INVESTIGACIONES ECONÓMICAS. vol. XXIV (1). 2000, 3-50

AUCTION THEORY; A GUIDED TOUR

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I present an overview of what has been the theoretical analysis of auctions in the last two decades. The goal is to offer a systematic exposition of the main issues addressed by this literature, with more emphasis on why than on what. I present a unified framework that takes the reader through the analysis of both design and positive questions dealing with the workings of several standard institutions. At the end of the tour, the reader will find him or herself inside the realm of market design and the theory of price formation.

Palabras clave: Bidding models, mechanisms, market design

(*JEL* D44)

1. Introduction

Since Vickrey's pioneer work in the early sixties, the economic analysis of auctions has known two waves of high activity. The first one, starting in the early eighties, had its roots in the invasion of economics by game theory. Soon, this old institution was the aim of game theoretic analysis. Not only did game theory make it possible to undertake a systematic analysis of the strategic interaction in auctions, but an auction also constituted a neatly defined set of rules and outcomes, one that immediately rendered itself as a game. Not surprisingly, the first years of that decade witnessed the laying down of what is still the basis of auction theory.[1]

József Sákovics is partially responsible for this work, since for the most part of my personal tour into auction theory I have had him as a companion I also acknowledge partial financial support from the Ministerio de Educación through CICYT Grant PB96-0897 and from the EC through HCM Network contract CHRX-CT94-0489.

[1] Surveys of this literature with a different emphasis from the one here can be found in McAfee and McMillan (1987b), Milgrom (1985), Milgrom (1989), or Wilson (1992), and a more recent one in Wolfstetter (1996).

The second wave is one that is perhaps still in its ascending phase and was caused by what has been referred to as the success story in game theory: the US Federal Communications Commission auctions of spectrum rights. These auctions, which started in 1994, put rights on radio spectrum that would become one of the supports of the new personal communications systems into the hands of private firms. Major auction theorists were in charge of designing these complex auctions as a substitute for administrative processes which were considered slow, allocationally inefficient, and weak at raising revenue. The first evaluations of the performance of these auctions were very positive and portrayed auction theory as a useful tool for designing economic reality. These auctions, however, have shown that many important issues are yet unexplored in auction theory.

But one could ask what is the real importance of the economic analysis of auctions in the wider territory of economic theory. A common answer to that question refers to the widespread use of one form or other of auctions for the trade of a good deal of commodities. Indeed, auctions are not only used for selling art or wine. Mineral rights and timber are usually allocated in this fashion, and public works contracts are typically awarded through competitive bidding as well. But so are treasury bills and other monetary instruments. Organized markets like stock exchanges can also be characterized as some type of auction, while recent reform in several electric generation markets, like that undertaken in the UK, Spain, or California, is heading in the direction of introducing competitive bidding as a way of assigning generation to plants.

I would like to emphasize yet another reason for studying auction theory: it offers appropriate grounds on which to build at least one theory of both market design and price formation. This is undoubtedly central to economic theory, and is for the most part still missing.[2] Indeed, the paradigmatic model of a market, the Walrasian model, postulates a large number of sellers and buyers, all of whom take prices as given, and then shows under what circumstances there is a price (vector) that clears the market. But there is no compelling story in this model as to who sets that price or how it becomes the market price. Auction theory takes, in a sense, the opposite approach. From

[2] Of course, auction theory is not the only way to do this. Search and bargaining theory are alternatives, if not rivals, to auction theory. See Osborne and Rubinstein (1990) for a good survey of these literatures.

the seminal work by Vickrey (1961), auction theory starts by defining the rules of trade, and then analyzes the behavior of agents under these rules. There is no reference to what would be an "equilibrium price", but rather the terms of trade and the trade itself are determined by the explicit interaction of agents acting under the specified set of rules.

An auction, it may be argued, is not a market in several respects. First, in a market there is the potential for trading a large number of homogeneous units of a good. Second, in a market there is no parallel to the "seller" or auctioneer, a peculiar side of a market. However, both these objections have been overruled by the development of auction theory. Indeed, if the early works in the area analyzed the auction of a single, indivisible good, it soon became clear that the same approach could be used to analyze the sale of multiple units of the good. Also, there was no conceptual need for the strong asymmetry between one side of the "market" and the other. The analysis of two sided bidding, where buyers and sellers both bid their maximum and minimum acceptable price, respectively, is feasible, if more difficult. Moreover, even decentralized competition, one in which sellers and buyers do not use the services of a "clearing house" or market maker, is suitable for analysis using the tools and approach of auction theory.

Auction theory has advanced not only in the positive analysis of trading institutions; in fact, most of the work in the area, at least in the eighties, has been of the normative type. Here too, Vickrey made the first contribution by defining what became known as the Vickrey or second price auction. The goal was to compare trading mechanisms or to look for what would be an appropriate one, where appropriate could mean optimal from the seller's point of view, or efficient from the social point of view. Thus, and with regard to the above paragraph, auction theory is becoming an important tool for the design of markets or trading institutions, as the already-mentioned FCC auctions or the new spot markets for electric generation have shown.

This exciting trajectory is what the tour presented here tries to illustrate. Thus, we start by analyzing the simplest case in which one unit of an indivisible good is to be sold to one of a set of potential buyers (Section 2). We then proceed to relax some of the assumptions made in this toy model, while still talking about a single good (Section 3). The next step is to discuss the sale of multiple units of a homogeneous good (Section 4). Finally, we bring competition to both sides of the trade, by considering multiple sellers and buyers (Section 5).

As one can infer from the early paragraphs of this introduction, game theory is the basic analytical tool in auction theory. The goal of the present tour is not to enumerate a list of results with a more or less compelling intuition attached, but rather to offer the blueprint of the machinery used to obtained these results. This intuition could then be better linked to assumptions and axioms of the models. Thus, while this tour is necessarily a little technical, a basic knowledge of game theory and basic differential calculus are perhaps all the necessary baggage for the tour. In exchange, this baggage should allow a researcher who is new to the field to understand the fundamental workings of auction theory. This is the aim of the tour that starts here.

2. The basic model: auctioning a single object

We begin the analysis by considering the simplest model: the auctioning of an indivisible, single object, like a Picasso painting or a unique bottle of wine. N potential buyers, each with their own willingness to pay for the piece, are seated in the room. Let us denote by v_i buyer i's willingness to pay (her valuation for the item), where $i = 1, 2, ..., N$. One of the important components of the problems we will be addressing is the information and beliefs that agents have with respect to these valuations. We will model this information and these beliefs in the standard game theoretical way. Thus, we will assume that there is a random process determining the vector $v = (v_1, v_2, ..., v_N)$, and that this random process is common knowledge. Alternatively (see Section 3.4), there may be some other information correlated with v that agents observe. We will then assume again that the stochastic process that determines this information is common knowledge. Beliefs will be formed in a Bayesian way, by using this common knowledge and any information the agent obtains. The nature of these random processes will be important for some of the results that we will be obtaining. We begin by looking at the simplest and most studied case, that in which all v_i's are independently and identically distributed, and in which buyer i and only buyer i observes the realization of v_i. This is the independent private values model. In addition, we assume that all agents are risk neutral. In subsequent subsections we will relax each of these assumptions.

Thus, assume that each v_i, which is private information to buyer i, is an independent realization of a continuous random variable with c.d.f. F and density f, which for simplicity of notation we assume positive in the interval (0,1). On the other hand, we assume that the seller's valuation is 0. The normalization is without loss of generality, but it is important to keep in mind that we assume that 0 is also the infimum in the distribution of any buyer's valuation. If this was not the case, some results would change in quite a straightforward way.

Given this setting, the seller's first problem is to decide which of the many available methods to use when auctioning her object. Among the most popular of these methods, we can count the English or oral ascending auction (the seller names prices in an increasing fashion until only one buyer remains who accepts that price); the Dutch or oral descending auction (the seller names prices in a descending fashion until one buyer accepts); the first price, sealed bid auction (buyers submit offers in a sealed envelope and the highest bid wins the object at a price equal to that bid); and the theoretical second price, sealed bid auction (equal to the first price auction, except that the winner pays the second highest bid, instead of her own). Another exotic auction would be the all-pay auction, similar to the sealed bid auctions, except that any buyer who submits a bid pays her bid, whether she wins the good or not. In order to choose the method, the seller should anticipate what the behavior of the buyers will be, and therefore what revenue to expect. Thus, let us look at the behavior of buyers in each of the auctions mentioned above.

Consider the second price, sealed bid auction. It is very easy to see that the best bid for buyer i is exactly her valuation v_i. Indeed, for buyer i and for any set of bids submitted by the rivals, winning the auction, i.e. bidding above the highest rival bid, means paying that highest rival bid. And buyer i is interested in winning under those conditions if and only if the highest rival bid is below her own valuation, which is what she guarantees by bidding $B(v_i) = v_i$.

Now, assume that the seller selects the first price method. Then bidding her own valuation is not a very clever strategy for buyer i. Indeed, by doing so she gets zero surplus (rents, from now on) both when winning and when loosing the auction, whereas by bidding less than her valuation she may still win the auction and secure some positive rents if this is the case. That is, a change in the trading rules induces a change in the traders' behavior. But, how much should buyer i reduce her bid? The answer to this question is going to be a little bit more involved. The reason is that, contrary to the second price case, now the best bid for buyer i is going to depend on what other buyers bid. That is, there is no dominant strategy in this game!

Thus, we should now look for a (Bayesian) equilibrium of the game, that is, a vector of bidding functions with the property that any buyer i with any valuation will find it optimal to bid according to her respective bidding function if she expects everybody else to do likewise. It is natural to expect that all buyers will use the same bidding function (not the same bid) and that this be strictly monotone. Indeed, the first price auction game treats buyers symmetrically, and buyers with higher valuation should be willing to pay more to increase the probability of winning the object. Thus, let us postulate a (common) monotone equilibrium bidding function, $b(v_i)$. From monotonicity, the winner will be the one with highest valuation and then buyer i's expected rents will be

$$[v_i - b(v_i)]F(v_i)^{N-1} \qquad [1]$$

But for $b(v_i)$ to actually be the best bid for buyer i, these rents should be higher than the rents she expects if bidding any other amount, for instance $b(z)$, for any arbitrary z; that is, by "pretending" to have any other valuation. Therefore, a necessary condition for equilibrium is

$$v_i = \arg\max_z\{[v_i - b(z)]F(z)^{N-1}\} \qquad [2]$$

for all v_i. The first order condition for this problem[3] defines a differential equation in v_i:

$$b'(v_i) = \frac{[v_i - b(v_i)]\,(N-1)f(v_i)}{F(v_i)} \qquad [3]$$

whose solution[4] is

$$b(v_i) = \int_0^{v_i} \frac{x(N-1)f(x)F(x)^{N-2}dx}{F(v_i)^{N-1}} \qquad [4]$$

Notice that this is the expected value of the first order statistic[5] of $N-1$ draws of the random variable with c.d.f. F conditional on all of them being below v_i.

[3]We are assuming differentiability of $b(v_i)$, but note that we were already assuming monotonicity and therefore, since $b(v_i)$ is obviously bounded, we were assuming differentiability almost everywhere.

[4]We are using the initial condition that the bid of a buyer with valuation 0 is 0. Indeed, it makes no sense for them to bid more, and bidding less would not form part of an equilibrium: note that in any monotone equilibrium a buyer with valuation 0 never wins the auction and therefore makes zero rents Now, if $b(0) < 0$ then the buyer could benefit from increasing her bid to, say $b(0)/2$, thus obtaining the object with some probability at a lower price than her valutuation; that is, obtaining some positive rents.

[5]The n^{th} order statistic is simply the n^{th} highest realization.

Obtaining the equilibrium $\beta(v_i)$ for an all-pay auction is a very similar exercise. Indeed, we only need to substitute

$$v_i = \arg\max_z \{v_i F(z)^{N-1} - \beta(z)\} \qquad [5]$$

for [2] above to obtain as a solution to the first order conditions of this problem

$$\beta(v_i) = \int_0^{v_i} x(N-1)f(x)F(x)^{N-2}dx. \qquad [6]$$

With respect to the oral auctions, we do not need much to obtain equilibrium strategies. Indeed, the Dutch auction is strategically equivalent to the first price auction: one can think of a strategy there as a stopping price, the price at which the buyer decides to jump in and accept the deal. The pay-off is given exactly by [1] above for a stopping price $b(v_i)$. On the other hand, the English auction has a dominant strategy in this setting: accepting any price the seller names as long as it is below the buyer's own valuation. Then, the buyer with the highest valuation wins and pays an amount equal to the second highest: the same result as in the second price auction.

Now that we have a prediction of how buyers will behave in the different auctions, we can analyze the expected outcomes of these auctions. That is, we can answer questions such as what auction gives higher expected revenues to the seller, or higher rents to the buyers, and which is more efficient. For instance, the answer to this last question is that all of them are equally efficient; in fact, fully efficient. Indeed, since the bidding functions in all of them are strictly monotone, the buyer with the highest valuation obtains the good in all the auctions analyzed. But this is the only condition for efficiency; all payments are transfers from one agent to another, and therefore the total surplus generated by the trade is simply the difference between the winning buyer's valuation and the seller's.

Turning to the seller's revenue, in a second price auction the winner pays the second highest bid, and then the seller's revenue is simply the second highest valuation, that is, the second order statistic of N draws from the random variable with c.d.f. F. In a first price auction, the revenue is the bid of the highest valuation bidder. Remember that this bid is the expected value of the highest of $N-1$ draws below her valuation. That is, in expected terms, we are talking about the expected value of the second order statistic of N draws again! Indeed, the expected value of the bid tendered by the buyer with the highest valuation is

$$\begin{aligned}
&\int_0^1 b(v)Nf(v)F(v)^{N-1}dv \\
= &\int_0^1\int_0^v x(N-1)f(x)F(x)^{N-2}dxNf(v)d(v) \qquad [7]\\
= &\int_0^1\int_x^1 Nf(v)dvx(N-1)f(x)F(x)^{N-2}dx \\
= &\int_0^1 xN(N-1)f(x)[1-F(x)]F(x)^{N-2}dx,
\end{aligned}$$

which is nothing[6] but the expected value of the second order statistic of N realizations of F.

Similarly, we can check that the all-pay auction gives the same revenues to the seller, in expected terms. For the sake of illustration, we can use an alternative approach this time. By definition, the seller's revenues are simply the difference between the surplus generated by the trade and the buyers' rents. Now, integrating by parts in [6] and substituting the new expression for $\beta(v)$ in [5], one obtains the rents a buyer with valuation v expects in an all-pay auction:

$$\int_0^v F(x)^{N-1}dv, \qquad [8]$$

which is exactly what one obtains by integrating by parts in [4] and substituting in [1]. Thus, each buyer expects the same rents for each possible valuation in both first price and all-pay auctions. Therefore, both the buyer's rents and the seller's revenues are the same under both trading mechanisms, even when conditioning on the valuation of the winning bid. And these identities can be extended as easily to the second price auction. The three of them are, in this sense, equivalent[7].

This result may seem surprising. On the face of it, we should wonder what is there to explain the coincidence. Also, one can be a little more

[6]We have used a little trick which is quite common in auction theory to obtain the second equality above, which is simply to change the order of integration Indeed, the right hand side of the first equality is simply the integral of a function in v and x in the triangle obtained by letting v range from 0 to 1 and then x range from 0 to v. This is the same triangle that one obtains by letting x range from 0 to 1 and then v range from x to 1.

[7]The realized price, however, is different. Given the buyers' valuation, how much they pay is deterministic in the all-pay auction and random in the other two, a randomness which, once one conditions on winning the auction, disappears in the first price auction and persists in the second price auction.

ambitious and try to find better ways (from the seller's point of view) of selling the good, perhaps even the best way (Myerson (1981), Riley and Samuelson (1981)). This last question seems out of reach, given that the set of possible trading mechanisms (think of any exotic set of rules that the seller can use to carry out the transaction) is infinite and, worse still, has no structure at all. But there is a way to tackle this question: invert the problem. Instead of considering the possible mechanisms one by one, then analyze the behavior, and finally obtain the results, as we were doing until now, one can look at the set of conceivable results first. Then one can ask which of these results can be obtained. There is hope in this, since the set of conceivable results does have some structure. Indeed, any result of a trading mechanism is a rule that assigns the object to a buyer and assigns payments to each party, all as a function of their valuations. Call these rules allocation rules. Now, one can hope that the set of all the feasible allocation rules, among those conceivable, still preserves a simple structure[8].

Thus, an allocation rule is a set of functions $\{\rho_i(v), \chi_i(v)\}_{i=1,2,\ldots,N}$, where the first is the payment i will make to the seller if v is the vector of valuations, and the second is the probability that i obtains the good in this case. Of course, for any v, $\Sigma_i\ \chi_i(v) \leq 1$, where the possibility of inequality arises from the fact that the seller can leave the object unsold. For most of this section, we will only need a summary of these functions. So, let $p_i(v_i) = E_{v_{-i}}[\rho_i(v)]$ and $x_i(v_i) = E_{v_{-i}}[\chi_i(v)]$, that is, respectively, how much buyer i expects to pay and the probability that she obtains the object given what she knows when entering the deal: her valuation.

If the allocation rule is implemented, a buyer i with valuation v_i expects rents $x_i(v_i)v_i - p_i(v_i)$. The goal for the seller is to maximize the expected revenue, that is,

$$\sum_i E_{v_i}[p_i(v_i)]. \qquad [9]$$

Now, there are (at least) two constraints that an allocation rule has to satisfy in order to be feasible (implementable). First, no buyer should be expected to enter into any trading mechanism if she expects to loose by doing so, and therefore (assuming the opportunity cost for the buyer is zero), we will not have an implementable allocation

[8]For the reader familiar with mechanism design, we will of course be using the logic behind the revelation principle (Myerson (1979) is the classic reference).

rule unless $p_i(v_i) \leq x_i(v_i)v_i$ for all. This is known as the individual rationality constraint. Second, in any trading mechanism there is no way to impose the choice of actions to a player. In particular, a buyer with certain valuation v_i can always choose to act as the allocation rule expects from a buyer with some other valuation, say z. Therefore, one can hope to implement an allocation rule (through a mechanism) only if one is certain that each buyer has the incentive to behave as predicted for her valuation. That is, if for all v_i

$$v_i = \arg\max_z \{x_i(z)v_i - p_i(z)\}, \qquad [10]$$

whose first order condition is the differential equation

$$x_i'(v_i)v_i = p_i'(v_i). \qquad [11]$$

By integrating both sides, we obtain the solution

$$p_i(v_i) = \int_0^{v_i} z x_i'(z)dz + p_i(0). \qquad [12]$$

Equation [12] already implies the equivalence of all five types of auctions analyzed above. It also constitutes the essence of what is known as the "revenue equivalence theorem" (Myerson (1981)). This theorem says that any two auctions that assign the object in the same way (that is with common functions $x_i(.)$), should have the same payment functions $p_i(.)$ too, provided these last functions coincide at at least one point (at $p_i(0)$, for instance). In all the auctions analyzed above, the object was assigned to the buyer with the highest valuation because all had symmetric, monotone equilibria. Also, a buyer with valuation 0 could be sure to pay 0. Thus, the expected payments for any buyer with any given valuation were the same in all five auctions, and the seller expected the same revenues.

It is useful to spend a little more time analyzing this result. Integrating by parts in [12], we get

$$x_i(v_i)v_i - p_i(v_i) = [x_i(0)0 - p_i(0)] + \int_0^{v_i} x_i(z)dz. \qquad [13]$$

The left hand side in [13] represents the rents for buyer i with valuation v_i. These (informational) rents are monotone in the valuation in a way which is standard in the literature on adverse selection: A buyer with valuation $v_i + \Delta$ can always pretend to be of valuation v_i. This would mean the same payment $p(v_i)$ but a value for the buyer equal to $(v_i + \Delta)x_i(v_i)$, instead of $v_i x_i(v_i)$. Therefore, the rents for the buyer with valuation $(v_i + \Delta)$ should at least be $\Delta x_i(v_i)$ higher than those for a buyer with valuation v_i. But a buyer with valuation v_i can also imitate a buyer with valuation $v_i + \Delta$ and therefore, by a similar argument, the difference in rents cannot exceed $\Delta x_i(v_i)$, for infinitesimal[9] Δ. This is what equation [13] states.

As we will see in later sections, this direct relationship of rents and allocation of the object is quite a powerful tool when comparing different auctions or searching for optimal ones. For the case we are analyzing, it allows us to find what the best trading mechanism will be. The crucial part is that we can basically disregard the functions $p_i(v_i)$ to concentrate on the way the mechanism allocates the object to potential buyers: given a vector of functions[10] $x_i(v_i)$ (and a set of initial conditions $p_i(0)$), there are payment functions $p_i(v_i)$ (virtually unique) which are compatible with that vector. This is all we need now. Indeed, by substituting [13] in [9], the seller's problem can be rewritten as[11]

$$Max \sum_i \left\{ \int_0^1 \left(x_i(v_i)v_i - \int_0^{v_i} x_i(z)dz \right) f(v_i)dv_i + p_i(0) \right\} =$$

$$Max \sum_i \left\{ \int_0^1 \left(x_i(v_i)v_i f(v_i) - \int_{v_i}^1 x_i(v_i) f(z)dz \right) dv_i + p_i(0) \right\} =$$

$$Max \sum_i \left\{ \int_0^1 \left[v_i - \frac{1 - F(v_i)}{f(v_i)} \right] x_i(v_i) f(v_i)dv_i + p_i(0) \right\} =$$

$$Max \left\{ \int_v \sum_i \left[v_i - \frac{1 - F(v_i)}{f(v_i)} \right] \chi_i(v) \Pi_{i=1}^N f(v_i)dv + p_i(0) \right\}.$$

Now, notice that an auction that solves this problem necessarily leaves zero rents to a buyer with valuation 0: in fact, since we have seen that rents need to be monotone, this is the only point at which the participation (or individual rationality) constraint need be considered. Finally, under the assumption that the "virtual valuation", defined as

$$J(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)} \qquad [14]$$

[9] We need Δ to be infinitesimal so as to be able to disregard the differences in probability of obtaining the object for the two types considered.

[10] This should be read as a necessary condition for the trading mechanism. If we add that $x_i(v_i)$ are monotone, then we have necessary and sufficient conditions for implementation, in fact

[11] The first equality is obtained by again using the little trick mentioned in footnote 6· by changing the order of variables in integrating over a triangle.

is monotone in v_i, which is known as the "regularity assumption", the solution to [14] is quite straightforward: for each value of the valuation vector v, the summation inside the integral is maximized by making $\chi_i(v) = 1$ for buyer i with the highest value of $J(.)$, which in the regular case means assigning the object to the buyer with highest valuation v_i, provided it is such that $J(v_i) \geq 0$. That is, whenever the highest valuation is above the value v^* defined by

$$J(v^*) = 0, \qquad [15]$$

and leaving the object unsold otherwise. An English or Dutch auction, a first or second price auction, each with a reserve price (minimum acceptable bid) of v^* is then an optimal auction in the regular case[12]. Notice that the reserve price that defines the optimal auction does not depend on N. It coincides with the best "take it or leave it" offer for a seller that faces a unique potential buyer.

This last result is also familiar in the literature on incentives. From our discussion on informational rents above, it should be clear that one way to reduce these rents is to manipulate the allocation (probabilities) of the object. This has a drawback, since it also reduces the surplus from the trade. The most successful way of manipulating the allocation is to reduce $x_i(v_i)$ for low values of v_i. Indeed, this has the largest impact on revenues (reducing $x_i(v_i)$ affects the rents of all buyers with valuations above v_i) and the lowest on surplus (the gain from trade for low values of v_i is low too). The value v^* can be defined as the point at which these two effects exactly balance.

3. Extensions

In this section, we study how the conclusions we have reached above change when we modify some of the assumptions of our basic model. In particular, we study the effects of endogenous entry, risk aversion, asymmetric buyers, and correlated values.

3.1 Entry

Up to now, we have assumed that the number of buyers was exogenously determined. In many cases, however, the number of buyers

[12] In the non-regular case, assigning the good to the buyer with the highest value of the function $J(.)$ is different from assigning it to the buyer with the highest value, which then goes against the monotonicity of the functions $x(.)$. The optimal auction is then a more complex one.

that a seller can attract depends on the trading mechanism she announces. Indeed, if there is any cost of participating in the auction, a potential buyer has to weigh these costs against the rents she can expect from taking part in the auction. As we have seen in the previous section, these rents, and therefore the entry decision, will depend on the auction rules.

But what are these entry costs? Taking part in an auction may imply costs related to actually "playing the game" (transportation, bid preparation), or costs related to even learning about the object (inspection). That is, one could distinguish between costs the buyer has to incur before even knowing her valuation and costs incurred after learning the valuation but before bidding. As we will see, some of the conclusions in the previous section will change when we consider any of these types of entry costs.

Thus, assume that, before learning her valuation, a bidder has to decide whether to incur a cost c that buys this information and any other resources needed to participate in the auction. We ask the question as to what would then be the optimal reserve price r in a first price auction.[13] One thing to notice is that, after all the buyers have taken their decisions and the bidding process starts (and assuming for convenience that everybody observes how many buyers have incurred the cost c and then are able to bid), the analysis in the previous section still applies: in particular, [4] still defines the equilibrium, symmetric, monotone bidding function (except that this time the boundary condition is $b(r) = r$). Then [13] defines the gross (that is, before deducting the entry cost c) buyers' rents, with

$$x_i(v_i) = \begin{cases} 0 & \text{if } v_i < r \\ F(v_i)^{n-1} & \text{if } v_i \geq r \end{cases} \qquad [16]$$

where n is the number of buyers who have entered (paid the cost c). And therefore, the gross rents that a buyer expects from entering before knowing her valuation but knowing that some other $n-1$ buyers will enter too, are

$$\begin{aligned} E\Pi(r;n) &= \int_r^1 \int_r^v F(z)^{n-1} dz f(v) dv = \\ \int_r^1 \int_z^1 F(z)^{n-1} f(v) dv dz &= \int_r^1 [1 - F(z)] F(z)^{n-1} dz. \end{aligned} \qquad [17]$$

[13] Everything would come out the same in English, Dutch, or second price auctions.

Notice that these expected rents are decreasing in n. In particular, for a large enough N, if every potential buyer enters the auction ($n = N$), the gross expected rents will be lower than the entry cost c. In this case, in equilibrium not all buyers will enter. Thus, any pure strategy equilibrium (in entry decisions) will have to be asymmetric in that some n buyers will enter and some $N - n$ will not, even though ex-ante all of them are symmetric. And any symmetric equilibrium will have to be in mixed strategies in that each buyer will enter with some probability $q < 1$. Both types of equilibria have been studied in the literature[14]. Consider the first. Disregarding the integer problem[15], equilibrium entry n will be defined by $E\Pi(r; n) = c$. That is, entry will occur until the net rents of entrants are driven down to zero. But then the seller's revenues are simply the expected surplus from trade minus entry costs, nc.

This immediately shows that the optimal reserve price is zero. Indeed, take any reserve price r and the induced entry level n. Now, consider as an alternative auctioning the object without a reserve price but charging an entry fee (a price a buyer has to pay the seller before submitting the bid) $e = E\Pi(0; n) - c$. It is clear that n is still the equilibrium entry level. Then the revenue for the seller is higher since the surplus is higher (the object is always sold), and the entry costs are the same as before.

Moreover, this also shows that the optimal entry level coincides with the efficient one. Indeed, since entry drives buyers' rents to zero, the seller's revenues coincide with the net surplus: surplus from trade minus entry costs. What is this efficient entry level? The entry of a new buyer contributes to the net surplus in that it represents an additional opportunity for high valuations, but it also detracts from this surplus in that it represents an additional entry cost of c. Thus, entry should continue until the two effects cancel, and not after that. So, once again ignoring integer problems, n^* is such that the last entry increases surplus from trade by an amount equal to c. That is, n^* is such that the difference in surplus when entry is n^* and when entry is

[14] See McAfee and McMillan (1987) for the first type and Samuelson (1985) for the second

[15] In general, this integer problem will matter, but only to the point that some marginal entry fees will be needed in order to extract marginal rents: with a zero reserve price and given that entry must be in integer numbers, buyers will expect some positive rents, which can be extracted by an entry fee equal to those rents

$n^* - 1$ equals c. That is,

$$c = \int_0^1 n^* x f(x) F(x)^{n^*-1} dx - \int_0^1 (n^* - 1) x f(x) F(x)^{n^*-2} dx = \int_0^1 [1 - F(x)] F(x)^{n^*-1} dx \quad [18]$$

where the last equality is obtained by integrating by parts. That is, according to [18], n^* is the same value that solves $EP(0; n^*) = c$, i.e., the level of entry that is obtained with a first price auction with no entry fee and no reserve price!

This result is in sharp contrast with the one obtained in Section 2. Indeed, here rent extraction is not an issue, since endogenous entry pushes these rents to zero. The interesting point is that buyers' incentives to enter are the correct ones from the social as well as from the seller's point of view. Put another way, when a seller considers whether to have higher competition that dissipates buyers' rents or to award to some buyers some "monopoly rents" that the seller then extracts, say, by using entry fees, the decision should always favor the first alternative[16,17].

If we consider the symmetric, mixed strategy equilibrium instead, the conclusions are the same: reserve prices are a dominated tool, efficient and optimal entry coincide, and this entry is the one obtained with no entry fee, that is, by using a standard first price auction.

Things are slightly different if buyers know their valuation before they have to decide whether to enter or not (see Levin y Smith (1994)). In this case, buyers can condition their entry decisions on their valuation. Then, rents cannot be completely extracted since buyers are better informed than the seller *before* joining the game. Yet some sort of zero rent condition will determine entry here too. Indeed, one can expect buyers to enter if their valuations are high enough (expecting high enough rents upon entry) to recover the entry cost, and therefore one

[16] Bulow and Klemperer (1996), in a closely related result, show that having one more buyer in one auction is better than not allowing this extra buyer to take part in the auction and bargain with him with a threat point equal to the hammered down price (selling to the winner of the auction is just an outside option in this case).

[17] This conclusion hinges on the private values assumption Indeed, if we assumed common values, as we do later in this same section, the efficient and optimal policies could be such as to curtail entry. Indeed, in this case the surplus from trade does not depend on N, but the entry costs still do (see French and McCormick (1981))

can expect that equilibrium entry will be characterized by some cut-off level v^*, the valuation of a buyer who expects to exactly break even, that is, expects gross rents exactly equal to c. Then buyers will enter if and only if their valuations are above v^*. If bidding is monotone after entry, a buyer with valuation v^* will not in equilibrium bid more that the reserve price r, since she only wins the auction if nobody else enters. Then

$$(v^* - r)F(v^*)^{N-1} = c \qquad [19]$$

defines this cut-off point v^* and entry (now understood as minimum valuation at which a buyer decides to participate) as a function of r. For buyers with valuation above v^*, [13] still defines the incentive constraint and the (net) rents are given by [17] with only v^* substituting for r and N for n.

Notice that v^* is monotone in r. Thus, choosing the optimal r is equivalent to choosing the entry level v^*. Then the optimal auction will maximize the difference between net surplus and rents, that is

$$\int_{v^*}^{1} Nzf(z)F(z)^{N-1}dz - N\int_{v^*}^{1}[1-F(z)]F(z)^{N-1}dz - N[1-F(v)]c, \qquad [20]$$

where the first term represents gross surplus from trade, the second is the buyers' rents and the third the entry costs. The first order conditions for maximization of [21] with respect to v^* is the following first order condition:

$$N[v^*f(v^*)F(v^*)^{N-1} - [1-F(v^*)]F(v^*)^{N-1} - cf(v^*)] = 0, \qquad [21]$$

which simplifies to:

$$v^* - c\frac{1-F(v^*)}{f(v^*)}. \qquad [22]$$

Notice that if $c = 0$ we are back at [16]. As c increases so does v^*. However, from [20] and for given v^*, r is decreasing in c. Then, the effect of entry costs on optimal r is ambiguous in general. However, by analyzing [22] we can guarantee that $v^* \to 1$ as $N \to \infty$ (with very high competition, only very high valuation bidders have any chance of winning the object and then have some positive rent, even without any reserve price), but $F(v^*)^{N-1} \to c$, which according to [20] implies that $r \to 0$! That is, for a large enough N, we conclude again that setting a positive reserve price is not in the seller's interest.

Also, as in the former case, one can easily check that efficient entry occurs when r is set at zero. That is, buyers once again have the right incentives to enter.

There is one more difference that endogenous entry introduces in auction design. Indeed, in this section we have been talking about optimal *first price* auctions instead of simply optimal auctions. The reason is that first price auctions are not optimal auctions in general in the presence of entry costs; the indirect approach taken in the previous section does not now guarantee that we are considering *all* possible allocation rules. It does not consider, for instance, auctions in which whether some buyers buy information or not depends on the information obtained by other buyers who did buy theirs[18]. How could the seller arrange things so that this type of contingent entry takes place? The answer takes us back to the use of reserve prices! Indeed, we have seen that a reserve price is not a very adequate instrument (and is never better than entry fees) for rent extraction when entry is endogenous. However, precisely in this case, reserve prices are a very convenient way to manage entry. Thus, in a recent paper (Burguet and Sákovics (1996)) we have shown that the seller can improve upon the static, optimal first price auction by announcing a (first price) auction with a reserve price that would be followed by an auction with żero reserve price if no buyer bids. In such a two stage auction, fewer buyers will enter at the first stage. However, if their valuation is low these buyers will not bid at that stage. Then buyers that did not find it profitable to buy information at the first stage can now infer that the buyers that did enter have low valuation, which makes entry more attractive at the second stage. Ex post, what the seller has implemented is restricted entry (and entry costs) if some buyers have high valuation and higher entry where they don't. The reserve price does not necessarily mean inefficiency (the object is always sold), but an instrument to balance entry costs with the search for higher valuations.

3.2 Asymmetry

We can now return to the basic model and this time drop the symmetry assumption. That is, we can assume that buyers are asymmetric even from an ex-ante view point. That simply means that each v_i is the realization of a random variable with distinct distribution F_i and

[18] For the reader familiar with implementation theory, the simplest version of the revelation principle does not apply here.

density f_i. This change in our assumptions has few conceptual implications. In particular, we could take the indirect approach used in the previous section with no changes; in fact we did not really use the symmetry assumption from equation [9] to equation [14]. The heart of that indirect approach, the incentive compatibility constraint on trading mechanisms according to which rents were virtually determined by the fact that a buyer can always pretend to have some other valuation, did consider each buyer in isolation. Then, the revenue equivalence theorem still holds in the sense that any two mechanisms that have common functions $x_i(.)$ (and give the same rents to buyers with zero valuation) are equivalent. Also, the solution to the maximization problem [14] will still have the buyer with highest virtual valuation winning the auction, provided this highest virtual valuation is above zero. However, in this case the virtual valuation functions are different for different buyers. That is, $J_i(v_i)$ does depend on i. Then, higher virtual valuation does not mean higher valuation even in the regular case. Also, our standard auctions (even complemented with a reserve price) will not generally be optimal auctions.

Consider a very simple example with $N = 2$, where both buyers have valuations distributed uniformly, but v_1 is uniform in [0,1] whereas v_2 is uniform in [0,2]. In this case $J_1(v_1) = 2v_1 - 1$ and $J_2(v_2) = 2v_2 - 2$. Then the optimal auction is to assign the object to buyer 1 if $v_1 > max\{v_2 - 1/2, 1/2\}$, to buyer 2 if $v_2 > max\{v_1 + 1/2, 1\}$, and not to sell the object otherwise. How could this be accomplished? By means of a second price auction with reserve price 1 and a bonus of 1/2 to buyer 1. That is, if she bids above buyer 2, buyer 1 has to pay buyer 2's bid minus the bonus of 1/2. It is simple to check that bidding the true valuation is still dominant for buyer 2 and bidding the true valuation plus 1/2 is dominant for buyer 1, which leads to the optimal auction assignment.

This example illustrates the two main points to be learnt about auctions for asymmetric buyers: in general it is in the seller's interest to discriminate in favor of (*ex-ante*) weaker buyers. This increases the otherwise soft competition that stronger buyers face, which helps reduce their informational rents (see McAfee and McMillan (1989) and Branco (1994), the latter for asymmetries arising from the seller's goals). The second point is a negative one: second price auctions, with or without bonuses, are not in general optimal. If the distribution is not uniform, there is no bonus that makes a second price auction replicate the optimal auction. That is unfortunate, since one would like to have optimal auctions that are robust in their design to non-observable parameters of the model. In the previous section we had the happy result that standard auctions with reserve prices were optimal for a very large class of distribution functions, any number of buyers, etc. That is, rules were invariant to these changes in parameters, whereas we do not have that invariance now. On the other hand, this makes the optimal auction approach less attractive, and one may prefer to know what is an optimal auction among a set of simple, invariant ones; for instance, whether first or second price auctions (respectively, Dutch or English auctions) give higher revenues to the seller. Unfortunately, even this simple question does not have a definite answer. As Maskin and Riley (1985) have shown, the answer depends once again on the particular distributions F_i.[19]

3.3 Risk aversion

We now drop the risk neutrality assumption and assume instead that buyer i has utility $u_i(v_i - p)$ if she obtains the object and pays p, and utility $u_i(-p)$ is she does not get the object and pays p, where we assume $u_i(.)$ to be concave[20]; and the rest of assumptions are as in Section 2. Then buyers will not be indifferent between two auctions that allow them the same expected rents but differ in the uncertainties they represent. There are two types of uncertainties a buyer can face when participating in an auction. First, her utility may be different when winning the object and when losing it. Second, even conditional on winning the object (or losing it), her payments, and once again her utility, may still be subject to some randomness (as in the second price auction).

Under risk aversion, buyers prefer auctions that imply lower risks, in the sense that the uncertainty about their utility is lower. In other words, buyers are less willing to pay for participating in a high uncertainty lottery for the object. This suggests that it is in a seller's interest to offer auctions hat insure buyers as much as possible.

[19] In more recent work, (Maskin and Riley (1995)), these same authors analyze more general conditions under which one or other auction is better If the question was efficiency, the answer would be clear; a second price auction does assign the object efficiently, whereas first price auctions do not, in general.

[20] Many more general types of utility functions could be considered See Maskin and Riley (1984).

However, when buyers are risk averse, risk may also be an interesting screening device.

Indeed, consider again any trading mechanism as described in Section 2 by way of the functions $x_i(.)$ and $p_i(.)$. One could consider these functions as a menu of contracts offered to the buyer; by choosing to report z as her valuation, the buyer chooses the contract $\{x_i(z), p_i(z)\}$. Different types of buyers obtain different expected values from the same contract: $x_i(z)v_i - p_i(z)$. That is the way the seller screened the buyer's type in Section 2. This need to screen buyers was the source of informational rents, and was represented by the incentive compatibility constraint. When buyers are risk averse, however, expected rents are not the only concern for the buyers. Then, by introducing more risk in the contract designed for z (that is, more variability in $p_i(z)$ or higher difference in the utility obtained when winning and when loosing if the true valuation is v_i), the seller may increase $p_i(.)$ for high v_i and still make sure that a buyer with that valuation prefers not to report a lower valuation z.

Maskin and Riley (1984) show that one of the types of uncertainty we mentioned above, the one regarding payments for given allocation of the good, never pays. However, they also show that the other type of uncertainty is indeed a useful tool for screening that helps to reduce informational rents. That is, an optimal auction would not completely insure buyers. In particular, buyers' payments conditional on winning and payments conditional on losing should be deterministic, but buyers should prefer winning the object to losing (except for the highest possible valuation). The difference in utility (risk) should be decreasing in the valuation. This last result, in fact, implies that high valuation buyers should be paid (subsidized) when they do not win the auction.[21]

To illustrate these points, let us compare first price and second price auctions in our case[22]. First, note that in a second price auction $B(v_i) = v_i$ is still a dominant strategy under risk aversion. Indeed, whatever risk is associated, any buyer prefers to win if the price is below her valuation and vice versa. Then, the expected payment by a winning buyer with valuation v_i is still given by the right hand side of [4] above. The derivative of this expected payment with respect to v_i

[21] Using subsidies as a screening device may also be interesting when faced with dynamic, endogenous entry (see Burguet (1996))

[22] We follow Maskin and Riley (1984) here

is equal to

$$\frac{dp_i(v_i \mid winning)}{dv_i} = \frac{(N-1)f(v_i)}{F(v_i)}[v_i - p_i(v_i \mid winning)]. \qquad [23]$$

On the other hand, a symmetric, monotone equilibrium bidding function for a first price auction should satisfy

$$v_i = \arg\max_z F(z)^{N-1} u_i(v_i - b(z)), \qquad [24]$$

where we are assuming (for economy in notation) that $u(0) = 0$. Then, the first order condition for this problem implies that

$$b'(v_i) = \frac{(N-1)f(v_i)}{F(v_i)} \frac{u(v_i - b(v_i))}{u'(v_i - b(v_i))}. \qquad [25]$$

Now, notice that in a first price auction $b(v_i) = p_i(v_i \mid winning)$, and notice that (if a symmetric, monotone equilibrium exists, which in our case is guaranteed; see Maskin and Riley (1984)), the same bidder wins in both auctions. Then, if we can show that $p_i(v_i \mid winning)$ is always higher in a first price auction than in a second price auction, we would conclude that the seller expects higher revenues (for any realization of the valuations vector v) in the first[23]. This illustrates the point that introducing uncertainty beyond the point of different utility when winning and when loosing is counterproductive

Now, to show that $p_i(v_i \mid winning)$ is always higher in a first price auction, we use a helpful Lemma (see Milgrom and Weber (1982), lemma 2, the present version of which is slightly reformulated):

LEMMA: *let g and h be two differentiable functions defined in [0,1] such that (i) $g(0) = h(0)$ and (ii) $g(x) = h(x)$ implies that $g'(x) > h'(x)$. Then $g(x) > h(x)$ for all x in [0,1].*

Applying this to our case, one can notice that both first price and second price auctions imply zero expected payments for a winning bidder with valuation 0. Second, assume that, for some value v_i, $p_i(v_i \mid winning)$ coincides in both first price and second price auctions. That

[23] Actually, this is also enough to conclude that the seller prefers the first price auction even if she herself is risk averse. Indeed, the first price auction is the same lottery with respect to who wins the auction, but the payments in the first price auction, given who wins, are not only higher in expected terms but also riskless For a study of a seller's preferences over different designs when the seller is risk averse, see Waehrer, Harstad, and Rothkopf (1998)

means that even in the second price auction $p_i(v_i \mid winning) = b(v_i)$, and then at that point

$$\frac{dp_i(v_i \mid winning)}{dv_i} = \frac{(N-1)f(v_i)}{F(v_i)}[v_i - b(v_i)] \qquad [26]$$

for the second price auction. Then, we only need to show that at such a point

$$v_i - b(v_i) < \frac{u_i(v_i - b(v_i))}{u_i'(v_i - b(v_i))} \qquad [27]$$

which follows from the facts that (i) considered as a function of b, the derivative of the left hand side is -1 whereas the derivative of the right hand side is $-1 + \frac{u_i'' u_i}{(u_i')^2} < -1$, and (ii), at $b = v_i$ the right hand side is equal to the left hand side (and equal to zero), but at any point v_i, $b(v_i) < v_i$.

As we mentioned before, this could be (wrongly) extended to conclude that the lower the risk for the buyers the higher the revenues for the seller. To show that this is wrong, consider an auction that would insure buyers perfectly. Such an auction would have fixed payments in the case that the bidder won the auction (call it $d(v_i)$) and fixed payments in the case that the bidder lost the auction (call it $a(v_i)$) which would satisfy $v_i - d(v_i) = -a(v_i)$, for all v_i. Then, incentive compatibility would require that

$$v_i = \arg\max_z x_i(z)u_i(v_i - d(z)) + (1 - x_i(z))u_i(-a(z)) \qquad [28]$$

whose first order conditions imply $(a' + 1 =)d' = 1 - x_i$. That is, in such an auction the revenue does not depend on the utility function u. In particular, the revenue would be the same if buyers were risk neutral. But under risk neutrality, a second price auction with a reserve price maximizes the (expected) revenues for the seller. Since the revenues in a second price auction are also independent of the buyers' attitude towards risk, we conclude that a perfect insurance auction does not represent more revenues for the seller than a second price auction (with reserve price), which in turn produces lower revenues than a first price auction (with reserve price). Hence, this risky auction is more profitable for the seller than a full insurance auction.

3.4 Correlated information; non-private values

Perhaps the most important assumption we made in Section 2, however, was that information was uncorrelated and that valuations were private. (In fact, in that section valuations and information were the same thing.) If information is not independent, a seller can use the (more informed) inferences a buyer can make about other buyers valuations to both extract this information and even screen the buyer's own information (proposing bets on other buyers' information; see Myerson (1981)). In some circumstances, the seller can extract the entire surplus in this way (see Crémer and McLean (1988)). Another implication of this is that the revenue equivalence results do not hold. And for very similar reasons to the ones discussed when we dealt with risk aversion; now the seller has an additional screening device, and two auctions that use the probability instrument in the same way may still differ in their use of the inference instrument. We will see this later in this section.

With respect to private values, the assumption is usually inappropriate in cases such as the auctioning of mineral rights, and goods with a resale market, like securities or real estate. In fact, in these cases, assuming that the valuation of the good is the same for all buyers is probably a better assumption. But if the value of the good is the same for every potential buyer, what is there that makes the problem interesting? The answer has to do with residual uncertainty about this common valuation at the bidding time. Indeed, when firms bid for mineral rights on a tract of land there is still a lot of uncertainty about the amount of mineral that can be extracted, the future prices for that mineral, and the costs involved. Moreover, different bidders may have different estimates of this common value. That is, they may have different information about this common value. Also, they should take into account how the auction processes this information and, therefore, what inferences can be made about the information that others have.

So, let us now assume that each buyer i receives some information about the common value v of some object. This information is formalized by a signal s_i which is correlated with v. The vector $(v, s_1, s_2, ..., s_N)$ is distributed according to some probability function, which is common knowledge. We assume that higher signals mean better news with respect to v and with respect to all other signals[24]. An example would have $v = s_1 + s_2 + ... + s_N$, with all s_i being independent draws of

[24] More formally, we will always assume that all random variables are affiliated, a generalization of positive correlation which roughly extends correlation to any subsets in the domains of the variables (see Milgrom and Weber (1982)).

some random variable. This would be a case of common values with independent information.

One of the most popular phenomena in common values auctions is the so-called *winner's curse*. Consider the simple case we have just mentioned in which the value is the sum of all signals, and assume these signals are uniformly distributed in [0,1], and $N = 5$. One could still hope that what we learnt about second price auctions in Section 2 applies here. That is, an equilibrium bidding strategy would be to bid $E[v|s_i]$, the expected value of v conditional on the information the buyer has. In our example, that means $B(s_i) = 2 + s_i$. But if everybody uses that strategy, then buyer i will win the auction only when her signal is highest. That is, when all other signals are below s_i, in which case the expected value of the object $E[v|s_i; \forall j, s_j \leq s_i] = 3s_i$. On the other hand, if every buyer uses the strategy $B(s_i) = 2 + s_i$, the price expected by buyer i when she wins is $2 + (4/5)s_i$. Thus, if $s_i = 1/2$, upon winning buyer i expects to pay 2.4 for an object that, precisely conditional on her winning, has an expected value of only 1.5! Thus, to the proposed strategy does not seem dominant, but rather disastrous.

The reason is that winning conveys information about the signals observed by others. When a buyer decides on her bid she has to take into account that this bid is only relevant if she wins. If the equilibrium bidding strategy is increasing in the signal, then winning means that her information has been the most optimistic. Thus, when bidding buyers should discount this excessive optimism when bidding.

As we mentioned above, two auctions that assign the good in the same fashion do not have to be equivalent from the seller's viewpoint. To illustrate this point, we will once again compare first price and second price auctions. So let us return to our second price auction. We have seen that $B(s_i) = E[v|s_i]$ is not a dominant strategy. In fact, it is not hard to see that there could be no dominant strategy equilibria in the common values case: winning always conveys information about the signals of others, but exactly what information will depend on the way others use theirs in forming their bids. What will then be a Bayesian equilibrium bidding strategy? The answer is

$$B(s_i) = E[v \mid \max_{j \neq i} s_j = s_i, s_i] \qquad [29]$$

(in our example above, $B(s_i) = \frac{7}{2}s_i$). Indeed, in a second price auction, the bid is the highest price a buyer is willing to pay for the object.

Now, assume that all bidders but i bid according to [30] and buyer i has a signal s_i. Is buyer i willing to accept a price $p < B(s_i)$? First we have to answer the question as to what the fact that the price (the highest rival bid) is p means. Given that all others are using [30], which is increasing in the signal, this means that the highest rival signal (call it s^*) is lower than s_i. In particular

$$p = b(s^*) = E[v \mid max_{j \neq i} s_j = s^*, s^*] < E[v | max_{j \neq i} s_j = s^*, s_i] \quad [30]$$

(the last inequality due to the fact that v is stochastically increasing in s_i). But the right hand side of the inequality is the expected value of v given that i has signal s_i and the highest rival bid is p. That is, buyer i is indeed willing to accept price p. A similar argument would show that buyer i would not be willing to take any price above $B(s_i)$, which means that this is indeed an equilibrium bidding strategy.

We could now compute the equilibrium bidding strategy in a first price auction (see Milgrom and Weber (1982)), but we will not even need this to conclude that a first price auction means lower revenue for the seller. The way to do it is to come back to our indirect approach of Section 2. Indeed, for any mechanism $\{\rho_i(s), \chi_i(s)\}_{i=1,2,...,N}$, where now the allocation and payments should depend on the vector of signals $s = (s_1, s_2, ..., s_N)$, rather than valuations, define again the functions $p_i(z; s_i) = E_{s_{-i}}[\rho_i(s_{-i}, z) \mid s_i]$ and $x_i(z; s_i) = E_{s_{-i}}[\chi_i(s_{-i}, z) \mid s_i]$. Remember that we use these functions to analyze the decision problem of a buyer, where she can decide how to behave (what signal she will pretend to have), while knowing what signal she really has. In the independent information case, this last piece of information was not important when computing the expected payment and probability of winning, since these data depended only on the buyer's behavior. Now, however, the probabilities that she assigns to the signals of others may depend on her true signal, and therefore expected payments and probability of winning depend on both the "reported signal" (z) and the true one. Thus, $x_i(z; s_i)$ is the probability that i obtains the good if she behaves as having observed signal z but has in fact observed signal s_i, and similarly for $p_i(z; s_i)$. We will also need to define what now plays the part of i's valuation. Thus, let

$$R_i(z; s_i) = E[v \mid \max_{j \neq i} s_j < z, s_i]. \quad [31]$$

Then, for any trading mechanism, i's incentive compatibility constraint can be written as

$$s_i = argmax_z \{R_i(z; s_i) x_i(z; s_i) - p_i(z; s_i)\}, \quad [32]$$

with first order condition

$$0 = R_i(s_i; s_i)x_{i(1)}(s_i; s_i) + R_{i(1)}(s_i; s_i)x_i(s_i; s_i) - p_{i(1)}(s_i; s_i) \quad [33]$$

where the subindex in brackets represents the variable with respect to which we are partially differentiating (e.g., $x_{i(1)}$ represents the partial derivative of x_i with respect to the first argument $-z-$). Again, the seller is interested in the payment functions evaluated on the diagonal, that is, taking into account that buyers do have incentives to behave according to predictions. So let us concentrate on the study of these functions $p_i(s_i; s_i)$. In particular, we will compare these functions for first price and second price auctions. We will prove that $p_i(s_i; s_i)$ for the second price auction is everywhere (weakly) above the corresponding function for the first price auction. That is, whatever her valuation, a buyer expects to pay more in the second price auction than in the first price auction. Thus, the expected revenues for the seller are higher in the former than in the latter.

The way to prove this result is to come back to the lemma in Section 3.3. Indeed, notice that $p_i(0;0) = R_i(0;0)$ in both second price and first price auctions. Then we will prove that whenever $p_i(s_i; s_i)$ coincide for the first price and second price auctions, the slope of this function (along the diagonal) is (weakly) higher for the second price auction than for the first price auction. To this end, first notice that

$$\frac{dp_i(s_i, s_i)}{ds_i} = p_{i(1)}(s_i, s_i) + p_{i(2)}(s_i, s_i). \quad [34]$$

Also, notice that, according to [34], the first term on the right hand side of [35] is common to the first price and second price auctions. Indeed, the functions $x_i(z; s_i)$ are identical in both auctions, and the function $R_i(z; s_i)$ does not depend on the trading mechanism. Therefore comparing the slopes of the functions $p_i(s_i; s_i)$ is the same as comparing the partial derivative $p_{i(2)}(s_i; s_i)$. Now, in the first price auction

$$p_i(z; s_i) = b(z)\Pr(z \geq s_j \forall j \mid s_i), \quad [35]$$

where Pr stands for probability, whereas in the second price auction

$$p_i(z; s_i) = E[B(max_j s_j) \mid s_i; max_j s_j \leq z]Pr(z \geq s_j \forall j | s_i), \quad [36]$$

and then, when $p_i(s_i; s_i)$ coincide for both auctions (which from [36] and [37] implies that $b(z) = E[B(max_j s_j) \mid s_i; max_j s_j \leq z]$), the difference between $p_{i(2)}(s_i; s_i)$ for the second price auction and the first price auction is simply

$$\Pr(s_i \geq sj \forall j \mid s_i) \left. \frac{\partial E[B(\max_j s_j) \mid s_i; \max_j s_j \leq z]}{\partial s_i} \right|_{z=s_i} \geq 0. \quad [37]$$

and then we conclude that indeed the function $p_i(s_i; s_i)$ in the second price auction is everywhere above the one for the first price auction. Thus, every buyer expects to pay more in the second price auction, and then the expected revenue for the seller is higher.

Notice that the source of this result is the fact that $p_{i(2)}(s_i; s_i)$ is higher in the second price auction. Also, notice that in obtaining the result we used the fact that the functions $x_i(z; s_i)$ and $R_i(z; s_i)$ where identical in both mechanisms. Now, generalize the definition of function $R_i(z; s_i)$ as follows:

$$R_i(z; s_i) = E[v \mid z \text{ is winning type, } s_i] \quad [38]$$

Then what we learnt by comparing first price and second price auctions can be generalized to one of the key results in auction theory, the *"Linkage Principle"*: for any two auctions that assign the good in the same fashion (which implies common $x_i(z; s_i)$ and common inferences $R_i(z; s_i)$) and leave the same rents to a bidder with the lowest possible signal, the revenue for the seller is higher in the auction at which the expected payment of a buyer *(given equilibrium behavior)* is more *linked* to her own signal (see Weber (1983)).

It is convenient to underline that the key ingredient in our comparison between auctions (what the linkage principle asked for) was not the fact that the value was common, but the fact that information across agents was correlated. Indeed, even under common values but with independent information (for instance, in our additive example above), first price and second price auctions are equivalent, since then [38] is satisfied with equality. The important feature is that when information is indeed correlated, the fact that buyer i has observed a high signal increases the likelihood of higher signals (and then higher bids) for i's rivals even conditioning on buyer i winning with the same bid!

In fact, everything we are discussing applies to a much more general model, which has the common value and private values models as particular cases. Indeed, one can assume that there is a random vector $(v_1, v_2, ..., v_N, s_1, s_2, ..., s_N)$ with some probability distribution that determines signals and valuations. If this probability distribution puts weight only on points where $v_1 = v_2 = ... = v_N$, then we are in the model we have just discussed. If the probability distribution puts all its weight on points for which $s_i = v_i$, for all i, then we are in the private values case. If, moreover, the different s_i (and therefore, the different v_i) are independent, we are in the independent private values model. In this more general model, and assuming that all random variables are affiliated (see Milgrom and Weber (1982)), the analysis we have just presented extends trivially. In particular, the revenue equivalence result of Section 2 is just a corollary of the linkage principle.

There are many other direct corollaries to the linkage principle. For instance, one can easily conclude from it that the English (oral) auction generates more revenue for the seller than the (sealed bid) second price auction. Indeed, in an oral, ascending auction all buyers adapt their behavior to their rivals' observed behavior. That means that the final price (the "bid" of the last rival to drop out of the auction) depends on all signals, and not only on this final rival's signal. But that means that the final price is more closely "linked" to the winner's signal (it is correlated to that signal through the correlation of all the signals with the winner's signal) than in a second price auction (where the price is linked to the winner's signal only through its correlation to the second highest bidder's signal). Similar arguments could be invoked to conclude that, if possible, the seller should commit to releasing any information she might have that is correlated to the buyers' information: this simply increases the correlation among the buyers' signals.

4. Multiple objects

In this section we extend our analysis in order to consider the auctioning of multiple units of a homogeneous good. Securities, oil tracts, and spectrum rights are all examples of (to a great extent) homogeneous goods that are auctioned to a common set of buyers. Considering multiple units opens a new range of questions (for example, whether a seller should sell the units simultaneously or in sequence, what is the optimal size of the lots for sale) and modeling possibilities (whether buyers have a declining inverse demand –willingness to pay–, whether synergies among the different units are a relevant phenomenon). All these questions and possibilities were absent or irrelevant when the seller had only one unit to sell.

We start the analysis of multiple-unit auctions by considering the simplest possible model, in which buyers have positive willingness to pay for only one unit of the good (unit demands). That is, a buyer i is willing to pay up to v_i for a unit of the good, but a second unit has no marginal value to her. This is, of course, a very special kind of demand. It may fit situations like the "bidding" for a CEO or some real estate auctions. But most importantly, it will allow us to derive many results as a direct extension of the last two sections.

Indeed, assume all v_i are i.i.d. Also, assume the seller has $K\,(< N)$ units of the good. Define the uniform price auction as a generalization of the second price auction, where bidders submit bids, and the K highest bidders get one unit of the good and pay the $K+1$ highest bid. Then it is a simple exercise to check that the strategy $B(v_i) = v_i$, is a dominant strategy in this auction. Also, define the discriminatory auction as a generalization of the first price auction, where the K highest bidders obtain one unit of the good but each pays his own bid. Then the same procedure used in Section 2 can be used to conclude that

$$b(v_i) = \frac{\int_0^{v_i} x(N-1)\begin{pmatrix} N-2 \\ K-1 \end{pmatrix} f(x)F(x)^{N-K-1}[1-F(x)]^{K-1}dx}{\int_0^{v_i}(N-1)\begin{pmatrix} N-2 \\ K-1 \end{pmatrix} f(x)F(x)^{N-K-1}[1-F(x)]^{K-1}dx}, \qquad [39]$$

that is, the expected value of the K^{th} order statistic of $N-1$ realizations of F conditional on being below v_i, is an equilibrium bidding strategy in this auction. Thus, a buyer with valuation v_i who wins this auction expects to pay the same as this same buyer would pay in the uniform price auction (the bid, i.e., the valuation of the K highest among the competitors, provided it is below i's bid, i.e., below v_i). Then these two auctions are equivalent for both the buyers and the seller.

Also, the reason why this revenue equivalence holds is the same we found in Section 2. The incentive compatibility constraint that an auction has to satisfy is still given by [10], where now the functions $x_i(z)$ are interpreted as the probability of obtaining one good. Also, equation [13] gives the expression for a buyer's informational rents, and implies that two auctions that assign the objects in the same fashion and give the same rents to the marginal buyer are also equivalent from the seller's point of view. Then [14] is the problem that defines the optimal auction and [16] gives the solution: an optimal auction is any auction that assigns one object to each of the K highest valuation bidders whose virtual valuation $J(v_i)$ exceeds zero (Maskin and Riley (1989), Engelbrecht-Wiggans (1988))[25]. For instance, both the uniform price and the discriminatory auctions are optimal when modified by the reserve price v^* as defined in Section 2.

Consider now auctioning the objects sequentially. That is, the seller auctions one object at a time using a first or second price auction at each stage. Of course, the bidding behavior described in Section 2 will not be equilibrium at each stage. However, applying the revenue equivalence result above we can conclude that, as long as the K highest valuation buyers end up obtaining one unit of the good (and buyers with zero valuation obtain no rents), the expected revenue for the seller is still the same. Moreover, as long as this is the case, we can compute the expected price at each of the stages without even looking at the equilibrium strategies (see Weber (1983) for the simple form of bidding functions in sequential auctions). The key is to consider the sequence of expected prices from the point of view of the winning bidder in, say, the first auction. Assume that a bidder expects that, conditional on her winning the first auction, the price in the next period is lower than what she would pay in the present period. In this case the bidder would not be behaving optimally in this period; she should reduce her bid today and try for the cheaper good tomorrow. A similar argument can be made to conclude that the price tomorrow should not be expected to be higher than today's, since then bidders today should bid more aggressively[26]. Thus, prices in the sequential auctions should follow a martingale. The ex-ante expected value of this martingale is fixed by the revenue equivalence result we have mentioned above.

Contrary to this theoretical result, the prices of sequential auctions of homogeneous goods have been observed to follow a declining pattern too frequently (see, for instance, Ashenfelter (1989), Ashenfelter and Genesove (1992), and Engelbrecht-Wiggans and Kahn (1992)). This is known as the *afternoon effect* or *the declining price anomaly.* What could explain this? Correlation does not seem to be a real answer: extending the intuition from the linkage principle discussed in the previous section, future prices should be more closely linked to a buyer's

[25] The definition of our functions $\chi_i(v)$ is now subject to the constraint that that they take values between 0 and 1 (as before) and, for any v, $\Sigma_i\chi_i(v)$ cannot exceed K (instead of 1).

[26] Here we implicitly use a continuity argument. expected prices tomorrow conditional on prices today are continuous. It is easy to check that this continuity has to be part of the equilibrium features.

signal or valuation than present ones, since buyers' bids in the future will depend on the inferences made from present bids. Thus, one would rather expect a pattern of increasing prices. Risk aversion, on the other hand, could probably do the job. Indeed, and also from the intuition gained in the previous section, a future price is more uncertain than the price today from today's point of view. For instance, in a first price auction the price is deterministic today conditional on winning, whereas conditional on losing future prices will depend on today's winning bid, which makes it random. Indeed, McAfee and Vincent (1993) show that any equilibrium in pure strategies for the sequential auction would induce declining prices under risk aversion. Existence of pure strategy equilibria, however, is problematic.

An alternative explanation for declining prices has to do with generalizing the demand functions considered. Black and de Meza (1992) show that downward sloping demands for more than one unit of the good could result in declining prices when the seller offers what is called the "buyer's option", a right for the winner of one round to buy as many units as she wants at the price hammered down in that round. This makes bidders more aggressive in the early stages in order to induce higher prices that would dissuade a rival winner from exerting the option[27].

The sub-martingale property of expected prices in sequential auctions is not the only change that results from considering more general demand functions. To start with, we need to reconsider the revenue equivalence theory (see Maskin and Riley (1989)). Thus, assume that a buyer now has an inverse demand function given by $h(q, v_i)$, where q represents the number of units of the good and v_i parameterizes the buyer's preferences. That is, $h(q, v_i)$ is a buyer's marginal willingness to pay for the q^{th} unit when her preferences are defined by the parameter v_i. We maintain the assumptions about the stochastic process generating the v_i's. We also assume that $h(q, v_i)$ is decreasing in q, and it is increasing and differentiable in v_i. Once again, a trading mechanism could be described by the vector $\{\rho_i(v), \chi_i(v)\}_{i=1,2,\ldots,N}$, where now $\chi_i(v)$ represents the number of units that buyer i obtains if v is

[27] In Burguet and Sákovics (1997a) we show that prices do decline if a buyer has flat demand for more than one unit, even though in this case the dissuasive effect of higher bids does not bite. Indeed, in the case analyzed, the "buyer's option" operates as endogenous uncertainty about the number of objects on sale. We also show that any uncertainty on this, whether endogenous or exogenous, will cause an expected declining pattern of realized prices even under risk neutrality

the realization of the taste parameters[28]. Then, similarly to Section 3.4, we can define the function $R_i(z, v_i)$ as the gross surplus (the area below her demand function $h(.,.)$) that i expects if her type is v_i and she behaves as if it was z. That is,

$$R_i(z, v_i) = E_{v_{-i}}\left[\int_0^{\chi(v_{-i},z)} h(q, v_i)dq\right]. \qquad [40]$$

Also, define $p_i(v_i)$ exactly as in Section 2. Then, the trading mechanism satisfies incentive compatibility if

$$v_i = \arg\max_z R_i(z, v_i) - p_i(z), \qquad [41]$$

whose first order condition is

$$R_{i(1)}(v_i, v_i) - p_i^{'}(v_i) = 0, \qquad [42]$$

using again the notation for partial derivatives discussed in the previous section. Notice that, here too, the allocation of the objects (plus N initial conditions) determines the payments buyers expect (and then the seller's revenues). Also, we can obtain the expression for the informational rents in terms of the allocation of the objects. Indeed, integrating in [43], taking into account that

$$\frac{dR_i(v_i, v_i)}{dv_i} = R_{i(1)}(v_i, v_i) + R_{i(2)}(v_i, v_i) \qquad [43]$$

we obtain the net consumer's surplus for buyers:

$$R_i(v_i, v_i) - p_i(v_i) = R_i(0, 0) - p_i(0) + \int_0^{v_i} R_{i(2)}(x, x)dx, \qquad [44]$$

and then by substituting [45] in the seller's problem, which is still represented by [9], we obtain:[29]

$$Max \sum_i E_{v_i}\left[R_i(v_i, v_i) - R_{i(2)}(v_i, v_i)\frac{1 - F(v_i)}{f(v_i)}\right] \qquad [45]$$

Notice that letting $h(q, v_i) = v_i$ for $q = 1$ and $h(q, v_i) = 0$ otherwise (and then $R_i(z, v_i) = x_i(z)v_i$ as defined in Section 2), we recover the unit demand case.

[28] For simplicity, we only consider deterministic mechanims, that is, we assume that, after all the buyers have made their moves, the number of units each receives is deterministic.

[29] Branco (1996) generalizes this to the non-private values case, where $h(.,.)$ depends on the whole vector v

The conclusions from [46] are quite different from the ones obtained in the unit demand case. Indeed, we could generalize the definition of a uniform price auction as one in which each bidder can submit up to K bids for units of the object, and the K units are awarded to the K highest bids at a price which is equal to the $K+1^{th}$ highest. That is, one buyer could win more than one unit. Similarly, the general definition of the discriminatory auction would coincide with that of the uniform price auction except that all bids that are accepted are paid. Both institutions have been used to auction securities and public debt. It is now the case that, in general, these two auctions do not solve [46], even when complemented with a reserve price. In general, the optimal auction is not a simple mechanism. It can be described as a non linear price rule with rationing: the seller fixes a payments function $T(q)$, and then each buyer demands a quantity k_i given this non linear pricing rule. If total demand does not exceed K, then all demands are satisfied. Otherwise these demands are rationed according to a method which is specified in advance (Maskin and Riley (1989)). In the special case that demands are linear, that is, $h(q, v_i) = v_i - aq$, with $a > 0$, and for the regular case, this rationing is a proportional one: If $\sum_j k_j \leq K$ then each buyer receives her "bid". Otherwise i receives $k_i - \frac{\sum_j k_j - K}{N}$ (payments are, of course, computed according to [45]).

As we have already mentioned, neither uniform price nor discriminatory auctions are optimal in general. Moreover, they are neither equivalent (in terms of the seller's revenues) or, more surprisingly, efficient. Indeed, for the unit demand case, efficiency was never a real issue: any of the most common auctions could guarantee this. Now, however, even efficiency will need special mechanisms. To illustrate this point in the simplest case, assume $K = 2$. Consider a discriminatory auction. A bid can be thought of as a demand function, thus stating how much the buyer is willing to pay for her first unit and how much for the second. Now, bidding a demand function equal to the true $h(q, v_i)$ cannot be an equilibrium, since that would mean zero rents (surplus) for the bidder. Thus, as in the one unit case, bidders will "shade" their bids. However, this shading will be different for the first and the second units. Indeed, the probability that a buyer obtains her first unit (i.e., the probability that her highest unit bid is among the highest 2 unit bids of all buyers) is higher than the probability of obtaining the second unit. Therefore, the trade-off (rents if winning versus probability of winning) is different for the two units and then the buyer will "shade" her two bids with different amounts (usually more for the first unit –her higher bid– than for the second unit). But this implies that a buyer's bid for a second unit may be higher than another buyer's highest bid even if the former values that second unit less than the latter values her first unit. Thus, inefficiencies will arise in a discriminatory auction with positive probability.

Perhaps more surprising than this result is the fact that similar inefficiencies arise when the seller uses a uniform price auction[30]. Again, submitting the true demand function as a "bid" (or set of two unit bids) is not the best response against virtually any rival strategy. Indeed, now there is a positive probability that the bid for the second unit (the buyer's second highest bid) is the third highest of all buyers' bids, that is, the one setting the price. Therefore, a bidder has an incentive to lower the bid for the second unit in order to decrease (with some probability) the price paid for her first unit, an incentive absent in her highest, first bid (and very similar to the incentive a monopolist has to increase the price above marginal cost). Thus, once again, a buyer can obtain her first unit even if she values it less than some other buyer values her second unit. In both cases, the inefficiency takes the form of a tendency towards disseminating the units across buyers more than what relative willingness to pay would indicate. Which of the two auctions is more efficient or gives more revenue to the seller depends on the particular parameters of the situation (see Ausubel and Cramton (1998); also, see Back and Zender (1993) for comparisons in the common vale case).

There are, however, relatively simple auctions that can guarantee efficiency in this context. In fact, Vickrey (1961) had already proposed an efficient auction, one that correctly generalized his second price auction to the multi-unit case. This auction has everybody submitting a demand function, as in the uniform price or the discriminatory auctions. However, buyer i pays a price equal to the k^{th} highest rival bid rejected for her k^{th} unit received. More recently, Ausubel (1997) has proposed what would be the oral, ascending version of it, just as the English auction can be considered the oral, ascending version of

[30] Actually, this should not be so surprising, since Vickrey (1961) had already underlined this fact However, even thirty years later, such reputed economists as Milton Friedman and Merton Miller still thought (and made it public) that in uniform price auctions truth was a dominant strategy (see Ausubel and Cramton (1998)).

the second price auction. In Ausubel's auction, bidders respond to the seller's (continuous and increasing) price announcements by declaring how many units they are willing to buy at that price. The process continues (with the restriction that a buyer cannot increase her demand as the price increases) until the first price at which total demand is equal to (or lower than) K. Demands are then satisfied. How much does a buyer pay for a unit? The answer is the price at which she "clinched" that unit. To illustrate the workings of the auction, let us come back to the $K = 2$ example, and assume $N = 2$. The price starts at zero and increases continually until one bidder, say bidder 1, reduces demand from 2 to 1 units. At that point bidder 2 has already clinched one unit. That is, whatever happens afterwards, she will receive one unit: bidder 1 has decreased demand to 1 and there are 2 units in total. Then we can consider that the first unit has been assigned to bidder 2, and the price is the one at which this has happened. The process continues, this time having one unit to assign and two bidders, each with a demand for one unit (the second, for bidder 2). The auction finishes when one of the two bidders drops out, and then the second unit is assigned to the other bidder at the going price. It is a simple exercise to prove that bidding "truthfully", that is, asking at each price for a number of units equal to the true demand at that price, is a dominant strategy, as in the Vickrey auction. So, once again, all the units will be allocated to those bidders who value them most (and seller's revenues are the same in both, oral and sealed bid, auctions, according to [46]).

So far we have been talking about "downward sloping" demand. In recent years, however, a new interest has arisen about what one could term as "upward sloping demand" auctions; that is, auctions of multiple objects with the property that buyers' valuation for two units is more than twice their valuation for one unit. The interest in these cases, referred to as auctions with synergies, increasing returns, or superadditive values, has to do with the much publicized and already mentioned auction of spectrum rights in the USA. This was conducted using a simultaneous, ascending auction[31]. What was new about the auctioning of a number of different licenses covering different territories and different bands of the radio spectrum was the fact that two licenses that were adjacent, both geographically and in terms of frequency, we-

[31] See, for instance, McAfee and McMillan (1996) and Cramton (1995) for a description and early analysis of these auctions.

re worth more than the addition of the two individual licenses, at least for some bidders. Indeed, they would allow the exploitation of economies of scale in territory covered by the new personal communication systems they could carry, and in the amount of spectrum that could be gained if frequencies did not have to be separated. The problem of designing optimal (or satisfactory) auction mechanisms for this type of goods is still very much open. After a first wave of excitement about the simultaneous ascending design used by the FCC in the US, some have started to question the efficiency (the main goal of the FCC) of such a design (see, for instance, Ausubel and Cramton (1998)). In fact, there are very few theoretical analyses of auctions with synergies. One exception is the work by Krishna and Rosenthal (1996), where they analyze equilibrium bidding behavior in a simultaneous, second price auction of two goods in the presence of "global bidders" (those with demand for more than one object and with synergies), and "local bidders" (those interested in only one of the objects for sale)[32]. The way the authors model synergies is by assuming that a global bidder has some (private and independent) valuation v for one good, but $2v + \alpha$ for both objects together. Apart from being complex (and discontinuous in general), the global bidders' behavior has some interesting properties. First, for very low values of v the global bidders bid less than their per unit value (i.e., $v + \alpha/2$) for each object. The reason is that the chances of obtaining both objects (i.e., beating local bidders for both objects) are very small, and then they bid as if expecting to gain only one object (which they value individually as v). However, for very high realizations of v global bidders bid more than their per unit valuation for each object. Indeed, in this case it is highly likely that they win both objects, but their marginal valuation for each of the objects conditional on having won the other is $v + \alpha$. Thus, global bidders have both regions of "overbidding" and regions of "underbidding". Krishna and Rosenthal also show that an increase in the number of bidders does not always lead to an increase in competition. In fact, an increase in global bidders always leads to less aggressive bidding by the global bidders. In terms of revenue comparison, they show that the ranking of sequential and simultaneous auctions is ambiguous.[33]

Considering multiple unit auctions poses yet another important question which was absent in the single unit auction; the possibility that a

[32] See Rosenthal and Wang (1996) for a similar analysis in a common value setting.
[33] See Branco (1997) for an analysis of sequential auctions with synergies.

seller also has different valuations for different units. Indeed, we have been assuming that the seller valued all units equally, which we normalized at zero. This need not be the case (governments may decide to sell more public debt if the "offers" to buy include lower prices or bids, for instance). That is, the seller's "supply" may have positive "price elasticity." Moreover, some recently created auction markets have these elasticities as an important component which should be taken into account. If we change the role of buyers and sellers, examples of this are some spot markets for electricity. Indeed, in some countries regulatory reform has introduced not only auction mechanisms to allocate power generation needed to satisfy demand, as in the British case, but also bidding on the part of the demand side (large consumers, electric utilities), as in California.

The study of such auction markets can be approached by first slightly generalizing our analysis of multiple object auctions.[34] Indeed, instead of a fixed amount K, assume now that the seller announces an increasing supply schedule $S(p)$. On the other hand, assume that buyers have flat demand schedules, that is, $h(q, v_i) = qv_i$. This is the model analyzed by Hansen (1988), which, if the roles of buyers and sellers are interchanged, can be interpreted as a model of bidding for the right to satisfy a downward sloping demand (in our case, to buy from an increasing supply $S(p)$), or an auction with endogenous quantity. Here we could restrict both uniform price and discriminatory auction rules so that buyers could only submit flat demand schedules, that is, offer a price. Then, in a uniform price auction the winner would be the one offering the highest bid or price p_i, but would pay the second highest bid p_j per unit and obtain $S(p_j)$ units of the good. In the discriminatory auction, the highest bidder would win again, but would buy an amount equal to $S(p_i)$ and pay her bid, p_i, per unit.

It is not difficult to see that bidding a price equal to the true valuation v_i is once again a dominant strategy for this version of the uniform price auction. For the version of the discriminatory auction and symmetric bidders, one could characterize a monotone symmetric equilibrium bidding function $p(v_i)$ in a way similar to that used in Section 2, when we analyzed first price auctions. That is, we ought to have

$$v_i = \arg\max_z [v_i - p(z)] F(z)^{N-1} S(p(z)), \qquad [46]$$

[34] This analysis will be considered in greater depth with the study of double auctions in the next section

with first order conditions that could be written as

$$p'(v_i) = \frac{[v_i - p(v_i)](N-1)f(v_i)}{F(v_i)(1 - [v_i - p(v_i)]S'/S)} \qquad [47]$$

Now, if we again use the Lemma in Section 3, we will conclude that the expected price in the discriminatory auction is higher in the endogenous quantity case than in the fixed quantity analyzed in Section 2. Indeed, a zero valuation buyer would have to bid zero in equilibrium, as in the fixed quantity case. Also, if the bidding function in the fixed quantity case $b(v_i)$ were at any point equal to $p(v_i)$, then the numerator of [48] and [3] would be the same, whereas the denominator for [48] would be lower (as long as S' is positive, as we are assuming), and therefore p' would be higher than b' at that point. According to our Lemma, this allows us to conclude that $p(v_i) > b(v_i)$ for all $v_i > 0$. But this also implies that in our endogenous quantity case the expected price (and the quantity sold) is higher in the discriminatory auction than in the uniform price one: in the latter it is the same as in the second price auction, which is in turn equal to the expected price in the first price (fixed quantity) auction; this one is lower, as we have just seen, than the expected price in the discriminatory auction with endogenous quantities.

The intuition behind this result is simple. Indeed, in the discriminatory auction, where one wins all or nothing, there is an incentive that works against shading the bid (which is represented by the extra term in the denominator of [48]): shading the bid makes the price fall, but the amount obtained is also reduced since $S(P)$ is increasing. On the other hand, in the uniform price auction in which a bidder wins all or nothing this incentive is non-existent, since the price, and therefore the quantity, is not directly affected by the winner's bid. Moreover, as a result of a higher expected price the quantity sold is also higher, which leads to efficiency gains; the marginal cost $S(P)$ is closer to marginal willingness to pay v_i.[35] Hansen shows that in general this gain in efficiency will make both buyers and seller better off.

The analysis above assumes constant marginal willingness to pay, that is, flat demand curves. Relaxing this assumption seems important when one is interested in studying such cases as the spot markets for electricity mentioned above (once again, interchanging the role of buyers and sellers). Indeed, in this case a plant is generally unable to

[35] We are using a concavity argument here, since the output in a uniform price auction would be more "stochastic" than the one from the discriminatory auction

supply the whole market (in our version, a buyer is unable or unwilling to absorb the whole supply). That is, they are capacity constrained. Moreover, a bidder usually owns more than one plant, perhaps with different costs. Thus, a better model for this case would assume that buyers (sellers, in power generation) have decreasing, maybe piecewise constant, marginal willingness to pay. This is the model analyzed by de Otto (1997) using the uniform price auction, except that the price is the bid made by the marginal plant that does produce (in general, not at full capacity), not the first one to be excluded. Thus, for inframarginal bidders, the situation is similar to the one faced by Hansen's bidders under the uniform price rule, whereas for the marginal bidder the situation is more like Hansen's bidders in a discriminatory auction. (Notice that if capacity constraints were absent, we would be dealing with this discriminatory auction.) Then firms are more aggressive (they ask for lower prices or, if it is the buyers' side that is bidding, offer higher prices) because they are capacity constrained. The final effect on the expected price, however, is ambiguous (now it is not necessarily the most efficient plant, or highest valuation buyer which sets the price). With respect to ownership structure, now bidders have an incentive to manipulate the bids of the less efficient units in order to affect the price for the inframarginal ones, an incentive very similar to those described by Ausubel and Cramton (1998).

We are now very close to the analysis of auctions as mechanisms for price formation in markets. The only thing that is missing in this regard is the presence of strategic agents in both sides of the market that take the mechanism as given. Indeed, so far we have only considered one strategic side of the market (buyers) while the other side's role, if it existed, was to define the rules of the game. In the next section we take the last step to entering into the analysis of auction games as markets.

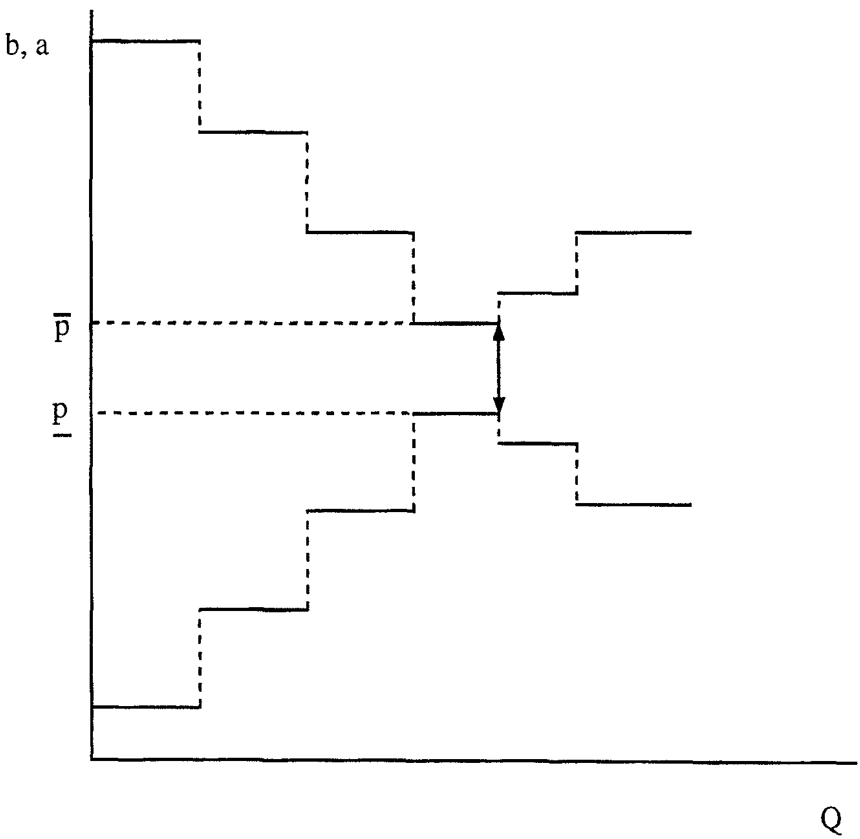
5. Auctions and markets

Some organized markets, like stock markets, are actually auctions in which both parties to the trade bid for exchange. This special type of auction is known as double auction.[36] One can distinguish oral double auctions from "sealed bid" double auctions, sometimes referred to

[36] This mechanism, for the case of one buyer and one seller, has received extensive attention as a model of bilateral bargaining. See Myerson and Satterthwaite (1983) for an early study of this case

as clearing houses or call markets. In the first, agents meet, buyers announce bids and sellers asks at any moment in time, and trade occurs whenever a seller accepts a bid or a buyer accepts an ask. In the "sealed bid" double auction, buyers and sellers simultaneously submit their bids and asks (perhaps for multiple units), then an "auctioneer" draws a demand function with the aggregation of the bids and a supply function with the aggregation of asks. A market clearing price for such supply and demand schedules is determined, and buyers who set bids above this price receive their corresponding units from sellers who set asks below that price (see Figure 1). Given that for indivisible units there generally will be a continuum (an interval) of market clearing prices, $[\underline{p}, \overline{p}]$, one usually refers to an auction picking a convex combination $k\overline{p} + (1-k)\underline{p}$ as the k-double auction.

FIGURE 1
Double Auctions



The analysis of oral, continuous time double auctions is almost hopelessly difficult. Most of the analysis of this institution has taken the form of controlled, laboratory experiments (see Friedman and Rust

(1993) for a collection of those studies)[37]. The analysis of the k-double auction is more tractable, under unit individual supplies and demands. Here one typically assumes that N buyers as characterized in Section 2 and M ($= N$, usually) sellers characterized by some i.i.d. valuation or reservation value c_j with some known c.d.f. and density function G and g, simultaneously decide their bids and asks, respectively. Even though no general results on existence are available, at least the first order conditions for symmetric equilibrium are not hard to obtain. An additional difficulty with double auctions, however, is the inherent multiplicity of (even symmetric, monotone) equilibria. The origin of this lies in the fact that initial conditions for what is now a differential equation system, that is, the first order conditions for the sellers' and the buyers' strategies, are somewhat endogenous. Indeed, knowing what is the lowest possible ask α a seller sets in equilibrium (for monotone equilibria, this is the ask of a seller with lowest possible valuation or reservation value), we have the initial conditions for the buyer's bidding strategy; a buyer with valuation α bids exactly α. Similarly, knowing the maximum possible bid β a buyer may make in equilibrium (the bid of a buyer with the highest possible valuation), we have the initial conditions for the seller's strategy; a seller with valuation or reservation value equal to β sets an ask of β. The problem is that what the highest equilibrium bid is depends on the initial conditions for the bidding strategy, which in turn we have seen depends on the lowest equilibrium ask; however, this one depends on the initial condition of the ask strategy. This circularity explains the generic multiplicity of equilibria (see Satterthwaite and Williams (1993)).

Even without closed forms for equilibrium strategies, there are interesting comments that can be made about k-double auctions. In particular, (see Rustichini, Satterthwaite, and Williams (1994)) trade is inefficient in this institution; buyers bid lower than their valuations and/or sellers' asks are above their reservation values, and then some trades that should take place do not. However, this misrepresentation and the inefficiency it entails converge to zero as the number of agents converges to infinity. Moreover, the convergence to efficiency is fast, at least for the uniform distributions case, in the sense that it attains the convergence speed of a theoretically optimal mechanism, computed in Gresik and Satterthwaite (1989). Thus, a double auction mechanism

[37] Easley and Ledyard (1993) study this institution from a non-Bayesian, non-game theoretic angle, postulating a set of plausible behavioral rules agents would use.

is a model of Walrasian price formation at least for a large number of traders, the case the Walrasian model studies, anyway.

There is, however, still another step to be taken if we want to get close to the idea of a market as a decentralized institution for the interaction of buyers and sellers. This is to dispense with the "center" needed in a double auction mechanism. Indeed, in a ("sealed-bid") double auction "someone" receives bids, compares them, and sets prices and trades according to some previously specified rule. This is only present in organized markets. Then one should question the potential of auction theory as a tool for analyzing price formation in decentralized markets.

A model of decentralized competition (in the auction tradition) would have sellers compete to attract buyers who would then bid for the sellers' goods.[38] Such is the model analyzed by McAfee (1993), Peters and Severinov (1997), and Peters (1997). Here, sellers announce the mechanism they will use to sell their (unit of the) good. After observing these announcements, buyers decide which auction to attend. Then each unit is assigned to one of the buyers attending the respective auction according to the previously announced rules. We could restrict our attention to second price auctions with reserve prices; then, a seller's mechanism would simply be an announced reserve price. All three articles cited conclude that the reserve prices offered by the sellers converge to their respective reservation value (cost) as the number of agents increses.

This could be interpreted as a Walrasian result (prices converge to marginal cost). Moreover, it seems to complement a previous paper by Bulow and Roberts (1989) which shows how the optimal auction design for a single seller (as studied in Section 2) is homomorphic to the monopolistic pricing problem. One would then have a parallel between auctions and markets, and in particular between reserve prices and market prices, for the two polar cases of perfect competition and monopoly. One would both have the intuition that the parallel might well extend to intermediate cases and pose the question as to why market prices and reserve prices should be so closely related. In a re-

[38] Some of the early literature on auctions, and some of the literature on financial markets deal with the question of price discovery in auction markets. That is, the old question on how "trade" can aggregate dispersed information (see for instance Wilson (1977), whose title is misleading in this regard). The question we are interested in here is rather how prices are set in a market which satisfies the atomistic, individualistic features of a Walrasian Market.

cent paper (Burguet and Sákovics 1999), we take these two points and show that the parallel is a mirage and consequently does not extend to "oligopolistic situations". Setting prices and setting reserve prices are quite different businesses if buyers cannot attend more than one auction.

More importantly, the "perfect competition" result is not a true Walrasian result for at least two reasons. First, the fact that reserve prices converge to marginal cost does not mean that the real price does the same. Second, neither does it mean that trade exhausts the potential gains from trade. As the number of agents grows, buyers' choices of what auction to attend approach a pure random choice. Then, even as the number of agents converge to infinity, the number of expected buyers in any one auction remains constant, (equal to the ratio between buyers and sellers), resulting in both price dispersion (realized prices, in general above marginal cost, still depend on the particular realizations of the valuations of the buyers who happen to attend one auction) and ineffiencies (a positive measure of sellers will happen not to have buyers, and some buyers who are not winners in the auction they attend could be winners in other auctions, including those that do not have buyers). Efficiency in trade and lack of price dispersion, both features of a Walrasian solution, are not attained at the limit.

Then, what type of auction model of competitive price formation could one envision? In my view, a satisfactory model of price formation in the vein of auction theory would have to satisfy the following two requirements. First, there should be no central agency imposing trades and prices. The double auction model fails in this. Second, buyer-seller interaction should be frictionless at least at the limit. The models of auction competition we have just discussed fail here. Indeed, buyers could not try to buy from two different sellers. That means a transaction cost which did not vanish as the number of agents gets larger. A model where buyers can bid for the goods of all sellers at a very low cost would be a better alternative. Even though this enterprise is not at all trivial, I hope that the journey that ends here has convinced the reader that the enterprise is a possible one.

6. Some closing remarks

We have come to the end of a tour, one among many possible ones, of the theory of auctions. The emphasis has been on understanding how agents behave strategically in trading situations, and on the effect of the rules of trade on the various outcomes. In this sense, the last part of our tour had to be reserved to what auction theory has to say about price formation in markets. We have taken both positive and normative approaches by looking at the design of optimal (or satisfactory) institutions, and also at the workings of well-known ones. Many issues things had to be left aside. To cite only a few, we have not touched on the issue of collusion and its effects on the workings of auctions. Indeed, we have always assumed that each buyer acted independently. The issue of buyers' collusion, however, is an important one, especially when the same set of buyers interact often (contract bidding, etc.). See McAfee and McMillan (1992) for one study of the effects of collusion.

Multidimensional auctions is another important topic we have not analyzed. This is of particular relevance in procurement or public works contracting, where the "goods" that several sellers may offer differ in quality or in other non-price dimensions. The auction design has to weigh up these different dimensions and provide the correct incentives for the mix, and at the same time limit informational rents (see Che (1993) and Branco (1997)). The list of other topics could continue with the inclusion of income effects, for instance due to financial constraints suffered by the bidders (see Che and Gale (1994)), or the analysis of auctions for heterogeneous (from the buyers' viewpoint) goods (see Bernhardt and Scoones (1994)), etc. But rather than providing a long list of references for everything that has been analyzed and discussed in the literature, the above pages try to synthesize the basic tools and themes of a theory that today, I believe, represents the most solid grounds on which to construct a theory of market and institution design in the presence of incomplete information.

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Resumen

Presento una resvista de lo que ha sido la el análisis teórico de las subastas en las dos últimas décadas. El objetivo es ofrecer una exposición sistemática de los asuntos principales que se han tratado por esta literatura con más énfasis en el por qué que en el qué. Para ello, presento un esquema unificado que lleva al lector a través del análisis tanto de cuestiones de diseño como de cuestiones positivas que tienen que ver con el funcionamiento de varias instituciones estándar. Al final del tour, el lector se encontrará en el territorio del diseño de mercados y la teoría de la formación de precios.

Palabras clave: Subastas, Mecanismos, Mercado.

Recepción del original, junio de 1998
Versión final, marzo de 1999