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Non-degenerate bilinear alternating maps $f : V \times V \to V$, $\dim(V) = 3$, over an algebraically closed field

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Abstract

Let V be a three-dimensional vector space over an algebraically closed field $\mathbb{F}$. Non-degenerate elements in $\bigwedge^2 V^* \otimes V$ under the tensorial action of the linear group $GL(V)$, are classified.

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1. Introduction

The goal of this paper is the classification of the non-degenerate bilinear alternating maps $f : V \times V \to V$, where V is a three-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic different from 2, under the tensorial action of the full linear group $GL(V)$ in $\bigwedge^2 V^* \otimes V$.

This classification is obtained in terms of a scalar invariant $\sigma_2(f)$ attached to each non-degenerate $f \in \bigwedge^2 V^* \otimes V$, which is defined in Section 4; the precise result

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about the classification is stated in Theorem 12. As $\dim GL(V) = \dim \bigwedge^2 V^* \otimes V = 9$, the existence of such an invariant could not be expected in principle, but it really appears due to the fact that the isotropy group of each non-degenerate bilinear alternating map is not discrete. In fact, as a consequence of the results obtained in Section 7, it follows that the isotropy group is generically an algebraic subgroup of dimension 1. The invariant $\sigma_2(f)$ is a rational function in terms of the matrix of f and, hence, algorithmically computable. Accordingly, the classification theorem allows one to determine explicitly whether two given non-degenerate bilinear alternating maps are equivalent or not. In Section 5, the algebraic structure of the corresponding moduli space is also obtained and, in Section 6, some examples are discussed.

The classification of bilinear alternating maps on the plane are trivial, as all non-vanishing bilinear alternating maps $f : V \times V \to V$ are equivalent when $\dim V = 2$. On the other hand, the classification problem in dimensions greater than three seems to be rather difficult and, in fact, the techniques that we use below are not applicable, as there is no canonical volume attached to a non-degenerate bilinear alternating map in higher dimensions. Hence the role of the dimension three in studying bilinear alternating maps, looks like being quite singular.

2. Notations and preliminaries

Let V be a finite-dimensional vector space over a field $\mathbb{F}$. We recall that the natural action of the full linear group $GL(V)$ on the tensor algebra of V is given by

$$(A \cdot t)(v_1, \ldots, v_p; w_1, \ldots, w_q) = t(A^{-1}v_1, \ldots, A^{-1}v_p; A^*w_1, \ldots, A^*w_q)$$

for all $v_i \in V$, $1 \leqslant i \leqslant p$; $w_j \in V^*$, $1 \leqslant j \leqslant q$, and $t \in (\otimes^p V^*) \otimes (\otimes^q V)$ is a tensor of covariant degree p and contravariant degree q, $A^* : V^* \to V^*$ being the dual map of $A \in GL(V)$; i.e., $A^*(w) = w \circ A$, $\forall w \in V^*$.

Two tensors t, t' of the same type are said to be equivalent if there exists $A \in GL(V)$ such that $A \cdot t = t'$. The isotropy group of t under this action is denoted by $\mathrm{Aut}(t) = \{A \in GL(V) : A \cdot t = t\}$.

The space $\bigwedge^2 V^* \otimes V$ is identified to the space of bilinear alternating maps $f : V \times V \to V$ by setting $t(v_1, v_2, w_1) = w_1(f(v_1, v_2))$, $\forall v_1, v_2 \in V$, $\forall w_1 \in V^*$. In terms of this identification, the $GL(V)$-action is given by the following formula: $(A \cdot f)(v_1, v_2) = A(f(A^{-1}(v_1), A^{-1}(v_2)))$.

Given a non-degenerate bilinear symmetric form $g : V \times V \to \mathbb{F}$, there exist isomorphisms induced from g, $\flat_g : V \to V^*$ and $\natural_g : V^* \to V$, which are inverse one each other. The first one is defined by $\flat_g(v)(x) = g(v, x)$, $\forall v, x \in V$ (e.g., see [1, 3.1.1]).

If we consider a non-degenerate contravariant bilinear symmetric form $g : V^* \times V^* \to \mathbb{F}$, as we shall do below, then V should be substituted for V^* and conversely in the previous formulas, by using the isomorphism $(V^*)^* \cong V$ induced by duality.

Hence in the contravariant case, the "flat" and "sharp" maps are defined as follows: $\flat_g : V^* \to V$ and $\sharp_g : V \to V^*$.

3. Bilinear alternating maps in dimension 3

3.1. The decomposition associated to a volume

From now onwards, we assume that V is a three-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic different from 2.

For every bilinear alternating map $f : V \times V \to V$ there exists a unique linear map $\tilde{f} : \bigwedge^2 V \to V$ such that $\tilde{f}(v_1 \wedge v_2) = f(v_1, v_2)$, $\forall v_1, v_2 \in V$. Moreover, the elements in $\mathrm{vol}(V) = \bigwedge^3 V - \{0\}$ are called the volumes of V. Every $\mathbf{v} \in \mathrm{vol}(V)$ induces an isomorphism

$$\phi_{\mathbf{v}} : V^* \to \bigwedge^2 V,$$
$$\phi_{\mathbf{v}}(w) = i_w \mathbf{v},$$

where i_w denotes the interior product with respect to the linear form $w \in V^*$ (see [3, III.11.7]). If we choose a basis (v_1, v_2, v_3) of V adapted to $\mathbf{v}$; i.e.,

$$\mathbf{v} = v_1 \wedge v_2 \wedge v_3, \tag{1}$$

then $i_w \mathbf{v} = w(v_1) v_2 \wedge v_3 - w(v_2) v_1 \wedge v_3 + w(v_3) v_1 \wedge v_2$. This formula proves that $\phi_{\mathbf{v}}$ is an isomorphism. If we consider another volume $\lambda\mathbf{v}$, $\lambda \in \mathbb{F}^*$, then we have

$$\phi_{\lambda\mathbf{v}} = \lambda \phi_{\mathbf{v}}. \tag{2}$$

By composing $\tilde{f}$ and $\phi_{\mathbf{v}}$ we obtain a linear map $\tilde{f}_{\mathbf{v}} = \tilde{f} \circ \phi_{\mathbf{v}} : V^* \to V$, which, in turn, induces a bilinear form

$$\bar{f}_{\mathbf{v}} : V^* \times V^* \to \mathbb{F},$$
$$\bar{f}_{\mathbf{v}}(w_1, w_2) = w_1(\tilde{f}_{\mathbf{v}}(w_2)).$$

Proposition 1. *Once a volume* $\mathbf{v} \in \mathrm{vol}(V)$ *has been fixed, the map* $f \mapsto \bar{f}_{\mathbf{v}}$ *induces an isomorphism between the vector space of the bilinear alternating maps* $f : V \times V \to V$ *and the vector space of bilinear forms* $\bar{f}_{\mathbf{v}} : V^* \times V^* \to \mathbb{F}$.

The proof of this proposition is straightforward and therefore it is omitted.

Proposition 2. *For every* $\mathbf{v} \in \mathrm{vol}(V)$, *the isomorphism* $f \mapsto \bar{f}_{\mathbf{v}}$ *is* $GL(V)$*-equivariant of weight* -1; *namely*,

$$(\overline{A \cdot f})_{\mathbf{v}} = \frac{1}{\det A}(A \cdot \bar{f}_{\mathbf{v}}), \quad \forall A \in GL(V).$$

Proof. In fact, if (v_1, v_2, v_3) is a basis of V adapted to $\mathbf{v}$ (see the formula (1)) from the definitions of $\bar{f}_{\mathbf{v}}$ and $\tilde{f}_{\mathbf{v}}$ we have

$$\begin{aligned}
&(\overline{A \cdot f})_{\mathbf{v}}(w_1, w_2)\\
&\quad = w_1((\widetilde{A \cdot f})_{\mathbf{v}}(w_2)) = w_1((\widetilde{A \cdot f})(\phi_{\mathbf{v}}(w_2)))\\
&\quad = w_1[(\widetilde{A \cdot f})(w_2(v_1)v_2 \wedge v_3 - w_2(v_2)v_1 \wedge v_3 + w_2(v_3)v_1 \wedge v_2)]\\
&\quad = w_1[w_2(v_1)(A \cdot f)(v_2, v_3) - w_2(v_2)(A \cdot f)(v_1, v_3)\\
&\qquad + w_2(v_3)(A \cdot f)(v_1, v_2)]\\
&\quad = w_1[w_2(v_1)A(f(A^{-1}v_2, A^{-1}v_3)) - w_2(v_2)A(f(A^{-1}v_1, A^{-1}v_3))\\
&\qquad + w_2(v_3)A(f(A^{-1}v_1, A^{-1}v_2))].
\end{aligned}$$

Furthermore, if $\mathbf{v}' = v_1' \wedge v_2' \wedge v_3'$, where $A(v_i') = v_i$, $i = 1, 2, 3$, then

$$\begin{aligned}
&(A \cdot (\bar{f}_{\mathbf{v}'}))(w_1, w_2)\\
&\quad = \bar{f}_{\mathbf{v}'}(A^*(w_1), A^*(w_2)) = (A^*w_1)(\tilde{f}_{\mathbf{v}'}(A^*w_2))\\
&\quad = (A^*w_1)(\tilde{f}(\phi_{\mathbf{v}'}(A^*w_2))) = (A^*w_1)\tilde{f}(i_{A^*w_2}\mathbf{v}')\\
&\quad = (A^*w_1)\tilde{f}((A^*w_2)(v_1')v_2' \wedge v_3' - (A^*w_2)(v_2')v_1' \wedge v_3' + (A^*w_2)(v_3')v_1' \wedge v_2')\\
&\quad = (w_1 \circ A)[w_2(Av_1')f(v_2', v_3') - w_2(Av_2')f(v_1', v_3') + w_2(Av_3')f(v_1', v_2')]\\
&\quad = w_1[w_2(v_1)A(f(A^{-1}v_2, A^{-1}v_3)) - w_2(v_2)A(f(A^{-1}v_1, A^{-1}v_3))\\
&\qquad + w_2(v_3)A(f(A^{-1}v_1, A^{-1}v_2))]
\end{aligned}$$

and we can conclude. □

Let $f : V \times V \to V$ be a bilinear alternating map. Given a volume $\mathbf{v}$, let $\bar{f}_{\mathbf{v}} = g^f_{\mathbf{v}} + h^f_{\mathbf{v}}$ be the decomposition of $\bar{f}_{\mathbf{v}}$ into a sum of a symmetric bilinear form $g^f_{\mathbf{v}} : V^* \times V^* \to \mathbb{F}$ and a skewsymmetric bilinear form $h^f_{\mathbf{v}} : V^* \times V^* \to \mathbb{F}$. The bilinear alternating map $f : V \times V \to V$ is said to be *non-degenerate* if its symmetric part is non-degenerate (e.g., see [5, III.6]). We remark that this notion does not depend on the volume chosen, as from the formula (2) we deduce $\bar{f}_{\lambda\mathbf{v}} = \lambda \bar{f}_{\mathbf{v}}$, and hence,

$$g^f_{\lambda\mathbf{v}} = \lambda g^f_{\mathbf{v}}, \quad h^f_{\lambda\mathbf{v}} = \lambda h^f_{\mathbf{v}}. \tag{3}$$

Accordingly, if $g^f_{\mathbf{v}}$ is non-degenerate, $g^f_{\lambda\mathbf{v}} = \lambda g^f_{\mathbf{v}}$ also is.

3.2. *Matrix expression*

Proposition 3. *Let $f : V \times V \to V$ be a bilinear alternating map. Given a basis (v_1, v_2, v_3) of V, let $M_f = (f_{hi})$ be the 3×3 matrix defined by the formulas*

$$f(v_1, v_2) = \sum_{j=1}^{3} f_{j3}v_j, \quad f(v_2, v_3) = \sum_{j=1}^{3} f_{j1}v_j, \quad f(v_3, v_1) = \sum_{j=1}^{3} f_{j2}v_j.$$

We have

$$M_{A\cdot f} = \frac{1}{\det A} A M_f A^t, \quad \forall A \in GL(V).$$

Proof. We write $(A \cdot f)(v_\alpha, v_\beta) = \sum_{j=1}^{3} f'_{j\gamma} v_j$, where $A = (a_{ij})$ with inverse $A^{-1} = (a^{ij})$, and the pair (α, β) runs the set $\{(1, 2), (2, 3), (3, 1)\}$ and $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. Then, we have

$$\begin{aligned}(A \cdot f)(v_1, v_2) &= A(f(A^{-1}(v_1), A^{-1}(v_2))) = a^{i1}a^{j2}A(f(v_i, v_j))\\ &= (a^{11}a^{22} - a^{21}a^{12})A(f(v_1, v_2))\\ &\quad + (a^{21}a^{32} - a^{31}a^{22})A(f(v_2, v_3))\\ &\quad + (a^{31}a^{12} - a^{11}a^{32})A(f(v_3, v_1))\\ &= \sum_{k=1}^{3}[A^{33} f_{k3} + A^{13} f_{k1} + A^{23} f_{k2}]A(v_k),\end{aligned}$$

where $A^{ij} = a_{ji}/\det A$ denotes the cofactor of the element a^{ij} in A^{-1}. Hence

$$\begin{aligned}\sum_{j=1}^{3} f'_{j3} v_j &= \frac{1}{\det A} \sum_{j,k=1}^{3} (a_{33} f_{k3} + a_{31} f_{k1} + a_{32} f_{k2}) a_{jk} v_j\\ &= \frac{1}{\det A} \sum_{i,j,k=1}^{3} a_{jk} f_{ki} a^t_{i3} v_j\end{aligned}$$

and then

$$f'_{j3} = \frac{1}{\det A} \sum_{i,k=1}^{3} a_{jk} f_{ki} a^t_{i3}, \quad 1 \leqslant j \leqslant 3.$$

The proof for the other pairs of indices (2, 3) and (3, 1) is the same. □

Throughout the paper, M_f is called the matrix associated to f in the basis (v_1, v_2, v_3). Note that such a matrix determines completely the bilinear alternating map.

Proposition 4. *If (v_1, v_2, v_3) is an adapted basis for $\mathbf{v} \in \mathrm{vol}(V)$, then the matrix of $\bar{f}_{\mathbf{v}}$ in the dual basis (v_1^*, v_2^*, v_3^*) is the same as the matrix associated to f; i.e., $M_{\bar{f}_{\mathbf{v}}} = M_f$.*

Proof. We have

$$\begin{aligned}\bar{f}_{\mathbf{v}}(v_1^*, v_2^*) &= v_1^*(\tilde{f}_{\mathbf{v}}(v_2^*)) = v_1^*(\tilde{f}(\phi_{\mathbf{v}}(v_2^*)))\\ &= v_1^*(\tilde{f}(v_3 \wedge v_1)) = v_1^*(f(v_3, v_1)) = f_{12}\end{aligned}$$

and similarly for the rest of pairs (v_i^*, v_j^*). □

3.3. The canonical volume

Proposition 5. *Every non-degenerate bilinear symmetric form canonically induces two volumes, which are opposite.*

Proof. Let $g : V^* \times V^* \to \mathbb{F}$ be a non-degenerate bilinear symmetric form. If (v_1^*, v_2^*, v_3^*) is an orthonormal basis for g, then one of these volumes is $\mathbf{v} = v_1 \wedge v_2 \wedge v_3$ and the other is $-\mathbf{v} = v_2 \wedge v_1 \wedge v_3$. The definition of $\{\pm\mathbf{v}\}$ does not depend on the orthonormal basis chosen, as the matrix of the basis change is an orthogonal matrix and, hence, its determinant equals ± 1. $\square$

In what follows, we denote by $\mathrm{vol}(g)$ the volumes $\{\pm\mathbf{v}\}$ induced by the non-degenerate bilinear symmetric form g according to the previous proposition.

Proposition 6. *If $f : V \times V \to V$ is a non-degenerate bilinear alternating map, then there exists a unique volume $\mathbf{v}$ such that* $\mathrm{vol}(g_{\mathbf{v}}^f) = \{\pm\mathbf{v}\}$.

Proof. If $\mathbf{v}_0$ is an arbitrary volume and (v_1, v_2, v_3) is a basis adapted to $\mathbf{v}_0$ (cf. (1)), then any other volume can be written as $\mathbf{v} = \alpha\mathbf{v}_0$, $\alpha \in \mathbb{F}^*$. If $(v_1'^*, v_2'^*, v_3'^*)$ is an orthonormal basis for $g_{\mathbf{v}_0}^f$, then $(\alpha^{-\frac{1}{2}} v_i'^*)$, $i = 1, 2, 3$, is orthonormal for $g_{\mathbf{v}}^f = \alpha g_{\mathbf{v}_0}^f$ (cf. (3)). Hence

$$\begin{aligned}
\mathrm{vol}(g_{\mathbf{v}}^f) &= \pm\left(\alpha^{\frac{1}{2}} v_1'\right) \wedge \left(\alpha^{\frac{1}{2}} v_2'\right) \wedge \left(\alpha^{\frac{1}{2}} v_3'\right) \\
&= \pm\alpha\alpha^{\frac{1}{2}} v_1' \wedge v_2' \wedge v_3' \\
&= \pm\alpha\alpha^{\frac{1}{2}} \det(A) v_1 \wedge v_2 \wedge v_3 \\
&= \pm\alpha\alpha^{\frac{1}{2}} \det(A)\mathbf{v}_0,
\end{aligned}$$

$A \in GL(V)$ being the automorphism given by $A(v_i) = v_i'$, $i = 1, 2, 3$. Therefore $\mathrm{vol}(g_{\mathbf{v}}^f) = \{\pm\mathbf{v}\}$ if and only if $\pm\alpha\alpha^{\frac{1}{2}} \det(A)\mathbf{v}_0 = \pm\alpha\mathbf{v}_0$; or equivalently $\alpha^{\frac{1}{2}} \det(A) = \pm 1$, and then, $\alpha = (\det A)^{-2}$. $\square$

4. The invariant $\sigma_2(f)$

Proposition 7. *If $f : V \times V \to V$ is a non-degenerate bilinear alternating map, then there exists a unique endomorphism $L_f : V^* \to V^*$ such that*

$$g_{\mathbf{v}}^f(w_1, L_f(w_2)) = h_{\mathbf{v}}^f(w_1, w_2), \quad \forall w_1, w_2 \in V^*. \tag{4}$$

In addition, L_f does not depend on the volume $\mathbf{v}$.

Proof. As $g_{\mathbf{v}}^f$ is non-degenerate, the map $\natural_{g_{\mathbf{v}}^f} : V^* \to V$, $\natural_{g_{\mathbf{v}}^f}(w)(w') = g_{\mathbf{v}}^f(w, w')$, $w, w' \in V^*$, is an isomorphism. Hence, for every $w_2 \in V^*$, the linear form on V^*, $w_1 \mapsto h_{\mathbf{v}}^f(w_1, w_2)$, $w_1 \in V^*$, is represented by a unique vector $w_2' \in V^*$; i.e., $g_{\mathbf{v}}^f(w_1, w_2') = h_{\mathbf{v}}^f(w_1, w_2)$, and we can define L_f by setting $L_f(w_2) = w_2'$. The linearity of L_f is readily checked. Finally, let us prove that L_f is independent of the volume chosen. In fact, if we repeat the construction above for another volume $\lambda\mathbf{v}$, then taking the formula (3) into account and by applying the Eq. (4) defining L_f, we can conclude. □

From (4) it follows that the matrix of L_f in the basis (v_1^*, v_2^*, v_3^*) is given by $M_{L_f} = M_{g_{\mathbf{v}}^f}^{-1} M_{h_{\mathbf{v}}^f}$. Let (v_1^*, v_2^*, v_3^*) be an orthonormal basis for $g_{\mathbf{v}}^f$ so that the matrix $M_{g_{\mathbf{v}}^f}$ is the identity matrix I in that basis. Then, the matrix of $h_{\mathbf{v}}^f$ is skewsymmetric,

$$M_{L_f} = M_{h_{\mathbf{v}}^f} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \tag{5}$$

where $h_{\mathbf{v}}^f(v_1^*, v_2^*) = a$, $h_{\mathbf{v}}^f(v_1^*, v_3^*) = b$, $h_{\mathbf{v}}^f(v_2^*, v_3^*) = c$. Therefore, the characteristic polynomial of L_f is

$$\det(xI - M_{L_f}) = \begin{vmatrix} x & -a & -b \\ a & x & -c \\ b & c & x \end{vmatrix} = x^3 + (a^2 + b^2 + c^2)x. \tag{6}$$

Let $\sigma_2(f) = a^2 + b^2 + c^2$ be the coefficient of the linear term in $\det(xI - M_{L_f})$.

Proposition 8. *If $f, f' : V \times V \to V$ are two equivalent non-degenerate bilinear alternating maps, then $\sigma_2(f) = \sigma_2(f')$.*

Proof. If $f' = A \cdot f$, $A \in GL(V)$, then from Proposition 2 we obtain

$$\begin{aligned} g_{\mathbf{v}}^{f'} + h_{\mathbf{v}}^{f'} &= (\overline{A \cdot f})_{\mathbf{v}} = \frac{1}{\det A}(A \cdot \bar{f}_{\mathbf{v}}) \\ &= \frac{1}{\det A}(A \cdot g_{\mathbf{v}}^f) + \frac{1}{\det A}(A \cdot h_{\mathbf{v}}^f). \end{aligned}$$

Hence $g_{\mathbf{v}}^{f'} = \det A^{-1}(A \cdot g_{\mathbf{v}}^f)$, $h_{\mathbf{v}}^{f'} = \det A^{-1}(A \cdot h_{\mathbf{v}}^f)$, and using the very definition of $L_{f'}$, we have $g_{\mathbf{v}}^{f'}(w_1', L_{f'}(w_2')) = h_{\mathbf{v}}^{f'}(w_1', w_2')$, $\forall w_1', w_2' \in V^*$. Substituting $g_{\mathbf{v}}^{f'}$ and $h_{\mathbf{v}}^{f'}$ for their corresponding expressions previously obtained, it yields $(A \cdot g_{\mathbf{v}}^f)(w_1', L_{f'}(w_2')) = (A \cdot h_{\mathbf{v}}^f)(w_1', w_2')$, or equivalently $g_{\mathbf{v}}^f(w_1, (A^* \circ L_{f'} \circ A^{*-1})(w_2)) = h_{\mathbf{v}}^f(w_1, w_2)$, with $w_i = A^* w_i'$, $i = 1, 2$. We thus have

$$L_{f'} = A^{*-1} \circ L_f \circ A^*, \tag{7}$$

and taking the characteristic polynomial on both sides, we can conclude. □

Let $g : V^* \times V^* \to \mathbb{F}$ be a non-degenerate bilinear symmetric form and $\mathbf{v} \in \mathrm{vol}(V)$ a volume. The cross product of two linear forms $w_1, w_2 \in V^*$ with respect to g and $\mathbf{v}$, is given by

$$\left(w_1 \underset{(g,\mathbf{v})}{\times} w_2\right)(v) = g(\natural_g(v), \natural_g(i_{w_2}(i_{w_1}\mathbf{v}))), \quad \forall v \in V.$$

If (v_1^*, v_2^*, v_3^*) is an orthonormal basis for g, i.e., $g = v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3$, then $\mathbf{v} = \lambda(v_1 \wedge v_2 \wedge v_3)$ with

$$\lambda = \mathbf{v}(v_1^*, v_2^*, v_3^*), \tag{8}$$

and we have $v_1^* \times_{(g,\mathbf{v})} v_2^* = \lambda v_3^*$, $v_2^* \times_{(g,\mathbf{v})} v_3^* = \lambda v_1^*$, and $v_3^* \times_{(g,\mathbf{v})} v_1^* = \lambda v_2^*$. Hence, the cross product of two arbitrary linear forms w_1, w_2, is computed by

$$w_1 \underset{(g,\mathbf{v})}{\times} w_2 = \lambda \begin{vmatrix} w_1(v_2) & w_1(v_3) \\ w_2(v_2) & w_2(v_3) \end{vmatrix} v_1^* + \lambda \begin{vmatrix} w_1(v_3) & w_1(v_1) \\ w_2(v_3) & w_2(v_1) \end{vmatrix} v_2^*$$
$$+ \lambda \begin{vmatrix} w_1(v_1) & w_1(v_2) \\ w_2(v_1) & w_2(v_2) \end{vmatrix} v_3^*.$$

Lemma 9. *Let $g : V^* \times V^* \to \mathbb{F}$ be a non-degenerate bilinear form, let $\mathbf{v}$ be a volume, and let (v_1^*, v_2^*, v_3^*) be an orthonormal basis for g. Then, the scalar $\Delta_{g,\mathbf{v}} = \mathbf{v}(v_1^*, v_2^*, v_3^*)^2$ does not depend on the orthonormal basis chosen.*

Proof. If $(v_1'^*, v_2'^*, v_3'^*)$ is another orthonormal basis for g, then the matrix $B \in GL(V)$ defined by $B(v_i) = v_i'$ is orthogonal and, hence $(\det B)^2 = 1$. We have $\mathbf{v}(v_1'^*, v_2'^*, v_3'^*)^2 = (\det B)^2 \mathbf{v}(v_1^*, v_2^*, v_3^*)^2 = \mathbf{v}(v_1^*, v_2^*, v_3^*)^2$. $\square$

Proposition 10. *Let $g : V^* \times V^* \to \mathbb{F}$ and $\mathbf{v} \in \mathrm{vol}(V)$ be as in the previous lemma. If $L : V^* \to V^*$ is an endomorphism such that*

$$g(w_1, L(w_2)) + g(L(w_1), w_2) = 0, \quad \forall w_1, w_2 \in V^*, \tag{9}$$

then a unique linear form $\xi \in V^*$ exists verifying $L(w) = \xi \times_{(g,\mathbf{v})} w$, $\forall w \in V^*$. Furthermore, the coefficient of the linear term in the characteristic polynomial of L, equals to $\Delta_{g,\mathbf{v}} \cdot g(\xi, \xi)$.

Proof. Let (v_1^*, v_2^*, v_3^*) be an orthonormal basis for g. By virtue of Eq. (9), the matrix of L is skewsymmetric; that is, it looks like the matrix M_L in the expression (5). By setting $\xi = \xi_1 v_1^* + \xi_2 v_2^* + \xi_3 v_3^*$ and using the formula above for the cross product, we have $\xi \times_{(g,\mathbf{v})} v_1^* = \lambda(\xi_3 v_2^* - \xi_2 v_3^*)$, $\xi \times_{(g,\mathbf{v})} v_2^* = \lambda(\xi_1 v_3^* - \xi_3 v_1^*)$, and $\xi \times_{(g,\mathbf{v})} v_3^* = \lambda(\xi_2 v_1^* - \xi_1 v_2^*)$, where λ is as in (8). Hence the matrix of the endomorphism $w \mapsto \xi \times_{(g,\mathbf{v})} w$ in this basis, is

$$\begin{pmatrix} 0 & -\lambda\xi_3 & \lambda\xi_2 \\ \lambda\xi_3 & 0 & -\lambda\xi_1 \\ -\lambda\xi_2 & \lambda\xi_1 & 0 \end{pmatrix}. \tag{10}$$

Therefore $a = -\lambda\xi_3$, $b = \lambda\xi_2$, $c = -\lambda\xi_1$. Moreover, we know that the coefficient of the linear term in the characteristic polynomial is $a^2 + b^2 + c^2 = \lambda^2(\xi_1^2 + \xi_2^2 + \xi_3^2) = \Delta_{g,\mathbf{v}} \cdot g(\xi, \xi)$. $\square$

5. The equivalence theorem

From the formula (6) it follows that $\sigma_2(f) = 0$ implies L_f is nilpotent. In this case, let n_f be the least positive integer such that $(L_f)^{n_f} = 0$.

Lemma 11. *If $\sigma_2(f) = 0$, then either $n_f = 1$ (i.e., $L_f = 0$) or $n_f = 3$.*

Proof. From the expression (5) of the matrix of L_f, taking the assumption into account, i.e., $0 = \sigma_2(f) = a^2 + b^2 + c^2$, it follows:

$$M_{L_f^2} = \begin{pmatrix} c^2 & -bc & ac \\ -bc & b^2 & -ab \\ ac & -ab & a^2 \end{pmatrix}.$$

Hence $(L_f)^2 = 0$ implies $L_f = 0$. $\square$

Theorem 12. *If $f, f' : V \times V \to V$ are two non-degenerate bilinear alternating maps, then they are equivalent if and only if, either $\sigma_2(f) = \sigma_2(f') \neq 0$, or $\sigma_2(f) = \sigma_2(f') = 0$ and $n_f = n_{f'}$.*

Proof. The necessity of the conditions follows from Proposition 8, remarking that from the formula (7) we deduce $n_f = n_{f'}$. Conversely, from Proposition 6 there exist volumes such that $\mathrm{vol}(g_{\mathbf{v}}^f) = \{\pm\mathbf{v}\}$, $\mathrm{vol}(g_{\mathbf{v}'}^{f'}) = \{\pm\mathbf{v}'\}$. Let ξ, ξ' denote the linear forms defined as follows (cf. Proposition 10):

$$L_f(w) = \xi \times_{(g_{\mathbf{v}}^f, \mathbf{v})} w, \quad L_{f'}(w) = \xi' \times_{(g_{\mathbf{v}'}^{f'}, \mathbf{v}')} w.$$

We are led to distinguish two cases:

1. If $\sigma_2(f) \neq 0$, then $g_{\mathbf{v}}^f(\xi, \xi) \neq 0$ and $g_{\mathbf{v}'}^{f'}(\xi', \xi') \neq 0$ as, from Proposition 10, we have $\sigma_2(f) = \Delta_{g_{\mathbf{v}}^f, \mathbf{v}} \cdot g_{\mathbf{v}}^f(\xi, \xi)$ and $\sigma_2(f) = \sigma_2(f')$ by virtue of the hypothesis. Set $k = \sigma_2(f)^{\frac{1}{2}} = \sigma_2(f')^{\frac{1}{2}}$ and

$$v_1^* = \frac{\xi}{g_{\mathbf{v}}^f(\xi, \xi)^{\frac{1}{2}}}, \quad v_1'^* = \frac{\xi'}{g_{\mathbf{v}'}^{f'}(\xi', \xi')^{\frac{1}{2}}}.$$

As $g_{\mathbf{v}}^f(v_1^*, v_1^*) = g_{\mathbf{v}'}^{f'}(v_1'^*, v_1'^*) = 1$, the vectors v_1^*, $v_1'^*$ can be included into orthonormal bases (v_1^*, v_2^*, v_3^*), $(v_1'^*, v_2'^*, v_3'^*)$ for $g_{\mathbf{v}}^f$ and $g_{\mathbf{v}'}^{f'}$, respectively. As $\xi =$

$g^f_{\mathbf{v}}(\xi,\xi)^{\frac{1}{2}}v_1^*$, $\xi' = g^{f'}_{\mathbf{v}'}(\xi',\xi')^{\frac{1}{2}}v_1'^*$, from the formula (10) it follows that the matrix of L_f in the basis (v_1^*, v_2^*, v_3^*) is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda g^f_{\mathbf{v}}(\xi,\xi)^{\frac{1}{2}} \\ 0 & \lambda g^f_{\mathbf{v}}(\xi,\xi)^{\frac{1}{2}} & 0 \end{pmatrix},$$

where λ is given in (8). Since $\mathbf{v} \in \mathrm{vol}(g^f_{\mathbf{v}})$, we have $\lambda = \pm 1$; hence $\Delta_{g^f_{\mathbf{v}},\mathbf{v}} = 1$ by Lemma 9. Therefore $k = g^f_{\mathbf{v}}(\xi,\xi)^{\frac{1}{2}}$. The same holds for f'. Making a permutation of the basis elements, if necessary, we can further assume that the matrices of L_f and $L_{f'}$ in the bases (v_1^*, v_2^*, v_3^*) and $(v_1'^*, v_2'^*, v_3'^*)$, respectively, are equal. Let A be the automorphism defined by $A^*(v_i'^*) = v_i^*$, or equivalently $A(v_i) = v_i'$, $i = 1, 2, 3$. We have $L_{f'} = A^{*-1} \circ L_f \circ A^*$ and hence $A \cdot g^f_{\mathbf{v}} = g^{f'}_{\mathbf{v}'}$, as A transforms an orthonormal basis for $g^f_{\mathbf{v}}$ into an orthonormal basis for $g^{f'}_{\mathbf{v}'}$. We also have $A \cdot h^f_{\mathbf{v}} = h^{f'}_{\mathbf{v}'}$, as follows from the definition of L_f in the formula (4). Hence $A \cdot \bar{f}_{\mathbf{v}} = \overline{f'}_{\mathbf{v}'}$.

Moreover, we have $\mathbf{v} = \pm v_1 \wedge v_2 \wedge v_3$ and $\mathbf{v}' = \pm v_1' \wedge v_2' \wedge v_3'$, so that $A \cdot \mathbf{v} = \pm \mathbf{v}' = (\det A)\mathbf{v}$. Taking the Proposition 2 and the formula (3) into account, we obtain $(\det A)(\overline{A \cdot f})_{\mathbf{v}} = A \cdot \bar{f}_{\mathbf{v}} = \overline{f'}_{\mathbf{v}'} = \pm(\det A)\overline{f'}_{\mathbf{v}}$. Hence $(\overline{A \cdot f})_{\mathbf{v}} = \pm\overline{f'}_{\mathbf{v}}$. If $(\overline{A \cdot f})_{\mathbf{v}} = \overline{f'}_{\mathbf{v}}$, then by virtue of the Proposition 1 we conclude that $A \cdot f = f'$. If $(\overline{A \cdot f})_{\mathbf{v}} = -\overline{f'}_{\mathbf{v}}$, then we can conclude similarly, taking account of the identity $(-A) \cdot f = -(A \cdot f)$.

2. If $\sigma_2(f) = 0$, then, as previously we have $g^f_{\mathbf{v}}(\xi,\xi) = 0$, and similarly $g^{f'}_{\mathbf{v}'}(\xi',\xi') = 0$. We distinguish two subcases:

(i) If $\xi = 0$, then $\xi' = 0$ since $n_f = n_{f'}$. Moreover, as $L_f = L_{f'} = 0$, we have $h^f_{\mathbf{v}} = h^{f'}_{\mathbf{v}'} = 0$. Let (v_1^*, v_2^*, v_3^*) (resp. $(v_1'^*, v_2'^*, v_3'^*)$) be an orthonormal basis for $g^f_{\mathbf{v}}$ (resp. $g^{f'}_{\mathbf{v}'}$). Let $A \in GL(V)$ be the automorphism defined by $A^*(v_i'^*) = v_i^*$, $i = 1, 2, 3$. Then $A \cdot g^f_{\mathbf{v}} = g^{f'}_{\mathbf{v}'}$, and hence $A \cdot \bar{f}_{\mathbf{v}} = \overline{f'}_{\mathbf{v}'}$. We can conclude behaving as in the case 1.

(ii) If $\xi \neq 0$, then $n_f = 3$ by virtue of Lemma 11; hence $\xi' \neq 0$ since $n_f = n_{f'}$. The isotropic vectors ξ, ξ' can be included into hyperbolic pairs, respectively (see [2, Définition 3.8]). Hence, there exist bases (v_1^*, v_2^*, v_3^*) and $(v_1'^*, v_2'^*, v_3'^*)$ such that,

$$\begin{aligned}
&v_1^* = \xi, && v_1'^* = \xi', \\
&g^f_{\mathbf{v}}(v_1^*, v_1^*) = 0, && g^{f'}_{\mathbf{v}'}(v_1'^*, v_1'^*) = 0, \\
&g^f_{\mathbf{v}}(v_2^*, v_2^*) = 0, && g^{f'}_{\mathbf{v}'}(v_2'^*, v_2'^*) = 0, \\
&g^f_{\mathbf{v}}(v_1^*, v_2^*) = 1, && g^{f'}_{\mathbf{v}'}(v_1'^*, v_2'^*) = 1, \\
&g^f_{\mathbf{v}}(v_3^*, v_3^*) = 1, && g^{f'}_{\mathbf{v}'}(v_3'^*, v_3'^*) = 1.
\end{aligned}$$

Again, let $A \in GL(V)$ be the automorphism $A^*(v_i'^*) = v_i^*$, $i = 1, 2, 3$. From the previous formulas we conclude $A \cdot g_{\mathbf{v}}^f = g_{\mathbf{v}'}^{f'}$. In order to state the equation $L_{f'} = A^{*-1} \circ L_f \circ A^*$, we proceed as follows. We set

$$\hat{v}_1 = \frac{1}{\sqrt{2}}(v_1 + v_2), \quad \hat{v}_2 = \frac{1}{\sqrt{-2}}(v_1 - v_2), \quad \hat{v}_3 = v_3,$$

and similarly, we define $(\hat{v}_1', \hat{v}_2', \hat{v}_3')$ in terms of the basis (v_1', v_2', v_3'). We have $A^*(\hat{v}_i'^*) = \hat{v}_i^*$, $i = 1, 2, 3$, $\mathbf{v} = \pm\hat{v}_1 \wedge \hat{v}_2 \wedge \hat{v}_3$, and $\mathbf{v}' = \pm\hat{v}_1' \wedge \hat{v}_2' \wedge \hat{v}_3'$, as the bases $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ and $(\hat{v}_1', \hat{v}_2', \hat{v}_3')$ are orthonormal for $g_{\mathbf{v}}^f$ and $g_{\mathbf{v}'}^{f'}$, respectively. As a simple calculation shows, the matrix of L_f in the basis $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ as well as that of $L_{f'}$ in the basis $(\hat{v}_1', \hat{v}_2', \hat{v}_3')$ both are equal to

$$\begin{pmatrix} 0 & 0 & \frac{-1}{\sqrt{-2}} \\ 0 & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{-2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

and we can conclude as in the previous cases. □

Once a basis (v_1, v_2, v_3) of V has been fixed, the ring of algebraic functions on the vector space $\bigwedge^2 V^* \otimes V$ is generated by the functions x_{ij} defined by $x_{ij}(f) = f_{ij}$, where $M_f = (f_{ij})$ is the matrix associated to f in the basis (v_1, v_2, v_3), according to Proposition 3. The matrices of the bilinear forms $g_{\mathbf{v}}^f$ and $h_{\mathbf{v}}^f$ in the basis (v_1^*, v_2^*, v_3^*) are, respectively $M_{g_{\mathbf{v}}^f} = \frac{1}{2}(M_f + M_f^t)$, $M_{h_{\mathbf{v}}^f} = \frac{1}{2}(M_f - M_f^t)$. Hence f is non-degenerate if and only if $\det(M_f + M_f^t) = \det(f_{ij} + f_{ji}) \neq 0$. Let $p \in \mathbb{F}[x_{ij}]_{i,j=1,\ldots,3}$ be the polynomial

$$p(x_{11}, x_{12}, \ldots, x_{32}, x_{33}) = \det(x_{ij} + x_{ji})_{i,j=1,\ldots,3}.$$

The non-degenerate bilinear alternating maps are the closed points of Zariski's open subset $\mathcal{O} \subset \bigwedge^2 V^* \otimes V$ defined as the complement of the zeroes of p; that is,

$$\mathcal{O} = \operatorname{Spec}\mathbb{F}[x_{ij}]_{i,j=1,\ldots,3} \setminus \{\mathfrak{p} \in \operatorname{Spec}\mathbb{F}[x_{ij}]_{i,j=1,\ldots,3} : (p) \subseteq \mathfrak{p}\}.$$

The ring of algebraic functions on $\mathcal{O}$ is obtained by localizing the ring of polynomials $\mathbb{F}[x_{ij}]_{i,j=1,\ldots,3}$ with the multiplicative systems of the powers of p. Hence, we have a morphism of affine varieties $\sigma_2 : \mathcal{O} \to \mathbb{A}^1(\mathbb{F})$ such that $\sigma_2(f)$ is the invariant attached to every closed point $f \in \mathcal{O}$. Denoting by x the coordinate in the affine line $\mathbb{A}^1(\mathbb{F})$ we have

$$\begin{aligned}(\sigma_2)^*(x) = \frac{2}{p}(&x_{12}^2x_{33} + x_{13}^2x_{22} + x_{21}^2x_{33} + x_{23}^2x_{11} + x_{31}^2x_{22} \\ &+ x_{32}^2x_{11} - 2x_{11}x_{23}x_{32} - x_{12}x_{13}x_{23} - x_{12}x_{13}x_{32} \\ &-2x_{12}x_{21}x_{33} + 3x_{12}x_{23}x_{31} - x_{12}x_{31}x_{32} - x_{13}x_{21}x_{23} \\ &+ 3x_{13}x_{21}x_{32} - 2x_{13}x_{22}x_{31} - x_{21}x_{23}x_{31} - x_{21}x_{31}x_{32}).\end{aligned}$$

According to Theorem 12, the fibre of σ_2 over a point $\alpha \in \mathbb{A}^1(\mathbb{F})$ different from the origin, coincides with an orbit of $GL(V)$, whereas the fibre of σ_2 over the origin decomposes into two orbits of the linear group. Hence we can conclude that the quotient $\mathcal{O}/GL(V)$ is "an affine line with the origin doubled"; for example, see [4, Example 2.3.6]. This is the global structure of the moduli space of non-degenerate bilinear alternating maps on V.

6. Some examples

6.1. Reduced matrices

According to Theorem 12, for every non-degenerate bilinear alternating map $f : V \times V \to V$ there exists a basis (v_1, v_2, v_3) such that the matrix of f in that basis is

1. If $\sigma_2(f) \neq 0$, then
$$M_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix},$$
where α runs over a set of representatives in the quotient group $\mathbb{F}^*/\{\pm 1\}$.
2. If $\sigma_2(f) = 0$ and $L_f = 0$, then
$$M_f = M_{g_{\mathbf{v}}^f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
3. If $\sigma_2(f) = 0$ and $L_f \neq 0$, then
$$M_f = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ \sqrt{-1} & 0 & 0 \\ -1 & 0 & \sqrt{-1} \end{pmatrix}.$$

6.2. Lie algebra structures

Let us study which non-degenerate bilinear alternating maps $f : V \times V \to V$ define a Lie algebra structure on V. We use a basis (v_1, v_2, v_3) on which the matrix of f is one of the three reduced matrices above. As the map

$$\begin{aligned} &V \times V \times V \to V, \\ &(x, y, z) \mapsto f(f(x, y), z) + f(f(y, z), x) + f(f(z, x), y) \end{aligned}$$

is trilinear and alternating, in order to check the Jacobi identity, we only need to prove the following:

$$f(f(v_1, v_2), v_3) + f(f(v_2, v_3), v_1) + f(f(v_3, v_1), v_2) = 0. \tag{11}$$

As a simple computation shows, the Eq. (11) does not hold for the matrix of types in items 1 and 3 above, and it does hold for the matrix in item 2. Hence we conclude that the only three-dimensional Lie algebra over $\mathbb{F}$ represented by a non-degenerate bilinear alternating map is $\mathfrak{sl}(2, \mathbb{F})$, up to isomorphisms. The relation between the basis (v_1, v_2, v_3) and the standard basis for $\mathfrak{sl}(2, \mathbb{F})$ (see [6, IV.1]), i.e.,

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is as follows: $v_1 = \frac{1}{\sqrt{2}}X$, $v_2 = \frac{1}{\sqrt{2}}Y$, $v_3 = \frac{1}{2}H$.

Another very important class of three-dimensional algebras is that of associative algebras but, unfortunately, none of them is defined by a non-degenerate bilinear alternating map, as a simple computation shows by imposing the condition $v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3$, which is necessary for associativity. In fact, no three-dimensional alternative algebra (in the sense of [3, III.Appendice]) may be represented by a non-degenerate bilinear alternating map.

Finally, by using the reduced matrices as above, it is readily proved that the center of the algebras defined by non-degenerate bilinear alternating maps, vanishes; i.e., the equation $f(x, y) = 0$ for all $y \in V$ implies $x = 0$.

7. Automorphisms

Lemma 13. *Let $f : V \times V \to V$ be a non-degenerate bilinear alternating map. A linear transformation $A \in GL(V)$ keeps f invariant if and only if it keeps invariant both $g_{\mathbf{v}}^f$ and $h_{\mathbf{v}}^f$, that is,* $\mathrm{Aut}(f) = \mathrm{SAut}(g_{\mathbf{v}}^f) \cap \mathrm{Aut}(h_{\mathbf{v}}^f)$, *where* $\mathrm{SAut}(g_{\mathbf{v}}^f) = \{A \in \mathrm{Aut}(g_{\mathbf{v}}^f) : \det A = 1\}$.

Proof. From Proposition 2 it follows that $A \in \mathrm{Aut}(f)$ if and only if $A \cdot g_{\mathbf{v}}^f = (\det A) g_{\mathbf{v}}^f$ and $A \cdot h_{\mathbf{v}}^f = (\det A) h_{\mathbf{v}}^f$. Taking determinants in both sides of the first equation we deduce that $\det A = 1$, as $\dim V = 3$. This completes the proof. □

In what follows we use the reduced matrices obtained in the previous section, in order to compute the group of isotropy of f. We also use a basis (v_1, v_2, v_3) on which the matrix of f is reduced as in Theorem 12. Maple 7 has been employed for computations.

1. If $\sigma_2(f) \neq 0$, then $\mathrm{Aut}(g_{\mathbf{v}}^f) = O(3, \mathbb{F})$ and we obtain $\mathrm{Aut}(f) \cong SO(2, \mathbb{F})$.
2. If $\sigma_2(f) = 0$ and $L_f = 0$, then $\mathrm{Aut}(f) = SO(3, \mathbb{F})$.
3. If $\sigma_2(f) = 0$ and $L_f \neq 0$, then $\mathrm{Aut}(f) \cong (\mathbb{F}, +)$.

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