a point \( X \) is an infinitesimal element of connection, that is, a horizontal subspace over \( X \).

Conversely, let \( \mathcal{P} \) be a global section of \( E \) and \( \mathcal{Q} \) a linear parallelism on \( B \). We obtain a non-holonomic parallelism \( \mathcal{P}^1(\mathcal{Q}) \) on \( E \) by defining \( S(X) \) to be
the “linear connection” at $\mathcal{P}(X)$ given by the horizontal subspace spanned by the tangent vectors \{d\mathcal{P}(X)(Q(X))\}.

**Definition 5.** (1) A non-holonomic parallelism $\mathcal{S}$ is called a prolongation if $\mathcal{S} = \mathcal{P}^1(\mathcal{Q})$. (2) $\mathcal{S}$ is called an integrable prolongation if $\mathcal{S} = \mathcal{P}^1(\mathcal{Q})$ and $\mathcal{Q}$ is integrable.

Suppose that $\mathcal{P}(x^i) = (x^i, \mathcal{P}^\alpha(x))$ and $\mathcal{Q}(x^i) = (x^i, Q^i_j(x))$. Hence

$$\mathcal{P}^1(\mathcal{Q})(x^i) = \left( x^i, \mathcal{P}^\alpha, Q^i_j, \frac{\partial \mathcal{P}^\alpha}{\partial x^k} \right).$$

Therefore $\mathcal{S}(x^i) = (x^i, \mathcal{P}^\alpha, Q^i_j, R^\alpha_j)$ is a prolongation if and only if

$$R^\alpha_j = Q^i_k \frac{\partial \mathcal{P}^\alpha}{\partial x^k}.$$

and, $\mathcal{S}$ is an integrable prolongation if and only if there exist local coordinates $(x^i)$ on $\mathcal{B}$ such that

$$Q^i_j = \delta^i_j,$$

$$R^\alpha_j = \frac{\partial \mathcal{P}^\alpha}{\partial x^j}.\tag{15}$$

$$R^\alpha_j = \frac{\partial \mathcal{P}^\alpha}{\partial x^j}.\tag{16}$$

If $\mathcal{S}$ is a non-holonomic parallelism on $\mathcal{E}$ then it defines three connections:
- A linear connection $\Gamma$ on $\mathcal{B}$ induced by the linear parallelism $\mathcal{Q}$ and with Christoffel components:

$$\Gamma^i_{jk} = -(Q^{-1})^i_k \frac{\partial Q^l_j}{\partial x^l}.\tag{17}$$

- A connection $\Lambda$ in the principal bundle $\pi : \mathcal{E} \to \mathcal{B}$ whose horizontal subspace at $\mathcal{P}(X)$ is obtained by transporting the tangent space $T_X \mathcal{B}$. Then we transport it by the action of the Lie group $\mathcal{G}$. The horizontal subspaces along $\mathcal{P}$ are locally spanned by

$$\left\{ \frac{\partial}{\partial x^i} + \frac{\partial \mathcal{P}^\alpha}{\partial x^i} \frac{\partial}{\partial X^\alpha} \right\}.\tag{18}$$

- A connection $\hat{\Lambda}$ in the principal bundle $\pi : \mathcal{E} \to \mathcal{B}$ whose horizontal subspaces along $\mathcal{P}$ are locally spanned by

$$\left\{ Q^i_j \frac{\partial}{\partial x^j} + R^\alpha_i \frac{\partial}{\partial X^\alpha} \right\}.\tag{19}$$

From (17), (18) and (19) we deduce the following.
THEOREM 2. A non-holonomic parallelism $\mathcal{S}$ is an integrable prolongation if and only if $\Gamma$ is symmetric and $\Lambda = \tilde{\Lambda}$.

Theorem 2 may be rephrased as follows. Denote by $T$ the torsion tensor of $\Gamma$. Hence we have.

THEOREM 3. A non-holonomic parallelism $\mathcal{S}$ is an integrable prolongation if and only if $T$ identically vanishes and $\mathcal{P}$ is parallel with respect to the connection $\tilde{\Lambda}$.

The result follows taking into account that $\Lambda$ and $\tilde{\Lambda}$ coincide if and only if

$$d\mathcal{P}(X)(Q_i) = (Q_i(X))^H, \quad \forall X \in \mathcal{B}, \quad 1 \leq i \leq n,$$

where $\{Q_1, \ldots, Q_n\}$ is the linear parallelism defined by $\mathcal{Q}$ and $U^H$ denotes the horizontal lift of a tangent vector $U \in T_X \mathcal{B}$ to $\mathcal{E}$.

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