RESEARCH ARTICLE

Dimension of the intersection of a pair of orthogonal groups

Seok-Zun Song\textsuperscript{a}, R. Durán Díaz\textsuperscript{b}, L. Hernández Encinas\textsuperscript{c}*, J. Muñoz Masqué\textsuperscript{c}, and A. Queiruga Dios\textsuperscript{d}

\textsuperscript{a}Department of Mathematics, Cheju National University, Jeju 690-756, Republic of Korea
\textsuperscript{b}Departamento de Automática, University of Alcalá, Campus Universitario N-II, km. 33,600, 28871-Alcalá de Henares, Madrid, Spain; \textsuperscript{c}Department of Information Processing and Coding, Applied Physics Institute, CSIC, C/ Serrano 144, 28006-Madrid, Spain;
\textsuperscript{d}Department of Applied Mathematics, E.T.S.I.I., University of Salamanca, Avda. Fernández Ballesteros 2, 37700-Béjar, Salamanca, Spain

(July 2008)

Let \( g, h : V \times V \rightarrow \mathbb{C} \) be two non-degenerate symmetric bilinear forms on a finite-dimensional complex vector space \( V \). Let \( G \) (resp. \( H \)) be the Lie group of isometries of \( g \) (resp. \( h \)). If the endomorphism \( L : V \rightarrow V \) associated to \( g, h \) is diagonalizable, then the dimension of the intersection group \( G \cap H \) is computed in terms of the dimensions of the eigenspaces of \( L \).

Keywords: Diagonalizable endomorphism, Isometry, Matrix exponential, Orthogonal group, Symmetric bilinear form.

AMS Subject Classification: 15A57, 15A63, 22E10, 22E60.

1. The group of isometries

This paper is an extended version of a preliminary statement of Theorem 2.1, which was presented [1] without proof. Here, we also include a counterexample showing that the hypothesis in the theorem cannot be improved.

Let \( V, W \) be two complex vector spaces of finite dimension and let \( \mathcal{L}(V,W) \) be the space of \( \mathbb{C} \)-linear mappings from \( V \) into \( W \). We write \( \mathfrak{gl}(V) = \mathcal{L}(V,V) \) and we denote by \( GL(V) \) the linear group of \( V \), i.e., the group of invertible elements in \( \mathfrak{gl}(V) \).

Definition 1.1 An element \( A \in GL(V) \) is said to be an isometry of a symmetric bilinear form \( g : V \times V \rightarrow \mathbb{C} \) if the following equation holds:

\[ g(A(x),A(y)) = g(x,y), \quad \forall x, y \in V. \tag{1} \]

Lemma 1.2 Let \( g : V \times V \rightarrow \mathbb{C} \) be a symmetric bilinear form on an \( n \)-dimensional complex vector space \( V \) and let \( V', V'' \) be vector subspaces such that,

(1) \( g|_{V'} \) is non-degenerate,
(2) \( g(v,v'') = 0, \forall v \in V, \forall v'' \in V'' \),
Then, every isometry \( A \in GL(V) \) of \( g \) can be written as

\[
A = \begin{pmatrix} A' & O \\ B & C \end{pmatrix}, \quad B \in \mathcal{L}(V', V''), \ C \in \mathfrak{gl}(V''),
\]

and \( A' \) is an isometry of \( g|_{V'} \).

**Proof** We set \( A = \begin{pmatrix} A' & D \\ B & C \end{pmatrix} \), with \( A' \in \mathcal{L}(V', V') \), \( B \in \mathcal{L}(V', V''), \ C \in \mathfrak{gl}(V''), \ D \in \mathcal{L}(V'', V') \). If \( v_1, \ldots, v_n \) is a basis for \( V \) such that \( v_1, \ldots, v_{n'}, n' = \dim V' \), is a basis for \( V' \) and \( v_{n'+1}, \ldots, v_n \) is a basis for \( V'' \), then \( A \) is an isometry if and only if the following equations hold:

\[
g(A(v_i), A(v_j)) = g(v_i, v_j), \quad i, j = 1, \ldots, n. \tag{2}
\]

If \( i = 1, \ldots, n' \), then \( A(v_i) = A'(v_i) + B(v_i) \), with \( A'(v_i) \in V', B(v_i) \in V'' \), and according to the item (2), for \( i, j = 1, \ldots, n' \) from the equation (2) we obtain

\[
g(A(v_i), A(v_j)) = g(A'(v_i), A'(v_j)) + g(B(v_i), B(v_j))
\]

\[
+ g(B(v_i), A'(v_j)) + g(B(v_j), B(v_i))
\]

\[
= g(A'(v_i), A'(v_j))
\]

\[
= g(v_i, v_j),
\]

thus proving that \( A' = A|_{V'} \) is an isometry for \( g|_{V'} \).

Similarly, if \( i = n'+1, \ldots, n \), then \( A(v_i) = D(v_i) + C(v_i) \) with \( D(v_i) \in V', C(v_i) \in V'' \). Again from the item (2), we obtain

\[
0 = g(v, v_i) = g(A(v), A(v_i)) = g(A(v), D(v_i) + C(v_i)) = g(A(v), D(v_i)),
\]

for every \( v \in V \). As \( A \) is an isomorphism this in particular implies \( g(v', D(v_i)) = 0 \) \( \forall v' \in V' \), and we can conclude \( D = 0 \) by applying the item (1).

Consequently, the structure of the set of isometries of a degenerate symmetric bilinear form \( g \) can be recovered from the non-degenerate part of \( g \). Because of this, below we confine ourselves to consider only non-degenerate symmetric bilinear forms. In this case, the equation (1) implies \( \det A = \pm 1 \), and the set of all isometries of \( g \) is a subgroup of \( GL(V) \), which is denoted by \( G \). By choosing an orthonormal basis in \( V \), every element of \( G \) is represented by an orthogonal matrix and an isomorphism holds, \( G \cong O(n, \mathbb{C}) \). The Lie algebra of \( O(n, \mathbb{C}) \) is denoted by \( \mathfrak{o}(n, \mathbb{C}) \). We also remark on the fact that \( G \) is a closed subgroup in \( GL(V) \) and hence, \( G \) is a Lie subgroup of the linear group of \( V \), the Lie algebra of which is denoted by \( \mathfrak{g} \).

2. The dimension of the intersection group

**Theorem 2.1** Let \( V \) be an \( n \)-dimensional complex vector space and let \( g, h \) be two non-degenerate symmetric bilinear forms on \( V \). Let \( G, H \) be the groups of isometries of \( g, h \), respectively and let \( L : V \to V \) be the endomorphism associated to \( g, h \), i.e.,


$$g(x, L(y)) = h(x, y), \forall x, y \in V.$$ If $L$ is diagonalizable, then

$$\dim(G \cap H) = \sum_{i=1}^{r} \left( \frac{m_i}{2} \right),$$

where $m_i$, $i = 1, \ldots, r$, are the dimensions of the eigenspaces of $L$.

**Proof** Let $\alpha_i$, $i = 1, \ldots, r$, be the distinct eigenvalues of $L$ and let $E(\alpha_i)$ be the eigenspace attached to $\alpha_i$. As $L$ is diagonalizable, we have $V = \oplus_{i=1}^{r} E(\alpha_i)$. We claim that $E(\alpha_i)$ and $E(\alpha_j)$ are orthogonal with respect to both metrics for $i \neq j$. In fact, if $v_i$ (resp. $v_j$) is a non-vanishing eigenvector for $\alpha_i$ (resp. $\alpha_j$), then taking account of the fact that $L$ is symmetric, we obtain

$$\alpha_j g(v_i, v_j) = g(v_i, L(v_j)) = g(L(v_i), v_j) = \alpha_i g(v_i, v_j).$$

Hence, $(\alpha_i - \alpha_j) g(v_i, v_j) = 0$. As $\alpha_i \neq \alpha_j$, we conclude $g(v_i, v_j) = 0$. In addition, from the definition of $L$, we have $h(v_i, v_j) = g(v_i, L(v_j)) = \alpha_j g(v_i, v_j) = 0$. Therefore $E(\alpha_i)$ and $E(\alpha_j)$ are also $h$-orthogonal.

As a consequence of the $g$-orthogonality of the eigenspaces we deduce that every $E(\alpha_i)$ is non-singular with respect to both bilinear forms $g$ and $h$.

By choosing a $g$-orthonormal basis for every subspace $E(\alpha_i)$ and collecting all these bases, we obtain a basis $(v_1, \ldots, v_n)$ of eigenvectors for $L$ which is also $g$-orthonormal. Hence the matrices of $g$ and $h$ in this basis are as follows:

$$M_g = I_n = n \times n \text{ identity matrix},$$

$$M_h = \text{diagonal} \left( \alpha_1, \ldots, \alpha_r \right), \quad m_1 + \ldots + m_r = n.$$ Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of $G$ (resp. $H$). The map $\exp: \mathfrak{g} \to G$ induces a diffeomorphism from a neighborhood of the origin in $\mathfrak{g}$ onto a neighborhood of the unit element in $G$ ([4, Theorem 3.31]). Hence $\dim(G \cap H) = \dim(\mathfrak{g} \cap \mathfrak{h})$, and we are led to determine the Lie algebra of the intersection subgroup. Moreover, as $\mathfrak{g} = \{ A \in \mathfrak{gl}(V): g(x, A(y)) + g(A(x), y) = 0, \forall x, y \in V \}$, and similarly for $\mathfrak{h}$, we conclude that $\mathfrak{g} \cap \mathfrak{h}$ can be identified to the subspace of $n \times n$ skew-symmetric matrices $A = (a_{ij})$ such that

$$A^t M_h + M_h A = 0. \quad (3)$$

We decompose $A$ in blocks as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{pmatrix},$$

each $A_{ij}$ being a $m_i \times m_j$ matrix for $i, j = 1, \ldots, r$, and the equation (3) transforms into the following system: $\alpha_i A_{ij} + \alpha_j A_{ji}^t = 0$, $i, j = 1, \ldots, r$. As $A$ is skew-symmetric, we have $A_{ij} + A_{ji}^t = 0$. Hence this system is equivalent to saying $(\alpha_i - \alpha_j) A_{ij} = 0$ for $1 \leq i < j \leq r$.

Accordingly, $A_{ij} = 0$, $i \neq j$, and the submatrices $A_{11}, \ldots, A_{rr}$ are arbitrary. As $\dim(\mathfrak{o}(m, \mathbb{C}) = \binom{m}{2}$, we can conclude.

**Corollary 2.2** Let $S^2 V^*$ be the space of symmetric bilinear forms on $V$ and let
$\mathcal{U} \subset S^2V^*$ be the subset of non-degenerate forms. The pairs $(g,h) \in \mathcal{U} \times \mathcal{U}$ for which the conclusion of the theorem above holds, is a dense subset in $\mathcal{U} \times \mathcal{U}$.

**Proof** The map $\theta : \mathcal{U} \times \mathcal{U} \to \mathfrak{gl}(V)$, $\theta(g,h) = L$, is analytic and the result follows taking [2, Chapter 7, Theorem 1] into account.

**Remark 1** According to the proof of the previous theorem, the matrices of the form $\exp(\tilde{A}_{11}) \cdots \exp(\tilde{A}_{r})$, with $A_{ii} \in \mathfrak{o}(m_i, \mathbb{C})$ for $1 \leq i \leq r$, and

$$\tilde{A}_{ii} = \begin{pmatrix}
O_{\mu_i,\mu_i} & O_{\mu_i,m_i} & O_{\mu_i,n-\mu_{i+1}} \\
O_{m_i,\mu_i} & A_{ii} & O_{m_i,n-\mu_{i+1}} \\
O_{n-\mu_{i+1},\mu_i} & O_{n-\mu_{i+1},m_i} & O_{n-\mu_{i+1},n-\mu_{i+1}}
\end{pmatrix},$$

where $\mu_i = m_1 + \ldots + m_{i-1}$, $O_{\mu \nu}$ denoting the null $\mu \times \nu$ matrix, span the intersection group $G \cap H$. Hence the problem of computing the intersection group is feasible; in fact, it reduces (up to polynomial time) to exponentiate skew-symmetric matrices of sizes $m_1, \ldots, m_r$ (see [3]).

**Example 2.3** Assume $\dim V = n = 5$, and that $L$ has two distinct eigenvalues $\alpha, \beta$ such that $\dim E(\alpha) = 2$, $\dim E(\beta) = 3$. In this case, $\mathfrak{g} \cap \mathfrak{h}$ is identified to the matrices of the form

$$A = \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$ 

According to Remark 1, the intersection group is generated by $\exp \tilde{A}_{11} \exp \tilde{A}_{22}$. Exponentiating, we obtain

$$\exp \tilde{A}_{11} \exp \tilde{A}_{22} = \begin{pmatrix}
\cos d & \sin d & O \\
-\sin d \cos d & O & [v]^{-2} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \end{pmatrix},$$

where $v = (a, b, c)$, and

$$\begin{align*}
\lambda_{11} &= c^2 + (a^2 + b^2) \cos(|v|), \\
\lambda_{12} &= a|v| \sin(|v|) + bc (\cos(|v|) - 1), \\
\lambda_{13} &= b|v| \sin(|v|) - ac (\cos(|v|) - 1), \\
\lambda_{21} &= -a|v| \sin(|v|) - bc (\cos(|v|) - 1), \\
\lambda_{22} &= b^2 + (a^2 + c^2) \cos(|v|), \\
\lambda_{23} &= c|v| \sin(|v|) + ab (\cos(|v|) - 1), \\
\lambda_{31} &= -b|v| \sin(|v|) + ac (\cos(|v|) - 1), \\
\lambda_{32} &= -c|v| \sin(|v|) - ab (\cos(|v|) - 1), \\
\lambda_{33} &= a^2 + (b^2 + c^2) \cos(|v|).
\end{align*}$$
3. A counterexample

The formula for the dimension of the intersection group $G \cap H$ given in the previous theorem is no longer true if the endomorphism $L$ is not diagonalizable.

We provide a counterexample in arbitrary dimension as follows: For the metrics $g, h$ with matrices given respectively by

$$M_g = \begin{pmatrix} (k) \\ 0 \ldots 1 \\ \vdots \\ 1 \ldots 0 \end{pmatrix}, \quad M_h = \begin{pmatrix} (k) \\ 0 \ldots 1 \\ \vdots \\ 1 \ldots 0 \end{pmatrix},$$

we obtain $\dim(g \cap h) = \min(k, n-k)$. In fact, assuming $k \leq n-k$, a computation shows that the $n \times n$ matrices $A$ such that $A^t M_g + M_g A = 0$ and $A^t M_h + M_h A = 0$ are

$$A = \begin{pmatrix} O & Y \\ Z & O \end{pmatrix},$$

where

$$Y = -\begin{pmatrix} (k) \\ 0 \ldots 1 \\ \vdots \\ 1 \ldots 0 \end{pmatrix}, \quad Z^t = \begin{pmatrix} (n-k) \\ 0 \ldots 1 \\ \vdots \\ 1 \ldots 0 \end{pmatrix}$$

and $Z$ is the $(n - k) \times k$ matrix given by

$$Z = \sum_{h=1}^{k} \sum_{i=0}^{h-1} z_h E_{n-k-i, h-i}, \quad z_1, \ldots, z_k \in \mathbb{C},$$

$(E_{ij})$ being the standard basis of the matrix vector space. Moreover, $\dim E(\alpha) = 2$, where $\alpha$ is the only eigenvalue of $L$. In fact,
\[ M_L = M_g^{-1} M_h = \begin{pmatrix} \alpha & 0 & 0 & \ldots & 0 & 0 \\ 1 & \alpha & 0 & \ldots & 0 & 0 \\ 0 & 1 & \alpha & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha & 0 \\ 0 & 0 & 0 & \ldots & 1 & \alpha \\ 0 & 0 & 0 & \ldots & 0 & 1 \alpha \end{pmatrix} \]

4. Conclusions

The dimension of the intersection group of the orthogonal complex groups corresponding to two non-degenerate symmetric bilinear forms \( g, h \) is seen to depend quadratically on the dimensions of the eigenspaces of the linear transformation \( L \) associated to \( g, h \), whenever \( L \) is semisimple. A computationally feasible procedure to obtain the intersection is provided. A counterexample in arbitrary dimension to the formula for the dimension of the intersection group in Theorem 2.1 when the nilpotent part of \( L \) does not vanish, is also included.

Acknowledgements

R. Durán Díaz and J. Muñoz Masqué are supported by Ministerio de Educación y Ciencia (Spain) under grant MTM2005–00173, and L. Hernández Encinas and Seok-Zun Song are supported by Korean Science and Engineering Foundation (Korea) under grant F01–2007–000–10047–0.

References