On an implication-free reduct of MV_n chains

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Abstract

Let $\mathbf{Ł}_{n+1}$ be the MV-chain on the $n+1$ elements set $\text{Ł}_{n+1} = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ in the algebraic language $\{\rightarrow, \neg\}$ [3]. As usual, further operations on Ł_{n+1} are definable by the following stipulations: $1 = x \rightarrow x$, $0 = \neg 1$, $x \oplus y = \neg x \rightarrow y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \wedge y = x \odot (x \rightarrow y)$, $x \vee y = \neg(\neg x \wedge \neg y)$. Moreover, we will pay special attention to the also definable unary operator ${}^*x = x \odot x$.

In fact, the aim of this paper is to study the $\{^*, \neg, \vee\}$-reducts of the MV-chains $\mathbf{Ł}_{n+1}$, that will be denoted as $\mathbf{Ł}^*_{n+1}$, i.e. the algebra on Ł_{n+1} obtained by replacing the implication operator $\rightarrow$ by the unary operation * which represents the square operator ${}^*x = x \odot x$ and which has been recently used in [4] to provide, among other things, an alternative axiomatization for the four-valued matrix logic $J_4 = \langle \mathbf{Ł}_4, \{1/3, 2/3, 1\}\rangle$. In this contribution we make a step further in studying the expressive power of the * operation, in particular we will focus on the question for which natural numbers n the structures $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. In other words, for which n the Łukasiewicz implication $\rightarrow$ is definable in $\mathbf{Ł}^*_{n+1}$, or equivalenty, for which n $\mathbf{Ł}^*_{n+1}$ is in fact an MV-algebra. We also show that, in any case, the matrix logics $\langle \mathbf{Ł}^*_{n+1}, F\rangle$, where F is an order filter, are algebraizable. What we present here is a work in progress.

Term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$

Let X be a subset of Ł_{n+1}. We denote by $\langle X\rangle^*$ the subalgebra of $\mathbf{Ł}^*_{n+1}$ generated by X (in the reduced language $\{^*, \neg, \vee\}$). For $n \geq 1$ define recursively $(^*)^n x$ as follows: $(^*)^1 x = {}^*x$, and $(^*)^{i+1}x = {}^*((^*)^i x)$, for $i \geq 1$.

A nice feature of the $\mathbf{Ł}^*_{n+1}$ algebras is that we can always define terms characterising the principal order filters $F_a = \{b \in \text{Ł}_{n+1} \mid a \leq b\}$, for every $a \in \text{Ł}_{n+1}$.

Proposition 1. *For each $a \in L_{n+1}$, the unary operation Δ_a defined as*

$$\Delta_a(x) = \begin{cases} 1 & \text{if } x \in F_a \\ 0 & \text{otherwise.} \end{cases}$$

is definable in $\mathbf{Ł}^*_{n+1}$. As a consequence, for every $a \in \text{Ł}_{n+1}$, the operation χ_a that corresponds to the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

Proof. The case $a = 1$ corresponds to the Monteiro-Baaz Delta operator and, as is well-known, it can be defined as $\Delta_1(x) = (^*)^n x$. For $a = 0$ define $\Delta_a(x) = \Delta_1(x) \vee \neg\Delta_1(x)$; then $\Delta_a(x) = 1$ for every x. Now, assume $0 < a = i/n < 1$. It is not difficult to show that one can always find a sequence of terms (operations) $t_1(x), \ldots, t_m(x)$ over $\{*, \neg\}$ such that $t_1(t_2(\ldots(t_m(x))\ldots)) = 1$

if $x \in F_a$ while $t_1(t_2(\ldots(t_m(x))\ldots)) < 1$ otherwise. Then $\Delta_a(x) = \Delta_1(t_1(t_2(\ldots(t_m(x))\ldots)))$ for every x.

As for the operations χ_a, define $\chi_1 = \Delta_1$, $\chi_0 = \neg\Delta_{1/n}$, and if $0 < a < 1$, then define $\chi_a = \Delta_a \wedge \neg\Delta_{a-1/n}$. □

It is now almost immediate to check that the following implication-like operation is definable in every $\mathbf{Ł}^*_{n+1}$: $x \Rightarrow y = 1$ if $x \le y$ and 0 otherwise. Indeed, $\Rightarrow$ can be defined as

$$x \Rightarrow y = \bigvee_{0\le i\le j\le n} (\chi_{i/n}(x) \wedge \chi_{j/n}(y)).$$

Actually, one can also define Gödel implication on $\mathbf{Ł}^*_{n+1}$ by putting $x \Rightarrow_G y = (x \Rightarrow y) \vee y$.

On the other hand, it readily follows from Proposition 1 that all the $\mathbf{Ł}^*_{n+1}$ algebras are simple. Indeed, if $a > b \in Ł_{n+1}$ would be congruent, then $\Delta_a(a) = 1$ and $\Delta_a(b) = 0$ should be so. Recall that an algebra is called *strictly simple* if it is simple and does not contain proper subalgebras. It is clear then that in the case of $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ algebras, they are strictly simple if $\{0, 1\}$ is their only proper subalgebra.

Remark 2. It is well-known that $\mathbf{Ł}_{n+1}$ is strictly simple iff n is prime. Note that, for every n, if $\mathbf{B} = (B, \neg, \to)$ is an MV-subalgebra of $\mathbf{Ł}_{n+1}$, then $\mathbf{B}^* = (B, \vee, \neg, *)$ is a subalgebra of $\mathbf{Ł}^*_{n+1}$ as well. Thus, if $\mathbf{Ł}_{n+1}$ is not strictly simple, then $\mathbf{Ł}^*_{n+1}$ is not strictly simple as well. Therefore, if n is not prime, $\mathbf{Ł}^*_{n+1}$ is not strictly simple. However, in contrast with the case of $\mathbf{Ł}_{n+1}$, n being prime is not a sufficient condition for $\mathbf{Ł}^*_{n+1}$ being strictly simple. In Lemma 7 below we will provide some examples of prime n for which $\mathbf{Ł}^*_{n+1}$ is not strictly simple, in view of Theorem 6.

Lemma 3. *$\mathbf{Ł}^*_{n+1}$ is strictly simple iff $\langle (n-1)/n\rangle^* = \mathbf{Ł}^*_{n+1}$.*

Proof. The 'only if' direction is trivial. In order to prove the converse, assume that $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$ for $a_1 = (n-1)/n$. For $i \ge 1$ let $a_{i+1} = t_i(a_i)$ such that $t_i(x) = {}^*x$ if $x > 1/2$, and $t_i(x) = \neg x$ otherwise. Since $\mathbf{Ł}^*_{n+1}$ is finite, there is $1 \le i < j$ such that $a_j = a_i$ and so $A_1 := \{a_i \mid i \ge 1\} = \{a_i \mid 1 \le i \le k\}$ for some k such that $a_i \ne a_j$ if $1 \le i, j \le k$. Let $A = A_1 \cup A_2 \cup \{0, 1\}$ where $A_2 = \{\neg a \mid a \in A_1\}$. Since ${}^*1 = 1$ and ${}^*x = 0$ if $x \le 1/2$, A is the domain of a subalgebra $\mathbf{A}$ of $\mathbf{Ł}^*_{n+1}$ over $\{^*, \neg, \vee\}$ such that $a_1 \in A$, hence $\langle a_1\rangle^* \subseteq \mathbf{A}$. But $\mathbf{A} \subseteq \langle a_1\rangle^*$, by construction. Therefore $\mathbf{A} = \langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$.

Fact: Under the current hypothesis (namely, $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$): if n is even then $n = 2$ or $n = 4$. Indeed, suppose that $\langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$ and n is even. If $n = 2$ or $n = 4$ then clearly $\mathbf{Ł}^*_{n+1}$ is strictly simple. Now, assume $n > 4$. Observe that: (1) for any $a \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$, ${}^*a = i/n$ such that i is even; and (2) if $i < n$ is even then $\neg(i/n) = (n-i)/n$ such that $n - i$ is even. That being so, if $i/n \in (A_1 \cup A_2) \setminus \{a_1, \neg a_1\}$ (recall the process described above) then i is even. But then, for instance, $3/n \notin \mathbf{A} = \langle a_1\rangle^* = \mathbf{Ł}^*_{n+1}$, a contradiction. This proves the Fact.

From the **Fact**, assume now that n is odd, and let $a = ((n+1)/2)/n$ and $b = ((n-1)/2)/n$. Since $\neg a = b$, $\neg b = a$ and $a, b \in A$ then, by construction of A, there is $1 \le i \le k$ such that either $a = a_i$ or $b = a_i$. If $a = a_i$ then $a_{i+1} = {}^*a = 1/n$ and so $a_{i+2} = \neg a_{i+1} = \neg 1/n = (n-1)/n = a_1$. Analogously it can be proven that, if $b = a_i$ then $a_1 = a_j$ for some $j > i$. This shows that $A_1 = \{a_1, \ldots, a_k\}$ is such that $a_{k+1} = a_1$ (hence $a_k = 1/n$). Now, let $c \in \mathbf{Ł}^*_{n+1} \setminus \{0, 1\}$ such that $c \ne a_1$. If $c \in A_1$ then the process of generation of A from c will produce the same set A_1 and so $\mathbf{A} = \mathbf{Ł}^*_{n+1}$, showing that $\langle c\rangle = \mathbf{Ł}^*_{n+1}$. Otherwise, if $c \in A_2$ then $\neg c \in A_1$ and, by the same argument as above, it follows that $\langle c\rangle = \mathbf{Ł}^*_{n+1}$. This shows that $\mathbf{Ł}^*_{n+1}$ is strictly simple. □

Lemma 4. *If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\odot$ is definable in $\mathbf{Ł}^*_{n+1}$, and hence $\langle (n-1)/n\rangle^* = \mathbf{Ł}^*_{n+1}$. Indeed, we can obtain $(n-i-1)/n = ((n-1)/n) \odot ((n-i)/n)$ for $i = 1, \ldots, n-1$, and $1 = \neg 0$. By Lemma 3 it follows that $\mathbf{Ł}^*_{n+1}$ is strictly simple. □

Corollary 5. *If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then n is prime.*

Proof. If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then $\mathbf{Ł}^*_{n+1}$ is strictly simple, by Lemma 4. By Remark 2 it follows that n must be prime. □

Theorem 6. *$\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ iff $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. The 'only if' part is Lemma 4. For the 'if' part, since $\mathbf{Ł}^*_{n+1}$ is strictly simple then, for each $a, b \in \mathrm{Ł}_{n+1}$ where $a \notin \{0,1\}$ there is a definable term $\mathbf{t}_{a,b}(x)$ such that $\mathbf{t}_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0,1\}$ and $b \in \mathrm{Ł}_{n+1}$ there is no such term then $\mathbf{A} = \langle a\rangle^*$ would be a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (since $b \notin \mathbf{A}$) different from $\{0,1\}$, a contradiction. By Proposition 1 the operations $\chi_a(x)$ are definable for each $a \in \mathrm{Ł}_{n+1}$, then in $\mathbf{Ł}^*_{n+1}$ we can define Łukasiewicz implication $\to$ as follows:

$$x \to y = (x \Rightarrow y) \vee \left(\bigvee_{n>i>j\geq 0} \chi_{i/n}(x) \wedge \chi_{j/n}(y) \wedge \mathbf{t}_{i/n,a_{ij}}(x) \right) \vee \left(\bigvee_{n>j\geq 0} \chi_1(x) \wedge \chi_{j/n}(y) \wedge y \right)$$

where $a_{ij} = 1 - i/n + j/n$. □

We have seen that n being prime is a necessary condition for $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ being term-equivalent. But this is not a sufficient condition: in fact, there are prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent.

Lemma 7. *If n is a prime Fermat number greater than 5 then $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent.*

Proof. Recall that a Fermat number is of the form $2^{2^k} + 1$, with k being a natural number. We are going to prove that if n is a prime Fermat number and $a_1 = (n-1)/n$, then $\langle a_1\rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (recall Theorem 6 and Lemma 3). Thus, let $n > 5$ be a prime Fermat number, that is, a prime number of the form $n = 2^m + 1$ with $m = 2^k$ and $k > 1$. The $(m-1)$-times iterations of $*$ applied to a_1 produce $((n+1)/2)/n$, that is: $(^*)^{m-1}(a_1) = ((n+1)/2)/n$. Since $^*(((n+1)/2)/n) = 1/n$, the constructive procedure for generating the algebra $\langle a_1\rangle^*$ described in the proof of Lemma 3 shows that $\langle a_1\rangle^* = \mathbf{A}$ has $2m+2$ elements: m elements in A_1, plus m elements in A_2 corresponding to their negations, plus 0 and 1. Since $2m+2 < 2^m+1 = n$ as $n > 5$, $\langle a_1\rangle^*$ is properly contained in $\mathrm{Ł}_{n+1}$, and it is different from $\{0,1\}$. □

The first Fermat prime number greater than 5 is $n = 17$. It is easy to see that

$$\langle 16/17\rangle^* = \{0, 1/17, 2/17, 4/17, 8/17, 9/17, 13/17, 15/17, 16/17, 1\}.$$

Actually, we do not have a full characterisation of those prime numbers n for which $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent. But computational results show that for prime numbers until 8000, about 60% of the cases yield term-equivalence.

Algebraizability of $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$

Given the algebra $\mathbf{Ł}^*_{n+1}$, it is possible to consider, for every $1 \leq i \leq n$, the matrix logic $Ł^*_{i,n+1} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$. In this section we will shown that all the $Ł^*_{i,n+1}$ are algebraizable in the sense of Blok-Pigozzi [1], and the quasivarieties associated to $Ł^*_{i,n+1}$ and $Ł^*_{j,n+1}$ are the same, for every i, j.

Observe that the operation $x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise is definable in $\mathbf{Ł}^*_{n+1}$. Indeed, it can be defined as $x \approx y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Also observe that $x \approx y = \Delta_1((x \Rightarrow_G y) \wedge (y \Rightarrow_G x))$ as well.

In order to prove the main result of this section, we state the following:

Lemma 8. *For every n, the logic $Ł^*_{n+1} := Ł^*_{n,n+1} = \langle \mathbf{Ł}^*_{n+1}, \{1\}\rangle$ is algebraizable.*

Proof. It is immediate to see that the set of formulas $\Delta(p, q) = \{p \approx q\}$ and the set of pairs of formulas $E(p, q) = \{\langle p, \Delta_0(p)\rangle\}$ satisfy the requirements of algebraizability. $\square$

Blok and Pigozzi [2] introduce the following notion of equivalent deductive systems. Two propositional deductive systems S_1 and S_2 in the same language are *equivalent* if there are translations $\tau_i : S_i \to S_j$ for $i \neq j$ such that: $\Gamma \vdash_{S_i} \varphi$ iff $\tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv\vdash_{S_i} \tau_j(\tau_i(\varphi))$. From very general results in [2] it follows that two equivalent logic systems are indistinguishable from the point of view of algebra, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This will be applied to $Ł^*_{i,n+1}$.

Lemma 9. *The logics $Ł^*_{n+1}$ and $Ł^*_{i,n+1}$ are equivalent, for every n and for every $1 \leq i \leq n-1$.*

Proof. It is enough to consider the translation mappings $\tau_1 : Ł^*_{n+1} \to Ł^*_{i,n+1}$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : Ł^*_{i,n+1} \to Ł^*_{n+1}$, $\tau_{i,2}(\varphi) = \Delta_{i/n}(\varphi)$. $\square$

Finally, as a direct consequence of Lemma 8, Lemma 9 and the observations above, we can prove the following result.

Theorem 10. *For every n and for every $1 \leq i \leq n$, the logic $Ł^*_{i,n+1}$ is algebraizable.*

As an immediate consequence of Theorem 10, for each logic $Ł^*_{i,n+1}$ there is a quasivariety $\mathcal{Q}(i, n)$ which is its equivalent algebraic semantics. Moreover, by Lemma 9 and by Blok and Pigozzi's results, $\mathcal{Q}(i, n)$ and $\mathcal{Q}(j, n)$ coincide, for every i, j. The question of axiomatising $\mathcal{Q}(i, n)$ is left for future work.

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