# The canonical 8-form on manifolds with holonomy group Spin(9) *

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**Abstract**

An explicit expression of the canonical 8-form on a Riemannian manifold with a Spin(9)-structure, in terms of the nine local symmetric involutions involved, is given. The list of explicit expressions of all the canonical forms related to Berger's list of holonomy groups is thus completed. Moreover, some results on Spin(9)-structures as $G$-structures defined by a tensor and on the curvature tensor of the Cayley planes, are obtained.

## 1 Introduction and Preliminaries

The group Spin(9) belongs to Berger's list [6] of restricted holonomy groups of locally irreducible Riemannian manifolds which are not locally symmetric. Manifolds with holonomy group Spin(9) have been studied by Alekseevsky [4], Brown and Gray [13], Friedrich [14, 15], and Lam [23], among other authors. As proved in [13, 4], a connected, simply-connected, complete non-flat Spin(9)-manifold is isometric to either the Cayley projective plane $\mathbb{O}\mathrm{P}(2) \cong F_4/\mathrm{Spin}(9)$ or its dual symmetric space, the Cayley hyperbolic plane $\mathbb{O}\mathrm{H}(2) \cong F_{4(-20)}/\mathrm{Spin}(9)$.

Moreover, $\Delta_9$ being the unique irreducible 16-dimensional Spin(9)-module, the Spin(9)-module $\Lambda^8(\Delta_9^*)$ contains one and only one (up to a non-zero factor) 8-form $\Omega_0^8$ which is Spin(9)-invariant and defines the unique parallel form on $\mathbb{O}\mathrm{P}(2)$. It induces a *canonical* 8*-form* $\Omega^8$ on any 16-dimensional manifold with a fixed Spin(9)-structure. This form is said to be canonical because (cf. [13, p. 48], Berger [7, p. 13]) it yields, for the compact case, a generator of $H^8(\mathbb{O}\mathrm{P}(2), \mathbb{R})$.

Some explicit expressions of $\Omega^8$ have been given. The first one by Brown and Gray in [13, p. 49] in terms of a Haar integral. Other expression was then given by Brada and Pécaut-Tison [12, pp. 150, 153], by using a "cross product." Unfortunately, their formula is not correct, as we explain in Appendix A. Another expression was then given by Abe and Matsubara in [2, p. 8] as a sum of 702 suitable terms (see also Abe [1]). Their formula contains some errors, see Appendix B below.

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In this paper we give (Theorem 1.1) an explicit expression of the canonical 8-form $\Omega^8$ on a Spin(9)-manifold, in terms of the nine local symmetric involutions involved.

On the one hand, this completes the list of canonical forms which are related to Berger's list of holonomy groups (for the Kraines form [22] for $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and the Bonan forms [10] for $G_2$ and Spin(7) see also, e.g. Salamon [28, pp. 126, 155, 173]). On the other hand, we furnish an explicit analogue to the Kähler 2-form $\Omega^2$ and quaternion-Kähler 4-form $\Omega^4$, which can in a sense be called their octonionic analogue, as follows.

We recall that a Spin(9)-structure on an connected, oriented 16-dimensional Riemannian manifold $(M, g)$ is defined as a reduction of its bundle of oriented orthonormal frames $\mathrm{SO}(M)$, via the spin representation $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$. Equivalently (Friedrich [14, 15]), a Spin(9)-structure is given by nine-dimensional subbundle $\nu^9$ of the bundle of endomorphisms $\mathrm{End}(TM)$ locally spanned by $I_i \in \Gamma(\nu^9)$, $0 \leqslant i \leqslant 8$, satisfying the relations $I_i I_j + I_j I_i = 0$, $i \neq j$, $I_i^2 = \mathrm{I}$, $I_i^T = I_i$, $\mathrm{tr}\ I_i = 0$, $0 \leqslant i, j \leqslant 8$. These endomorphisms define 2-forms $\omega_{ij}$, $0 \leqslant i < j \leqslant 8$, on $M$ locally by $\omega_{ij}(X, Y) = g(X, I_i I_j Y)$. Similarly, using the skew-symmetric involutions $I_i I_j I_k$, $0 \leqslant i < j < k \leqslant 8$, one can define 2-forms $\sigma_{ijk}$. The 2-forms $\{\omega_{ij}, \sigma_{ijk}\}$ are linearly independent and a local basis of the bundle $\Lambda^2 M$.

The main purpose of the present paper is to prove

**Theorem 1.1.** *The canonical* 8*-form on the* Spin(9)*-manifold* $(M, g, \nu^9)$ *is given by*

$$\Omega^8 = \sum_{\substack{0 \leqslant i,j \leqslant 8 \\ 0 \leqslant i',j' \leqslant 8}} \omega_{ij} \wedge \omega_{ij'} \wedge \omega_{i'j} \wedge \omega_{i'j'},$$

*where* $\omega_{ij} = -\omega_{ji}$ *if* $i > j$ *and* $\omega_{ij} = 0$ *if* $i = j$.

On the other hand, some expressions for the curvature tensors of the Cayley planes have been given (cf. Brown and Gray [13], Brada and Pécaut-Tison [11, 12], and [25, 26]). As an application of our Theorem 1.1 we give one expression in terms of the nine local symmetric operators and relate it to the other expressions.

The importance of the Cayley planes in geometry is well known. Moreover, both the group Spin(9) and the Spin(9)-structures do appear in some questions of Physics, and we now recall some of them. The space $\mathbb{O}\mathrm{H}(2)$ is the only solution to $N = 9$, $d = 16$, 3-dimensional supergravity (cf. de Wit, Tollstén, and Nicolai [31]). The group Spin(9) appears in M-theory (see Banks et al. [5]), related to 16 fermionic superpartners, transforming as spinors under SO(9), linked to the very short strings connecting a system of D0 branes. Furthermore, Sati [29, 30] has recently studied the relation of Spin(9)-structures with M-theory fields, proving that the massless fields of M-theory are encoded in the spinor bundle of $\mathbb{O}\mathrm{P}(2)$ and that the massless multiplet of 11-dimensional supergravity is related to $\mathbb{O}\mathrm{P}(2)$ bundles over eleven-manifolds. In addition, the canonical 8-form $\Omega^8$ is there used to define a term of the action functional given in the theory. We remark that, besides the theoretical expression of $\Omega^8$ given in [13], the flawed expressions in [12, 2] are mentioned in [30].

As for the contents of this paper, in §2, after recalling some properties of Spin(9)-manifolds and the nine local symmetric involutions involved, we obtain the aforementioned expression for $\Omega^8$ and then some corollaries. In §3 we apply the previous results to the definition of a Spin(9)-structure as a structure defined by a tensor. We deduce in §4 some results on the curvature tensor of the Cayley planes. Finally, the aforementioned appendices A and B follow.

# 2 The canonical 8-form in terms of the nine local symmetric involutions

In order to prove Theorem 1.1, we first study the action of the group Spin(9) on $\mathbb{R}^{16} \equiv \mathbb{O}^2$ in terms of the nine local symmetric involutions $I_i$.

## 2.1 The action of $\mathrm{Spin}(9)$ on $\mathbb{R}^{16} \equiv \mathbb{O}^2$

The isotropy representation of either $\mathbb{O}\mathrm{P}(2)$ or $\mathbb{O}\mathrm{H}(2)$ is known to be isomorphic to the 16-dimensional spin representation $\rho$ of Spin(9).

Let $V^9$ be a real vector space of dimension nine endowed with a positive definite bilinear form $Q$. Let $e_0, \dots, e_8$ be an orthonormal basis of $V^9$. The Clifford algebra $\mathrm{Cl}_+(9)$ in terms of this basis is defined as the real associative algebra with unit 1, generators $e_0, \dots, e_8$, and defining relations

$$e_i \cdot e_j + e_j \cdot e_i = 0, \quad i \neq j, \qquad e_i^2 = 1, \quad 0 \leqslant i, j \leqslant 8.$$

Let $\mathrm{Pin}_+(9)$ be the multiplicative subgroup of the group of all the invertible elements of $\mathrm{Cl}_+(9)$ generated by the vectors of length one in $V^9$. If $Q(v,v) = 1$ then $v \cdot v = 1$, so $v \in \mathrm{Pin}_+(9)$. The Lie group $\mathrm{Spin}_+(9)$, which we denote simply by Spin(9), as they are isomorphic (cf. Postnikov [27, Lect. 13, Rem. 2]), is the subgroup of $\mathrm{Pin}_+(9)$ consisting of even elements, i.e.

$$\mathrm{Spin}(9) = \{\, v_1 \cdot v_2 \cdot \ldots \cdot v_{2k},\ Q(v_i, v_i) = 1,\ i = 1, \dots, 2k,\ k \in \mathbb{N} \,\}.$$

Moreover, the group Spin(9) preserves under conjugation the space $V^9$, that is, $sV^9s^{-1} = V^9$ for all $s \in \mathrm{Spin}(9)$ (cf. [27, Lect. 13]). We denote by $\pi$ the corresponding representation of the group Spin(9) on $V^9$. Then $\pi(\mathrm{Spin}(9)) = \mathrm{SO}(9)$ and $\pi\colon \mathrm{Spin}(9) \to \mathrm{SO}(9)$ is the usual two-fold covering homomorphism (cf. [27, Lect. 13]).

There exists a faithful representation $\rho$ of $\mathrm{Pin}_+(9)$ by orthogonal matrices (cf. [27, Lect. 13]). In other words, $\rho(\mathrm{Pin}_+(9)) \subset \mathrm{O}(16)$ and $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$. Therefore, there exist nine orthogonal linear transformations $I_i$ of $\Delta_9 = \mathbb{R}^{16}$ satisfying the relations

$$I_i I_j + I_j I_i = 0,\ i \neq j, \quad I_i^2 = \mathrm{I}, \quad I_i^T = I_i, \quad \mathrm{tr}\, I_i = 0, \qquad 0 \leqslant i, j \leqslant 8. \tag{2.1}$$

The set $\{I_iI_j, 0 \leqslant i < j \leqslant 8\}$ is a basis of the Lie algebra $\rho_*(\mathfrak{spin}(9)) \subset \mathfrak{so}(16)$. Indeed, since

$$[I_iI_j, I_k] = \begin{cases} 0, & \text{if } k \neq i, j, \\ -2I_j, & \text{if } k = i, \\ 2I_i, & \text{if } k = j, \end{cases}$$

the operators $I_iI_j$ are linearly independent and generate a space of dimension equal to $\dim \mathfrak{so}(9)$. Taking into account that each operator $I_iI_j$ is the tangent vector at $t = 0$ to the curve

$$s(t) = \big(\cos(t/2)I_i - \sin(t/2)I_j\big)\big(\cos(t/2)I_i + \sin(t/2)I_j\big) = \cos t \cdot \mathrm{I} + \sin t \cdot I_iI_j$$

in $\rho(\mathrm{Spin}(9))$ passing through the identity I, we obtain that the operators $I_iI_j$ generate the Lie algebra $\rho_*(\mathfrak{spin}(9))$ and, consequently, by the connectedness of the Lie group Spin(9) the following proposition holds

**Proposition 2.1.** *The Lie group* $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$ *is generated by the one-parameter families of endomorphisms*

$$\exp(tI_iI_j) = \cos t \cdot \mathrm{I} + \sin t \cdot I_iI_j, \qquad 0 \leqslant i < j \leqslant 8, \quad t \in \mathbb{R}.$$

In the sequel, we shall denote $I_iI_j$ simply by $I_{ij}$ and so on.

Let $(M, g, \nu^9)$ be a Spin(9)-manifold, $p \in M$ and $I_i, 0 \leqslant i \leqslant 8$, a local basis of sections of $\nu^9$ around $p$ satisfying the relations (2.1). Then, there exists an isomorphism between $\mathbb{O}^2 \equiv \mathbb{R}^{16}$ and $T_pM$ such that the restriction of $g$ at $p \in M$ induces the standard scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{O}^2$, given by

$$(2.2) \qquad \langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle, \quad \langle x_a, y_a \rangle = \frac{1}{2}(x_a\bar{y}_a + y_a\bar{x}_a),$$

for $a = 1, 2$, and the endomorphisms $I_0, \dots, I_8$ of $\mathbb{O}^2 \equiv T_pM$ read

$$(2.3) \qquad I_i(x_1, x_2) = (u_i\bar{x}_2, \bar{x}_1u_i), \quad I_8(x_1, x_2) = (-x_1, x_2), \qquad (x_1, x_2) \in \mathbb{O}^2,$$

where $u_0 = 1 \in \mathbb{O}$ and $u_i$, $i = 1, \dots, 7$, stand for the imaginary units of $\mathbb{O}$. One can easily check that these endomorphisms satisfy the appropriate relations (2.1) (see Postnikov [27, Lect. 15] and [26, (3),(4)]).

Moreover, as seen in Proposition 2.1, the group $\rho(\mathrm{Spin}(9))$ acting on $\mathbb{R}^{16} \equiv \mathbb{O}^2$ is generated by the endomorphisms $M^t_{kl} = \cos t \cdot \mathrm{I} + \sin t \cdot I_{kl}$, for $0 \leqslant k < l \leqslant 8$, and it is a subgroup of the group SO(16) determined by the standard scalar product (2.2) of $\mathbb{O}^2$.

## 2.2 Proof of Theorem 1.1

We must prove that the 8-form $\Omega^8_0 = \Omega^8|T_pM$, for an arbitrarily fixed point $p \in M$, is Spin(9)-invariant and non-trivial.

*The* 8*-form* $\Omega^8_0$ *is* Spin(9)*-invariant.* Fix a pair $kl$, $0 \leqslant k < l \leqslant 8$ and consider the action of the endomorphism $M^t_{kl}$ on the set of forms $\{\varpi_{ij} = \omega_{ij}|T_pM :$

$0 \leqslant i,j \leqslant 8\}$. Remark that $\varpi_{ij} = 0$ if $i = j$. Denote by $\overline{D}$ the set of all the ordered pairs $ij$, where $1 \leqslant i,j \leqslant 8$ and $i \neq j$. We will call a subset $r_i = \{i'j' \in \overline{D} : i' = i\}$ of the set $\overline{D}$ (resp. $c_j = \{i'j' \in \overline{D} : j' = j\}$) an $i$-row (resp. a $j$-column). We also consider the short $k$-row $r_k^* = r_k \setminus \{kl\}$ and the short $k$-column $c_k^* = c_k \setminus \{lk\}$ (this time $\#(c_k^*) = \#(r_k^*) = 7$). Similarly one determines the short $l$-row and the short $l$-column. Put

$$\begin{aligned}
A_0 &= \{ij \in \overline{D} : \{k,l\} \cap \{i,j\} = \emptyset\},\\
A_2 &= \big\{ij \in \overline{D} : \{k,l\} \cap \{i,j\} = \{k,l\}\big\} = \{kl, lk\},\\
A_1^+ &= \{ij \in \overline{D} : \{k,l\} \cap \{i,j\} = \{k\}\} = r_k^* \sqcup c_k^*,\\
A_1^- &= \big\{ij \in \overline{D} : \{k,l\} \cap \{i,j\} = \{l\}\big\} = r_l^* \sqcup c_l^*,\\
P_{kl} &= r_k \cup c_k \cup r_l \cup c_l = A_1^+ \sqcup A_1^- \sqcup A_2,
\end{aligned}$$

where we denote the union of two sets $A$ and $B$ by $A \sqcup B$ if $A \cap B = \emptyset$. It is clear that $\overline{D} = A_0 \sqcup A_2 \sqcup A_1^+ \sqcup A_1^-$. Given a pair $ij \in \overline{D}$, we denote by $\widehat{ij}$ the new pair obtained by replacing the element $k$ (if it occurs in $ij$) by $l$ and the element $l$ (if it occurs in $ij$) by $k$. The correspondence $ij \mapsto \widehat{ij}$ defines a bijection $\mu \colon \overline{D} \to \overline{D}$. It is clear that $\mu(A_1^{\pm}) = A_1^{\mp}$ and this mapping is an involutive automorphism of the set $\overline{D}$. In particular, $\widehat{ij} = ij$ for $ij \in A_0$ and $\widehat{kl} = lk$.

By definition, for arbitrary $X, Y \in \mathbb{O}^2$, we have

$$\big((M_{kl}^t)^* \varpi_{ij}\big)(X,Y) = \langle (\cos t + \sin t \cdot I_{kl})X, (\cos t + \epsilon \sin t \cdot I_{kl}) I_{ij} Y\rangle\,,$$

where $\epsilon = 1$ if the number of common elements in the sets $\{i,j\}$ and $\{k,l\}$ is even, and $\epsilon = -1$ if it is odd. Taking into account that all the operators $I_i$ are orthogonal and that the operator $I_{kl}$ is skew-symmetric, it is easily seen that

$$(M_{kl}^t)^* \varpi_{ij} = \begin{cases} \varpi_{ij}, & \text{if} \quad ij \in A_0 \cup A_2,\\ \cos 2t \cdot \varpi_{ij} + \sin 2t \cdot \varpi_{\widehat{ij}}, & \text{if} \quad ij \in A_1^+,\\ \cos 2t \cdot \varpi_{ij} - \sin 2t \cdot \varpi_{\widehat{ij}}, & \text{if} \quad ij \in A_1^-. \end{cases} \tag{2.4}$$

Therefore, for all $ij, i'j' \in A_1^+$ we obtain

$$\begin{aligned}
(M_{kl}^t)^*(\varpi_{ij} \wedge \varpi_{i'j'} + \varpi_{\widehat{ij}} \wedge \varpi_{\widehat{i'j'}}) &= \varpi_{ij} \wedge \varpi_{i'j'} + \varpi_{\widehat{ij}} \wedge \varpi_{\widehat{i'j'}},\\
(M_{kl}^t)^*(\varpi_{ij} \wedge \varpi_{\widehat{i'j'}} - \varpi_{\widehat{ij}} \wedge \varpi_{i'j'}) &= \varpi_{ij} \wedge \varpi_{\widehat{i'j'}} - \varpi_{\widehat{ij}} \wedge \varpi_{i'j'}.
\end{aligned} \tag{2.5}$$

Consider now the commutative polynomial ring $RD = \mathbb{R}[x_{ij}; ij \in \overline{D}, i < j]$. Put $x_{ij} = -x_{ji}$ for $i > j$ and $x_{ii} = 0$. Denote by $RD_I$ the subring of $RD$ generated by the family of polynomial functions

$$\begin{aligned}
X_I = &\{x_{ij} : ij \in A_0 \cup A_2\}\\
&\cup \{x_{ij}x_{i'j'} + x_{\widehat{ij}}x_{\widehat{i'j'}},\ x_{ij}x_{\widehat{i'j'}} - x_{\widehat{ij}}x_{i'j'} : ij, i'j' \in A_1^+\}.
\end{aligned}$$

Since all the 2-forms $\varpi_{ij}$ commute, $\Omega_0^8$ is invariant with respect to the one-parameter group $M_{kl}^t$ if the polynomial function $F = \sum_{ij,i'j' \in \overline{D}} x_{ij}x_{ij'}x_{i'j}x_{i'j'}$

is an element of the subring $RD_I$. To prove this fact, note that the sequence $ij, ij', i'j, i'j' \in \overline{D}$ is a sequence of vertices of either a rectangle or a degenerate rectangle made of entries of a square $9 \times 9$ matrix without the diagonal. This sequence originates an either 4- or 2- or 1-element subset of $\overline{D}$. So it is natural to consider the following sets:

$$\overline{D}_4 = \big\{\{ij, ij', i'j, i'j'\} \subset \overline{D} : i \neq i',\ j \neq j'\big\},$$
$$\overline{D}_2 = \big\{\{ij, i'j'\} \subset \overline{D} : i = i' \text{ or } j = j',\ ij \neq i'j'\big\}.$$

Using these sets we can rewrite the polynomial $F$ as a sum $F = F_1 + F_2 + F_4$ of three polynomials

$$F = \sum_{ij \in \overline{D}} x_{ij}^4 + 2 \sum_{\{ij, i'j'\} \in \overline{D}_2} x_{ij}^2 x_{i'j'}^2 + 4 \sum_{\{ij, ij', i'j, i'j'\} \in \overline{D}_4} x_{ij} x_{ij'} x_{i'j} x_{i'j'}. \tag{2.6}$$

Consider the polynomial $F_1 + F_2$. Using the decomposition $\overline{D} = A_0 \sqcup A_2 \sqcup A_1^+ \sqcup A_1^-$, we can write the first polynomial $F_1$ as a sum $F_1 = F_{1,0} + F_{1,2} + F_{1,1}^+ + F_{1,1}^-$ (replacing the set $\overline{D}$ in the formula for $F_1$ by $A_0$, $A_2$, $A_1^+$ and $A_1^-$, respectively). We also consider the decomposition $\overline{D}_2 = \overline{D}_{2,0} \sqcup \overline{D}_{2,1} \sqcup \overline{D}_{2,2}$ of the set $\overline{D}_2$, where each $\{ij, i'j'\} \in \overline{D}_{2,\alpha}$ has $\alpha$ common elements with the subset $P_{kl} \subset \overline{D}$; and the corresponding decomposition $F_2 = F_{2,0} + F_{2,1} + F_{2,2}$ of $F_2$.

By definition, $F_{1,0} + F_{1,2} \in RD_I$. Since $A_1^{\pm} \subset P_{kl}$, we have $F_{2,0} \in RD_I$. Taking into account that in any pair $\{ij, i'j'\} \in \overline{D}_{2,1}$ one element belongs to the subset $A_0 \subset \overline{D}$ and the other to the subset $A_1^+ \sqcup A_1^-$ (i.e. either $\widehat{ij} = ij$ or $\widehat{i'j'} = i'j'$), we conclude that

$$F_{2,1} = 2 \sum_{\{ij, i'j'\} \in \overline{D}_{2,1}} x_{ij}^2 x_{i'j'}^2 = \sum_{\{ij, i'j'\} \in \overline{D}_{2,1}} (x_{ij}^2 x_{i'j'}^2 + x_{\widehat{ij}}^2 x_{\widehat{i'j'}}^2) \in RD_I.$$

Taking into account that $x_{ij}^2 = x_{ji}^2$ and $P_{kl} = r_k^* \sqcup c_k^* \sqcup r_l^* \sqcup c_l^* \sqcup A_2$, we can rewrite the polynomial $F_{2,2}$ as

$$4 \sum_{j, j' \notin \{k,l\}, j < j'} (x_{kj}^2 x_{kj'}^2 + x_{lj}^2 x_{lj'}^2) + 4 \sum_{j \notin \{k,l\}} x_{kl}^2 (x_{kj}^2 + x_{lj}^2) + 4 \sum_{j \notin \{k,l\}} x_{kj}^2 x_{lj}^2.$$

But

$$F_{1,1}^+ + F_{1,1}^- = \sum_{ij \in A_1^+} (x_{ij}^4 + x_{\widehat{ij}}^4) = 2 \sum_{j \notin \{k,l\}} (x_{kj}^4 + x_{lj}^4).$$

Therefore the component

$$4 \sum_{j \notin \{k,l\}} x_{kl}^2 (x_{kj}^2 + x_{lj}^2) + 4 \sum_{j \notin \{k,l\}} x_{kj}^2 x_{lj}^2 + 2 \sum_{j \notin \{k,l\}} (x_{kj}^4 + x_{lj}^4)$$

of the polynomial $F_{1,1}^+ + F_{1,1}^- + F_{2,2}$ is an element of $RD_I$ because $\widehat{kj} = lj$ and, consequently, $x_{kl}$, $(x_{kj}^2 + x_{lj}^2) \in RD_I$.

Denote the first term (polynomial) in the above expression of $F_{2,2}$ as $F_2^*$. It only remains to be proved that $F_2^* + F_4 \in RD_I$.

For any pair $ij, i'j' \in \overline{D}$ with $\{i,j\} \cap \{i',j'\} = \emptyset$ denote by $q(ij,i'j')$ the quadruple $\{ij, i'j, ij', i'j'\}$. It is clear that $q(ij,i'j') = q(i'j',ij)$ and $q(ij,i'j') = q(ij',i'j)$. Moreover, the involution $\mu$ on $\overline{D}$ induces on the set $\overline{D}_4$ a well-defined involution $\mu_4\colon \{ij, i'j, ij', i'j'\} \mapsto \{\widehat{ij}, \widehat{i'j}, \widehat{ij'}, \widehat{i'j'}\}$ (it is easy to verify that the image of the rectangle is a rectangle). In particular, $q(ij,i'j') \mapsto q(\widehat{ij}, \widehat{i'j'})$.

Taking into account that the set $P_{kl} = A_1^+ \sqcup A_1^- \sqcup A_2$ is a union of two rows and two columns, any quadruple $q \in \overline{D}_4$ has either zero, or two, or three or four common points with the set $P_{kl}$. Denote the corresponding subsets of $\overline{D}_4$ by $\overline{D}_{4,0}$, $\overline{D}_{4,2}$, $\overline{D}_{4,3}$, $\overline{D}_{4,4}$, respectively. Then $F_4 = F_{4,0} + F_{4,2} + F_{4,3} + F_{4,4}$, where the polynomial $F_{4,\alpha}$ corresponds to the subset $\overline{D}_{4,\alpha} \subset \overline{D}_4$, $\alpha = 0,2,3,4$. We claim that $F_{4,0} + F_{4,2} + F_{4,3} \in RD_I$. To prove this fact, consider the sets $\overline{D}_{4,0}$, $\overline{D}_{4,2}$, $\overline{D}_{4,3}$ in more detail.

If $q \in \overline{D}_{4,0}$ then the four elements of $q$ belong to the set $A_0$, i.e. $F_{2,0} \in RD_I$. If $q \in \overline{D}_{4,2}$ then two elements of $q$ belong to $A_0$ and two elements of $q$ belong to $A_1^+$ or $A_1^-$, i.e. the set $\overline{D}_{4,2}$ is invariant under the natural action of the involution $\mu_4$ on $\overline{D}_4$ and a fixed point set for this action on $\overline{D}_{4,2}$ is empty. Therefore

$$F_{4,2} = 2 \sum_{\{ij,ij',i'j,i'j'\} \in \overline{D}_{4,2}} (x_{ij}x_{ij'}x_{i'j}x_{i'j'} + x_{\widehat{ij}}x_{\widehat{ij'}}x_{\widehat{i'j}}x_{\widehat{i'j'}}) \in RD_I.$$

In the third case, each quadruple $q \in \overline{D}_{4,3}$ contains precisely one element of the set $A_2$, so $\overline{D}_{4,3} = \overline{D}_{4,3}^{kl} \sqcup \overline{D}_{4,3}^{lk}$, where $\overline{D}_{4,3}^{kl}$ (resp. $\overline{D}_{4,3}^{lk}$) is the set of all elements from $\overline{D}_{4,3}$ containing the pair $kl$ (resp. $lk$). This decomposition of the set $\overline{D}_{4,3}$ determines the decomposition $F_{4,3} = F_{4,3}^{kl} + F_{4,3}^{lk}$ of the polynomial $F_{4,3}$. Each quadruple $q \in \overline{D}_{4,3}^{kl}$ is uniquely defined by the pair $\{kl, ij\}$, i.e. by some element $ij \in \overline{D}$, so $q = \{kl, kj, il, ij\}$. It is clear that in this element, $ij \in A_0$ whilst $kj \in A_1^+$ and $il \in A_1^-$. But for an arbitrary $ij \in \overline{D}$ the quadruple $q(kl,ij) = \{kl, kj, il, ij\}$ belongs to $\overline{D}_4$ if and only if $\{k,l\} \cap \{i,j\} = \emptyset$. Since this unique relation defining the quadruples is invariant under interchange of $i$ and $j$, the quadruple $q(kl,ij) \in \overline{D}_4$ if and only if $q(kl,ji) = \{kl, ki, jl, ji\} \in \overline{D}_4$. Therefore the correspondence $q(kl,ij) \mapsto q(kl,ji)$ determines an involutive automorphism on the set $\overline{D}_{4,3}^{kl}$. Taking into account that $x_{ji} = -x_{ij}$ and $q(kl,ij) \neq q(kl,ji)$ we obtain that

$$F_{4,3}^{kl} = 2 \sum_{\substack{ij \in \overline{D} \\ i,j \notin \{k,l\}}} x_{kl}(x_{kj}x_{il}x_{ij} + x_{ki}x_{jl}x_{ji}) = 2 \sum_{\substack{ij \in \overline{D} \\ i,j \notin \{k,l\}}} x_{kl}x_{ij}(x_{kj}x_{il} - x_{lj}x_{ik}).$$

Since $kl, ij \in A_0 \sqcup A_2$, $kj \in A_1^+$, $il \in A_1^-$ and $lj = \widehat{kj}$, $ik = \widehat{il}$, we have $F_{4,3}^{kl} \in RD_I$. Similarly, $F_{4,3}^{lk} \in RD_I$.

If a quadruple $q \in \overline{D}_4$ has four common points with the set $P_{kl}$, then either $q = \{kj, kj', lj, lj'\}$ or $q = \{jk, j'k, jl, j'l\}$, i.e. two elements of $q$ belong to the short $k$-row (or column) and another two to the short $l$-row (or column). Since

$x_{ij} = -x_{ji}$, we have $F_{4,4} = 8\sum_{0\leqslant j<j'\leqslant 8} x_{kj}x_{kj'}x_{lj}x_{lj'}$, where $j, j' \notin \{k, l\}$, so

$$F_2^* + F_{4,4} = 4 \sum_{j,j'\notin\{k,l\}, j<j'} (x_{kj}^2 x_{kj'}^2 + x_{lj}^2 x_{lj'}^2 + 2x_{kj}x_{kj'}x_{lj}x_{lj'}) \in RD_I,$$

because $(x_{kj}x_{kj'} + x_{lj}x_{lj'})^2 \in RD_I$ by definition.

In conclusion, the form $\Omega_0^8$ is invariant with respect to the action of each of the subgroups $M_{kl}^t$ generating the Lie group Spin(9), i.e. the form $\Omega_0^8$ is Spin(9)-invariant.

*The* 8*-form* $\Omega_0^8$ *is not trivial.* To this end we consider the eight vectors $X_i = (u_i, 0)$ and two vectors $X = (x, 0)$ and $Y = (y, 0)$ belonging to the space $\mathbb{O}^2$. Using the expressions (2.3) for the endomorphisms $I_i$, we obtain that for $0 \leqslant i, j \leqslant 7$, $i \neq j$, one has

$$\varpi_{ij}(X, Y) = g(X, I_{ij}Y) = \langle (x, 0), (u_i(\bar{u}_j y), 0)\rangle = \langle x, u_i(\bar{u}_j y)\rangle$$

and

$$\varpi_{i8}(X, Y) = 0, \tag{2.7}$$

because the vector $X = (x, 0)$ is orthogonal to $I_{i8}Y = (0, -\bar{y}u_i)$. We can rewrite the expression for $\varpi_{ij}(X, Y)$ as

$$\varpi_{ij}(X, Y) = \langle x, u_i(\bar{u}_j y)\rangle = \langle \bar{u}_i x, \bar{u}_j y\rangle = \langle \bar{x}u_i, \bar{y}u_j\rangle, \tag{2.8}$$

because (cf. [13, Sect. 2]) for arbitrary octonions $a, b, c \in \mathbb{O}$, one has

$$\langle ab, c\rangle = \langle b, \bar{a}c\rangle = \langle a, c\bar{b}\rangle \quad \text{and} \quad \langle a, b\rangle = \langle \bar{a}, \bar{b}\rangle. \tag{2.9}$$

Since $\Omega_0^8$ is a sum of the 8-forms $W(i, i'; j, j') = \varpi_{ij} \wedge \varpi_{ij'} \wedge \varpi_{i'j} \wedge \varpi_{i'j'}$, it is sufficient to show that $W^0(i, i'; j, j') = W(i, i'; j, j')(X_0, \ldots, X_7) < 0$. It is clear that the 8-form $W(i, i'; j, j')$ is determined by the unordered pairs $\{i, i'\}$ and $\{j, j'\}$ of rows and columns, so $W(i, i'; j, j') = W(i', i; j, j')$ and $W(i, i'; j, j') = W(i, i'; j', j)$. Moreover, since $\varpi_{ij} = -\varpi_{ji}$ and all these 2-forms commute, we have

$$W(i, i'; j, j') = W(j, j'; i, i'). \tag{2.10}$$

Let $S_8$ be the permutation group acting on the set $B = \{u_0, \ldots, u_7\}$ and let $B^{\pm} = \{\pm u_0, \ldots, \pm u_7\}$. For arbitrary $v, v', w, w' \in B^{\pm}$, put

$$\widetilde{W}^0(v, v'; w, w') = 2^{-4} \sum_{\sigma \in S_8} A_\sigma(v, v'; w, w'),$$

where $A_\sigma$, for $\sigma = (u_{i_0}, \ldots, u_{i_7})$, is given by

$$A_\sigma(v, v'; w, w') = \varepsilon(\sigma)\langle u_{i_0}, v(wu_{i_1})\rangle\langle u_{i_2}, v(w'u_{i_3})\rangle\langle u_{i_4}, v'(wu_{i_5})\rangle\langle u_{i_6}, v'(w'u_{i_7})\rangle.$$

As the elements $v, v', w, w'$ occur in this expression twice, we have

$$\widetilde{W}^0(v, v'; w, w') = \widetilde{W}^0(\pm v, \pm v'; \pm w, \pm w'). \tag{2.11}$$

By definition $W^0(i, i'; j, j') = \widetilde{W}^0(u_i, u_{i'}; \bar{u}_j, \bar{u}_{j'})$, but as $\bar{u}_l = \pm u_l$, it follows that

$$W^0(i, i'; j, j') = \widetilde{W}^0(u_i, u_{i'}; u_j, u_{j'}). \tag{2.12}$$

We now prove two lemmas.

**Lemma 2.2.** *For an arbitrary automorphism $\Phi$ of the algebra $\mathbb{O}$ preserving the set $B^{\pm}$, one has $\widetilde{W}^0(v, v'; w, w') = \widetilde{W}^0(\Phi(v), \Phi(v'); \Phi(w), \Phi(w'))$.*

*Proof.* It is clear that $\Phi(u_k) = \varepsilon^{\Phi}_{u_k}\sigma^{\Phi}(u_k)$, where $\varepsilon^{\Phi}_{u_k} = \pm 1$ and $\sigma^{\Phi}$ is some permutation in $S_8$. Moreover, since $\Phi$ is an element of the exceptional connected Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$, we have $\prod_{k=0}^{7} \varepsilon^{\Phi}_{u_k} \cdot \varepsilon(\sigma^{\Phi}) = 1$ and, consequently, we have $A_{\sigma^{\Phi}\sigma}(\Phi(v), \Phi(v'); \Phi(w), \Phi(w')) = A_{\sigma}(v, v'; w, w')$, because $\varepsilon(\sigma^{\Phi}\sigma) = \varepsilon(\sigma^{\Phi})\varepsilon(\sigma)$ and $\sigma^{\Phi}\sigma(u_k) = \varepsilon^{\Phi}_{\sigma(u_k)}\Phi(\sigma(u_k))$. Noting that $\sigma^{\Phi}S_8 = S_8$, we conclude. $\square$

**Lemma 2.3.** *For any $u \in B^{\pm}$, one has $\widetilde{W}^0(v, v'; w, w') = \widetilde{W}^0(vu, v'u; wu, w'u)$.*

*Proof.* Since the lemma is obvious for $u = \pm u_0$, assume that $u \neq \pm u_0$. Due to the relations (2.8) and the fact that $\bar{u}_k = \pm u_k$, we can rewrite the expression for $A_{\sigma}(v, v'; w, w')$ as $\varepsilon(\sigma)\langle vu_{i_0}, wu_{i_1}\rangle\langle vu_{i_2}, w'u_{i_3}\rangle\langle v'u_{i_4}, wu_{i_5}\rangle\langle v'u_{i_6}, w'u_{i_7}\rangle$ (the elements $v, v', w, w'$ occur in this expression twice). But for arbitrary octonions $a, b, c$, their associator $(a, b, c) = (ab)c - a(bc)$ is skew-symmetric with respect to the second and third arguments, i.e. $(ab)c + (ac)b = a(bc + cb)$ (cf. [13, Sect. 2]). Thus, if $u_k u = -uu_k$ then $(au_k)u = (-au)u_k$. Since $u \neq \pm u_0$, one has $u_k u \neq -uu_k$ if and only if either $u_k = u_0$ or $u_k = \pm u$. It is clear that in these two cases one has $(au)u_k = (au_k)u$. Noting then that precisely six elements of the set $B$ anticommute with $u$ and that by (2.9), one has $\langle au, bu\rangle = \langle a, (bu)\bar{u}\rangle = \langle a, b|u|^2\rangle = \langle a, b\rangle$, we conclude. $\square$

Suppose now as usual that the basis $B$ coincides with the set $\{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}, \mathbf{e}, \mathbf{ie}, \mathbf{je}, (\mathbf{ij})\mathbf{e}\}$, where $\mathbf{i} = u_1$, $\mathbf{j} = u_2$ and $\mathbf{e} = u_4$, so that for instance $u_5 = u_1u_4$. Each element of the algebra $\mathbb{O}$ admits a unique expression as $q_1 + q_2\mathbf{e}$ with $q_1, q_2 \in \mathbb{H}$, where $\mathbb{H}$ is the quaternion algebra generated by $\mathbf{i}, \mathbf{j}$. Then the multiplication in $\mathbb{O}$ is defined by the standard multiplication relations in $\mathbb{H}$ and by the relations

$$q_1(q_2\mathbf{e}) = (q_2q_1)\mathbf{e}, \quad (q_1\mathbf{e})q_2 = (q_1\bar{q}_2)\mathbf{e}, \quad (q_1\mathbf{e})(q_2\mathbf{e}) = -\bar{q}_2q_1. \tag{2.13}$$

Put $B^0 = B \setminus u_0$. Let $\mathbf{i}', \mathbf{j}', \mathbf{e}'$ be three arbitrary distinct elements of the set $B^0 \cup (-B^0)$ such that $\mathbf{e}' \neq \pm\mathbf{i}'\mathbf{j}'$. Then there exists a unique automorphism $\Phi$ of the octonion algebra $\mathbb{O}$ such that $\Phi(\mathbf{i}') = u_1$, $\Phi(\mathbf{j}') = u_2$ and $\Phi(\mathbf{e}') = u_4$ (cf. [27, Lect. 15]). It is evident that $\Phi(u_0) = u_0$. Now, taking into account Lemmas 2.2 and 2.3, and the relations (2.10), (2.11) and (2.12), we have to calculate only the four numbers

$$\widetilde{W}^0(u_0, u_0; u_1, u_1),\ \widetilde{W}^0(u_0, u_0; u_1, u_2),\ \widetilde{W}^0(u_0, u_1; u_2, u_3),\ \widetilde{W}^0(u_0, u_1; u_2, u_4).$$

Indeed, calculating $\widetilde{W}^0(u_i, u_{i'}; u_j, u_{j'})$, by Lemma 2.3 we can suppose that $u_i = u_0$. If the sequence $(ij, ij', i'j, ij')$ originates a 1-element subset of $\overline{D}$, i.e. $i = i' = 0$ and $j = j'$, then $\Phi(u_j) = u_1$ for some automorphism $\Phi$ of $\mathbb{O}$; if $(ij, ij', i'j, ij')$ originates an 2-element subset of $\overline{D}$, for instance $i = i' = 0$ and $j \neq j'$, then $\Phi(u_j) = u_1$ and $\Phi(u_{j'}) = u_2$ for some automorphism $\Phi$ (when $j = j'$ we can suppose by Lemma 2.3 that $j = 0$ and use (2.10)); if this sequence originates an 4-element subset of $\overline{D}$, i.e. all $i = 0, i', j, j'$ are distinct, then according to either $u_{j'} = \pm u_{i'} u_j$ or $u_{j'} \neq \pm u_{i'} u_j$, we can obtain as image of the triple $u_{i'}; u_j, u_{j'}$ under $\Phi$ the triple $u_1; u_2, u_3$ or $u_1; u_2, u_4$, respectively.

First of all we consider the restriction $\varpi'_{ij}$ of the form $\varpi_{ij}$ to the subspace $V \subset \mathbb{O}^2$ generated by the vectors $X_k$, for $k = 0, \ldots, 7$. Let $\{x_0^*, \ldots, x_7^*\}$ be the dual basis of $V^*$. Using the relations (2.13) it is easy to verify that

$$\varpi'_{01} = x_0^* \wedge x_1^* + x_2^* \wedge x_3^* + x_4^* \wedge x_5^* - x_6^* \wedge x_7^*.$$

Therefore we have $\varpi'_{01} \wedge \varpi'_{01} \wedge \varpi'_{01} \wedge \varpi'_{01} = -24\, x_0^* \wedge x_1^* \wedge \cdots \wedge x_7^*$, that is, $\widetilde{W}^0(u_0, u_0; u_1, u_1) = -24$. Thus $\widetilde{W}^0(u_i, u_i; u_j, u_j) = -24$ for arbitrary $0 \leqslant i, j \leqslant 7$, $i \neq j$, because $u_j u_i \neq \pm u_0$ and, consequently, there exists some automorphism $\Phi$ such that $\Phi(\pm u_j u_i) = u_1$. In other words,

$$\varpi'_{ij} = \varepsilon_0 x_{i_0}^* \wedge x_{i_1}^* + \varepsilon_2 x_{i_2}^* \wedge x_{i_3}^* + \varepsilon_4 x_{i_4}^* \wedge x_{i_5}^* + \varepsilon_6 x_{i_6}^* \wedge x_{i_7}^*,$$

where $\sigma_{ij} = (i_0, \ldots, i_7)$ is some permutation of the set $\{0, \ldots, 7\}$, $\varepsilon_{2k} = \pm 1$, and $\prod_{k=0}^{3} \varepsilon_{2k} \cdot \varepsilon(\sigma_{ij}) = -1$. Consider also the form

$$\varpi'_{ij'} = \varepsilon'_0 x_{j_0}^* \wedge x_{j_1}^* + \varepsilon'_2 x_{j_2}^* \wedge x_{j_3}^* + \varepsilon'_4 x_{j_4}^* \wedge x_{j_5}^* + \varepsilon'_6 x_{j_6}^* \wedge x_{j_7}^*,$$

where $i \neq j'$ and $j' \neq j$.

We now show two more lemmas.

**Lemma 2.4.** *For arbitrary distinct elements $i, j, j' \in \{0, \ldots, 7\}$, the 4-form $\varpi'_{ij} \wedge \varpi'_{ij'}$ is a sum of at most eight linearly independent terms (4-forms) $\varpi'_{k,ij,ij'}$, $k = 0, \ldots, 7$, of type $\pm x_{k_0}^* \wedge x_{k_1}^* \wedge x_{k_2}^* \wedge x_{k_3}^*$. For each such term $\varpi'_{k,ij,ij'}$, there is a unique term $\varepsilon_{2p}\, x_{i_{2p}}^* \wedge x_{i_{2p+1}}^*$ of $\varpi_{ij}$ and a unique term $\varepsilon'_{2p'}\, x_{j_{2p'}}^* \wedge x_{j_{2p'+1}}^*$ of $\varpi_{ij'}$ such that their exterior product is proportional to $\varpi'_{k,ij,ij'}$ (and, consequently, it is equal to $\varpi'_{k,ij,ij'}$).*

*Proof.* Put $u_l = \pm u_i u_j$ and $u_{l'} = \pm u_i u_{j'}$. It is clear that $u_l$ and $u_{l'}$ are two distinct imaginary units of $\mathbb{O}$. Therefore if $\varpi'_{ij}(u_{i_0}, u_{i_1}) = \pm\langle u_{i_0}, u_l u_{i_1}\rangle \neq 0$ then $u_l = \pm u_{i_0} u_{i_1}$ and $u_{l'} \neq \pm u_{i_0} u_{i_1}$, i.e. $\varpi'_{ij'}(u_{i_0}, u_{i_1}) = 0$. So precisely two terms of $\varpi'_{ij'}$ contain $x_{i_0}^*$ and $x_{i_1}^*$ as a factor. Therefore there exists precisely two terms of $\varpi'_{ij'}$ such that their exterior product with $x_{i_0}^* \wedge x_{i_1}^*$ is not zero. Since the form $\varpi'_{ij}$ contains four terms, the number of linearly independent terms of $\varpi'_{ij} \wedge \varpi'_{ij'}$ is at most eight.

Assume that the product of the terms $x_{i_0}^* \wedge x_{i_1}^*$ and $x_{j_0}^* \wedge x_{j_1}^*$ of the forms $\varpi'_{ij}$ and $\varpi'_{ij'}$ respectively, is not trivial, i.e. $\{i_0, i_1\} \cap \{j_0, j_1\} = \emptyset$. The forms $\varpi'_{ij}$ and $\varpi'_{ij'}$ contain a unique term with the factor $x_{i_0}^*$. As we show above, in the form $\varpi'_{ij'}$ the second factor of this term is not equal to $x^*_{i_1}$. Assume that this factor is equal to $x^*_{j_k}$, $k = 0, 1$. Then $\varpi'_{ij'}(u_{i_0}, u_{j_k}) \neq 0$, i.e. $u_{i_0} = \pm u_{l'} u_{j_k}$. But $u_{j_0} = \pm u_{l'} u_{j_1}$, i.e. $\{i_0, i_1\} \cap \{j_0, j_1\} \neq \emptyset$. This contradicts our non-triviality assumption. We can proceed similarly in the case of the factor $x^*_{i_1}$. $\square$

**Lemma 2.5.** *For arbitrary distinct elements $i, j, j' \in \{0, \dots, 7\}$ and for $0 \leqslant i' \leqslant 7$, the expression $\widetilde{W}^0(u_i, u_{i'}; u_j, u_{j'}) = 2^{-4} \sum_{\sigma \in S_8} A_\sigma(u_i, u_{i'}; u_j, u_{j'})$ contains at most $2^4 \cdot 8$ non-zero terms.*

*Proof.* By the previous lemma, each term of $\varpi'_{ij} \wedge \varpi'_{ij'}$ is the exterior product of a uniquely defined pair of terms of the forms $\varpi'_{ij}$ and $\varpi'_{ij'}$. On the other hand, this term of $\varpi'_{ij} \wedge \varpi'_{ij'}$ determines a unique complementary factor in $x^*_0 \wedge \dots \wedge x^*_7$ which belongs to $\varpi'_{i'j} \wedge \varpi'_{i'j'}$. If such a factor exists, then $i' \notin \{j, j'\}$ and by the previous lemma this factor is the exterior product of a uniquely defined pair of terms of the forms $\varpi'_{i'j}$ and $\varpi'_{i'j'}$. Since the number of terms of $\varpi'_{ij} \wedge \varpi'_{ij'}$ equals at most 8 and due to the skew-symmetry of the 2-forms, the Lemma follows. $\square$

Suppose that $i, j, j' \in \{0, \dots, 7\}$ and $i', j, j' \in \{0, \dots, 7\}$ are two triples containing three distinct elements. Due to the skew-symmetry of the 2-forms, one has $\widetilde{W}^0(u_i, u_{i'}; u_j, u_{j'}) = \sum_{[\sigma] \in S'_8} A_\sigma(u_i, u_{i'}; u_j, u_{j'})$, where $S'_8 = S_8/S'$ and the subgroup $S' \subset S_8$ is generated by the 4 transpositions $(0, 1)$, $(2, 3)$, $(4, 5)$, and $(6, 7)$. By Lemma 2.5 this sum contains at most 8 non-zero terms. Let us describe these terms. To this end, using (2.8) we can rewrite the expression for $A_\sigma(v, v'; w, w')$ as

$$-\varepsilon(\sigma)\langle u_{i_0} v, u_{i_1} w\rangle\langle u_{i_2} v, u_{i_3} w'\rangle\langle u_{i_4} v', u_{i_5} w\rangle\langle u_{i_6} v', u_{i_7} w'\rangle,$$

as $\bar{u}_k = -u_k$ for all of the seven imaginary units and the elements $v, v', w, w'$ occur in this expression twice. Let $u \in B$ and $a \in B^\pm$. Applying the same arguments as in the proof of Lemma 2.3, we obtain that if $au = -ua$ then $(u_k a)u = (-u_k u)a$. But $au \neq -ua$ if and only if $a = \pm u$ or $a = \pm u_0$ or $u = \pm u_0$. In all these cases $(u_k a)u = (u_k u)a$. Since $\langle au, bu\rangle = \langle a, b\rangle$, we obtain the following expression for $A_\sigma(v, v'; w, w')$:

$$-\varepsilon(\sigma)\big\langle (u_{i_0} u)v, (u_{i_1} u)w\big\rangle\big\langle (u_{i_2} u)v, (u_{i_3} u)w'\big\rangle\big\langle (u_{i_4} u)v', (u_{i_5} u)w\big\rangle\big\langle (u_{i_6} u)v', (u_{i_7} u)w'\big\rangle$$

(the elements $v, v', w, w'$ occur in this expression twice).

Suppose now that $A_\sigma(u_i, u_{i'}; u_j, u_{j'}) \neq 0$ for some $\sigma \in S_8$. Right multiplication by $u$ determines the permutation $\sigma^u$ of the set $B$: $u_k u = \varepsilon^u_{u_k} \sigma^u(u_k)$ ($\varepsilon^u_{u_k} = \pm 1$). This permutation is even since if $u \neq u_0$ then $u^2 = -u_0$ and $\sigma^u$ is a product of four independent transpositions. The sequence $(\varepsilon^u_{u_0}, \dots, \varepsilon^u_{u_7})$ contains an even number of $-1$. One can easily verify this fact for $u = u_1$ using (2.13) and for the other imaginary units $u_l$ using an automorphism $\Phi$ for which $\Phi(u_1) = u_l$:

$$\Phi(u_k)\Phi(u_1) = \varepsilon^\Phi_{u_k} \sigma^\Phi(u_k) \cdot u_l = \varepsilon^\Phi_{u_k} \varepsilon^{u_l}_{\sigma^\Phi(u_k)} \sigma^{u_l}(\sigma^\Phi(u_k)),$$

$$\Phi(u_k u_1) = \Phi(\varepsilon^{u_1}_{u_k} \sigma^{u_1}(u_k)) = \varepsilon^{u_1}_{u_k} \varepsilon^\Phi_{\sigma^{u_1}(u_k)} \sigma^\Phi(\sigma^{u_1}(u_k)).$$

Taking into account that $\prod_{k=0}^{7}\varepsilon^{\Phi}_{u_k}=\prod_{k=0}^{7}\varepsilon^{\Phi}_{\sigma^{u_1}(u_k)}$, we have

$$\prod_{k=0}^{7}\varepsilon^{u_l}_{u_k}=\prod_{k=0}^{7}\varepsilon^{u_l}_{\sigma^{\Phi}(u_k)}=\prod_{k=0}^{7}\varepsilon^{u_1}_{u_k}.$$

Thus $A_\sigma(u_i,u_{i'};u_j,u_{j'})=A_{\sigma^{u_k}\sigma}(u_i,u_{i'};u_j,u_{j'})$ for all of the eight even permutations $\sigma^{u_k}$, $k=0,\dots,7$. It only remains to be proved that the permutations $\sigma^{u_k}\sigma$ determine distinct classes in the quotient group $S_8'$.

Suppose that $\sigma^{u_k}\sigma=\sigma^{u_p}\sigma\cdot s$ for some element $s\in S_8'$ and $k\neq p$. Taking into account that $\sigma^{u_p}\sigma^{u_k}=\sigma^{u_k}\sigma^{u_p}=\sigma^{u_q}$, where $u_q\in B$ and $u_q=\pm u_k u_p=\pm u_p u_k$, we can assume that $u_p=u_0$ and $\sigma(u_0)=u_0$. But for $u\in B$ we have $\{\pm u_0 u,\pm u_{i_1}u\}=\{\pm u_0,\pm u_{i_1}\}$ if and only if $u\in\{u_0,u_{i_1}\}$. Since $A_\sigma(u_i,u_{i'};u_j,u_{j'})\neq 0$, we have $u_{i_1}=u_l$ and $u_{i_3}=\pm u_{l'}u_{i_2}$, where $u_l=\pm u_i u_j$ and $u_{l'}=\pm u_i u_{j'}$. Taking into account that $u_l\neq u_{l'}$, we obtain that $u_{i_3}\neq\pm u_l u_{i_2}=\pm u_{i_1}u_{i_2}$, i.e. $u_k=u_0$, a contradiction. Thus the permutations $\sigma^{u_k}\sigma$ determine 8 distinct classes in $S_8'$. So if the sequences $(i,j,j')$ and $(i',j,j')$ contain 3 distinct elements then $W^0(i,i';j,j')=8A_\sigma(u_i,u_{i'};u_j,u_{j'})$, where $\sigma\in S_8$ is an arbitrary permutation such that $A_\sigma(u_i,u_{i'};u_j,u_{j'})\neq 0$. Using now the relations (2.13), we can describe such permutations for the following sequences $(i,i';j,j')$:

$$\begin{aligned}
(0,0;1,2):&\quad \sigma=(0,1,4,6,2,3,5,7),\quad \varepsilon(\sigma)=-1,\\
(0,1;2,3):&\quad \sigma=(0,2,4,7,5,6,1,3),\quad \varepsilon(\sigma)=-1,\\
(0,1;2,4):&\quad \sigma=(0,2,1,5,4,7,3,6),\quad \varepsilon(\sigma)=1.
\end{aligned}$$

For all these cases $A_\sigma(u_i,u_{i'};u_j,u_{j'})=-1$. Thus, if the sequences $i,j,j'$ and $i',j,j'$ or the sequences $i,i',j$ and $i,i',j'$ from the set $\{0,\dots,7\}$ contain three distinct elements (i.e. a sequence $ij,ij',i'j,i'j'$ generates either a rectangle or an interval) then $W^0(i,i';j,j')=-8$. We also proved that $W^0(i,i;j,j)=-24$ for all $0\leqslant i\neq j\leqslant 7$.

Let $\overline{D}'$, $\overline{D}_2'$ and $\overline{D}_4'$ be sets defined for the index set $\{0,\dots,7\}$ as $\overline{D}$, $\overline{D}_2$ and $\overline{D}_4$ were defined for the index set $\{0,\dots,8\}$. Then $\overline{D}_2'\subset\overline{D}_2$ and $\overline{D}_4'\subset\overline{D}_4$. Taking into account that $\#(\overline{D}_2')=(8\cdot7)(6\cdot2)/2$ and $\#(\overline{D}_4')=(8\cdot7)(6\cdot5)/4$ (for each pair $ij\in\overline{D}'$ there exist $6\cdot5$ ordered pairs $i'j'\in\overline{D}'$ such that $\{i,j\}\cap\{i',j'\}=\emptyset$), from (2.6) it follows that

$$\Omega_0^8(X_0,\dots,X_7)=-24(8\cdot7)-8(8\cdot7\cdot12)-8(8\cdot7\cdot30)=-14\cdot1440,$$

hence $\Omega_0^8$ is not trivial.

We must finally prove that the canonical 8-form on any Spin(9)-manifold $(M^{16},g,\nu^9)$, given in the statement, is globally defined. In other words, we must prove that the definition of the form $\Omega_0^8$ is independent of the choice of the basis $\{I_j\}$ of the space $V^9=\nu^9(p)$, $p\in M$, satisfying the relations (2.3). Indeed, given one such basis $\{I_j\}$, any other basis $\{I_j'\}$ is obtained as $I_i'=\sum_{0\leqslant j\leqslant 8}m_i^j I_j$, for $i=0,\dots,8$, and $(m_i^j)\in\mathrm{SO}(9)$. From this fact it follows in particular that the Spin(9)-groups associated with these two bases coincide. But as we remarked above, $\pi(\mathrm{Spin}(9)) = \mathrm{SO}(9) = SO(V^9)$, i.e. there exists some element $s \in \mathrm{Spin}(9)$ such that $sI_js^{-1} = I'_j$, for all $j = 0, \ldots, 8$. Now since the group Spin(9) preserves the scalar product $g_p = \langle \cdot, \cdot \rangle$ on $T_pM \equiv \mathbb{O}^2$ and the form $\Omega_0^8$ is Spin(9)-invariant, the form $\Omega_0^8$ does not depend on the chosen basis $\{I_j\}$.

### 2.3 Some Corollaries to Theorem 1.1

We can get some consequences of the proof of Theorem 1.1. By (2.5) with $ij = i'j' \in A_1^+$, the 4-form $\sum_{0\leqslant i,j\leqslant 8} \varpi_{ij} \wedge \varpi_{ij}$ on the space $T_pM \equiv \mathbb{O}^2$ is invariant with respect to the action of each of the subgroups $M_{kl}^t$ generating the Lie group Spin(9). It is Spin(9)-invariant hence trivial ([13, Sect. 5]) so it defines a global (trivial) 4-form on $M$. We thus obtain the next corollary to Theorem 1.1.

**Corollary 2.6.** *The 4-form $\sum_{0\leqslant i<j\leqslant 8} \omega_{ij} \wedge \omega_{ij} = 0$, vanishes, i.e. we have*

$$\sum_{0\leqslant i<j\leqslant 8} \big\{\omega_{ij}(X,Y)\omega_{ij}(Z,W) - \omega_{ij}(X,Z)\omega_{ij}(Y,W) + \omega_{ij}(Y,Z)\omega_{ij}(X,W)\big\} = 0,$$

*or, equivalently,*

$$\underset{XYZ}{\mathfrak{S}} \sum_{0\leqslant i<j\leqslant 8} \omega_{ij}(X,Y)W^\flat(I_{ij}Z) = 0, \qquad X,Y,Z,W \in \mathfrak{X}(M). \tag{2.14}$$

Moreover, since the 8-form $(\sum_{0\leqslant i,j\leqslant 8} \varpi_{ij} \wedge \varpi_{ij}) \wedge (\sum_{0\leqslant i',j'\leqslant 8} \varpi_{i'j'} \wedge \varpi_{i'j'})$ vanishes, we can rewrite the expression of the canonical form as

**Corollary 2.7.**

$$\Omega^8 = -\frac{1}{2} \sum_{\substack{0\leqslant i,j\leqslant 8\\ 0\leqslant i',j'\leqslant 8}} (\omega_{ij} \wedge \omega_{i'j'} - \omega_{i'j} \wedge \omega_{ij'}) \wedge (\omega_{ij} \wedge \omega_{i'j'} - \omega_{i'j} \wedge \omega_{ij'}).$$

Furthermore, given a triple $ijq$, we denote by $\widehat{ijq}$ the new triple obtained by replacing the element $k$ (if it occurs in $ijq$) by $l$ and the element $l$ (if it occurs in $ijq$) by $k$. It is easy to verify that for the restriction $\bar\sigma_{ijq} = \sigma_{ijq}|T_pM$, one has

$$(M_{kl}^t)^*\bar\sigma_{ijq} = \begin{cases} \bar\sigma_{ijq}, & \text{if } \ \{k,l\} \cap \{i,j,q\} = \emptyset, \\ \bar\sigma_{ijq}, & \text{if } \ \{k,l\} \subset \{i,j,q\}, \\ \cos 2t \cdot \bar\sigma_{ijq} + \sin 2t \cdot \bar\sigma_{\widehat{ijq}}, & \text{if } \ \{k,l\} \cap \{i,j,q\} = \{k\}, \\ \cos 2t \cdot \bar\sigma_{ijq} - \sin 2t \cdot \bar\sigma_{\widehat{ijq}}, & \text{if } \ \{k,l\} \cap \{i,j,q\} = \{l\}, \end{cases}$$

and, consequently, the 4-form $\sum_{0\leqslant i,j,q\leqslant 8} \bar\sigma_{ijq} \wedge \bar\sigma_{ijq}$ on the space $T_pM \equiv \mathbb{O}^2$ is invariant with respect to the action of each of the subgroups $M_{kl}^t$ generating the Lie group Spin(9). It is Spin(9)-invariant and, consequently, it is also trivial ([13, Sect. 5]), so we obtain

**Corollary 2.8.** *The* 4*-form* $\sum_{0\leqslant i<j<k\leqslant 8}\sigma_{ijk}\wedge\sigma_{ijk}$*, vanishes, i.e. we have*

$$\sum_{0\leqslant i<j<k\leqslant 8}\big\{\sigma_{ijk}(X,Y)\sigma_{ijk}(Z,W)-\sigma_{ijk}(X,Z)\sigma_{ijk}(Y,W) \\ +\sigma_{ijk}(Y,Z)\sigma_{ijk}(X,W)\big\}=0.$$

*Remark* 2.9. Using the method of the proof of Theorem 1.1 one could obtain the expression for the canonical form $\Omega^8$ in terms of the 2-forms $\sigma_{ijp}$. But since the proof is technically more complicated, we state it as the next

**Conjecture.** *The canonical* 8*-form* $\Omega^8$ *on the* Spin(9)*-manifold* $(M,g,\nu^9)$ *is given by*

$$\Omega^8=\frac{1}{4}\sum_{\substack{0\leqslant i,j\leqslant 8\\ 0\leqslant i',j'\leqslant 8}}\;\sum_{0\leqslant p,p'\leqslant 8}\sigma_{ijp}\wedge\sigma_{ijp'}\wedge\sigma_{i'j'p}\wedge\sigma_{i'j'p'}.$$

# 3 Spin(9)-structures as $G$-structures defined by a tensor

The concept of *$G$-structure defined* (or *characterized*) *by a tensor* is well known (see Bernard [8, pp. 210–212], Fujimoto [17, p. 24], Marín and de León [24, p. 377], and Salamon [28, p. 11]; cf. also [28, pp. 127, 175]). We now focus our attention to the case where $G=\mathrm{Spin}(9)$.

We would like to remark firstly that in this case the tensor used to define a Spin(9)-structure will never be a stable tensor (cf. Friedrich [14, p. 2], [16, p. 2]). A tensor on $\mathbb{R}^n$ is said to be stable if its orbit under the action of $\mathrm{GL}(n,\mathbb{R})$ is an open subset (see Hitchin [21, p. 2], Witt [32, §§3.2]). These special structures play an interesting role in the theory of $G$-structures. But for $G=\mathrm{Spin}(9)$ a simple computation of dimensions shows that the interior of any orbit on the space of 8-forms is void.

On the other hand, Friedrich's local bases $\{\omega_{ij},\sigma_{ijk}\}$ of $\Lambda^2M$ given in Section 1 are related to the decomposition of $\Lambda^2(\Delta_9)$, which we now recall (cf. e.g. Adams [3, Th. 4.6, (ii)]). Let $\lambda^r$ denote the representation arising from the $r$th exterior power representation of SO(9) via the homomorphism $\pi\colon \mathrm{Spin}(9)\to\mathrm{SO}(9)$. Then one has $\Delta_9\otimes\Delta_9=\sum_{r=0}^{4}\lambda^r$. Moreover, as $\Delta_9$ is self-dual, we have the decomposition of $\Delta_9\otimes\Delta_9\cong\Delta_9^*\otimes\Delta_9\cong\mathfrak{gl}(\mathbb{R},16)$ into symmetric and skew-symmetric components,

$$S^2(\Delta_9)=\lambda^0\oplus\lambda^1\oplus\lambda^4,\quad \Lambda^2(\Delta_9)=\lambda^2\oplus\lambda^3, \tag{3.1}$$

where $\lambda^0$ is the center of $\mathfrak{gl}(16,\mathbb{R})$.

We have proved in Theorem 1.1 that $\Omega_0^8$ is Spin(9)-invariant and non-trivial. We now prove that $\rho(\mathrm{Spin}(9))\subset\mathrm{GL}(16,\mathbb{R})$ is actually the stabilizer group of $\Omega_0^8$ in the group $\mathrm{GL}(16,\mathbb{R})$, showing that this group is no bigger than $\rho(\mathrm{Spin}(9))$.

We have

**Theorem 3.1.** *The stabilizer group of the canonical 8-form $\Omega_0^8$ on $\mathbb{R}^{16}$, under the natural action of the group* $\mathrm{GL}(16,\mathbb{R})$*, is the Lie group* $\rho(\mathrm{Spin}(9))$*.*

*Proof.* To simplify notation in this proof, we will write simply Spin(9) and $\mathfrak{spin}(9)$ instead of $\rho(\mathrm{Spin}(9))$ and $\rho_*(\mathfrak{spin}(9))$, respectively. Let $G$ be the stabilizer group of $\Omega_0^8$ and $\mathfrak{g}$ its Lie algebra.

As $\mathfrak{spin}(9)$ is a subalgebra of $\mathfrak{gl}(16,\mathbb{R})$, the adjoint representation of $\mathfrak{gl}(16,\mathbb{R})$ induces the representation of $\mathfrak{spin}(9)$ on $\mathfrak{gl}(16,\mathbb{R})$. The set $\{I_{i_1\dots i_r}, 0 \leqslant i_1 < \dots < i_r \leqslant 8\}$ is a basis of the $\mathfrak{spin}(9)$-invariant subspace $\lambda^r$ of $\mathfrak{gl}(16,\mathbb{R})$ in (3.1), for $r = 1,\dots,4$, respectively. Moreover, all the operators in each $\lambda^r$ are traceless (for example, $2I_{i_1i_2i_3i_4} = [I_{i_1}, I_{i_2i_3i_4}]$). As the submodules in (3.1) are mutually not isomorphic, if $\mathfrak{g} \neq \mathfrak{spin}(9)$, then $\lambda^r \subset \mathfrak{g}$ for some $0 \leqslant r \leqslant 4$. We know that $\mathfrak{so}(16) = \lambda^2 \oplus \lambda^3$ and $\mathfrak{spin}(9) = \lambda^2$ and it is clear that $\lambda^0 \not\subset \mathfrak{g}$.

Suppose then that $\lambda^1 \subset \mathfrak{g}$. Then the one-parameter subgroup

$$M_8^t = \cosh t \cdot \mathrm{I} + \sinh t \cdot I_8 \subset \mathrm{GL}(16,\mathbb{R})$$

generated by the vector $I_8 \in \mathfrak{gl}(16,\mathbb{R})$, would be a subgroup of $G$. It is easy to verify (see the proof of (2.4)) that for any $0 \leqslant i < j \leqslant 8$,

$$(M_8^t)^*\varpi_{ij} = \begin{cases} \varpi_{ij}, & \text{if} \quad 8 \in \{i,j\}, \\ \cosh 2t \cdot \varpi_{ij} + \sinh 2t \cdot \bar{\sigma}_{ij8}, & \text{if} \quad i,j < 8. \end{cases}$$

Let $V \subset \mathbb{O}^2$ be (as in the proof of Theorem 1.1) the subspace with basis $X_i = (u_i, 0)$, $i = 0,\dots,7$. Then by (2.7), we have $\varpi_{i8}|V = 0$. Further, $\bar{\sigma}_{ij8}|V = -\varpi_{ij}|V$, because by (2.3) one has $I_8 v = -v$ for all $v \in V$. Now taking into account the expression for the 8-form $\Omega_0^8$, we obtain that

$$((M_8^t)^*\Omega_0^8)|V = \sum_{\substack{0\leqslant i,j\leqslant 7\\ 0\leqslant i',j'\leqslant 7}} (\cosh 2t - \sinh 2t)^4 (\varpi_{ij} \wedge \varpi_{ij'} \wedge \varpi_{i'j} \wedge \varpi_{i'j'})|V,$$

i.e. $((M_8^t)^*\Omega_0^8)|V = (\cosh 2t - \sinh 2t)^4 \Omega_0^8|V$. Thus $\lambda^1 \not\subset \mathfrak{g}$, because $\Omega_0^8|V \neq 0$.

The form $\Omega_0^8$ is not SO(16)-invariant. In the opposite case, it would determine a non-trivial SO(17)-invariant harmonic differential 8-form on the 16-dimensional sphere $S^{16}$, but since $H^8(S^{16},\mathbb{R}) = 0$, we would get a contradiction. Hence $\lambda^3 \not\subset \mathfrak{g}$.

So if $\mathfrak{g} \neq \mathfrak{spin}(9)$ then $\mathfrak{g} = \lambda^4 \oplus \mathfrak{spin}(9)$. It is clear that $[\lambda^4, \lambda^4] \subset \mathfrak{so}(16)$ and, consequently, the subspace $\lambda^4 \oplus \mathfrak{spin}(9)$ is a Lie algebra if and only if $[\lambda^4, \lambda^4] \subset \mathfrak{spin}(9)$. But since $[I_k I_{i_1i_2i_3}, I_k I_{j_1j_2j_3}] = -[I_{i_1i_2i_3}, I_{j_1j_2j_3}]$ for any 4-element subsets $\{k, i_1, i_2, i_3\}$ and $\{k, j_1, j_2, j_3\}$ of the set $\{0,\dots,8\}$, we have $[\lambda^3, \lambda^3] \subset [\lambda^4, \lambda^4]$. As the homogeneous space SO(16)/Spin(9) is not a symmetric space (cf. Helgason [20, p. 518]), i.e. $[\lambda^3, \lambda^3] \not\subset \mathfrak{spin}(9) = \lambda^2$, we obtain that $[\lambda^4, \lambda^4] \not\subset \mathfrak{spin}(9)$, that is, $\mathfrak{g} = \mathfrak{spin}(9)$.

It only remains to be proved that the group $G$ is connected. To this end, similarly to Brown and Gray in [13, Prop. 5.3], we shall find the normalizer (containing $G$) of the group Spin(9) in $\mathrm{GL}(16,\mathbb{R})$. Suppose that $A \in \mathrm{GL}(16,\mathbb{R})$

normalizes Spin(9). Since Spin(9) has no outer automorphisms there exists an element $B \in \mathrm{Spin}(9)$ such that $AB^{-1}$ is in the centralizer in $\mathrm{GL}(16, \mathbb{R})$ of Spin(9). The complexification of the 16-dimensional representation of Spin(9) is irreducible so $AB^{-1}$ is a scalar operator $tI$, $t \in \mathbb{R}$. But the operator $tB$ preserves the 8-form if and only if $t^8 = 1$. Since by definition Spin(9) contains $I_1 I_2 I_1 I_2 = -I$, we have $G = \mathrm{Spin}(9)$. This completes the proof. □

As a consequence of Theorem 3.1 we have

**Corollary 3.2.** *A reduction of the structure group of the bundle of oriented orthonormal frames of a connected, oriented* 16*-dimensional Riemannian manifold $M$ to* Spin(9) *is characterized by a parallel* 8*-form $\Omega^8$ which is linearly equivalent at each point $p \in M$ to the* Spin(9)*-invariant* 8*-form $\Omega_0^8$ on $\mathbb{R}^{16}$.*

*Proof.* According to [13, Props. 5.2, 5.4, 5.5] we must only prove that $\Omega_0^8$ is Spin(9)-invariant but not SO(16)-invariant. We have proved the first fact in Theorem 1.1 and the second one in the proof of Theorem 3.1. □

# 4 The curvature tensor of the Cayley planes

We now apply our previous conclusions to obtain an expression of the curvature tensor of the Cayley planes in terms of the nine local symmetric involutions involved and then to relate it to the well-known expression in terms of triality given by Brown and Gray [13], to the one in terms of the brackets of the Lie algebra $\mathfrak{f}_4$ of $F_4$, furnished by Brada and Pécaut-Tison [11, 12], and also to the expression given in [26].

First recall ([4, 13]) that the curvature tensor $R$ of a non-flat Spin(9)-manifold is a non-zero multiple of the curvature tensor $R^{\mathbb{O}\mathrm{P}(2)}$ of $\mathbb{O}\mathrm{P}(2)$. Further, as duality reverses curvature, in the next formulas we can take a constant $c \in \mathbb{R}\backslash\{0\}$, being understood that $c > 0$ (resp. $c < 0$) in the compact (resp. noncompact) case.

Then we have

**Proposition 4.1.** *The curvature tensor $R_{XY}Z$ of the Cayley planes is given by*

$$R_{XY}Z = -\frac{c}{4} \sum_{0 \leqslant i < j \leqslant 8} \omega_{ij}(X,Y) I_{ij} Z, \qquad c \in \mathbb{R}\backslash\{0\}. \tag{4.1}$$

*Proof.* The form $\lambda \sum_{i<j} \omega_{ij} \otimes I_{ij}$, $\lambda \in \mathbb{R}$, is a $\rho_*(\mathfrak{spin}(9))$-valued 2-form. Moreover, the necessary algebraic conditions are clearly satisfied by $\lambda \sum_{i<j} \omega_{ij} \otimes \omega_{ij}$, except for the Bianchi identity, but this is immediate from equation (2.14).

As the curvature tensor is a non-zero multiple of $R^{\mathbb{O}\mathrm{P}(2)}$, it only rests to find the coefficient of the right-hand side of (4.1). To compute the sectional curvature we take two orthonormal vectors $v = (x_1, x_2), w = (y_1, y_2) \in S^{15} \subset T_pM \equiv \mathbb{O}^2$. Now, the map $(v, w) \mapsto -\lambda \sum_{0 \leqslant i,j \leqslant 8} \omega_{ij}^2(v, w)$ is easily seen from (2.4) to be invariant under each endomorphism $M_{kl}^t$, hence under Spin(9). Consider then the orthonormal basis $e_1 = (u_0, 0), \ldots, e_8 = (u_7, 0), e_9 = (0, u_0), \ldots, e_{16} = (0, u_7)$ of $\mathbb{O}^2 \equiv T_pM$. As Spin(9) acts transitively on $S^{15}$, there exists an element of Spin(9) mapping $v$ to $(u_0, 0)$ and $w$ to a vector $w' = \sum_{k=0}^{7} \big(\mu_k(u_k, 0) + \nu_k(0, u_k)\big)$ with $\mu_0 = 0$. So for certain $\lambda \in \mathbb{R}\backslash\{0\}$, as a computation using (2.2) and (2.3) shows, we have for $R_{vwvw} = g(R_{vw}v, w)$ that

$$R_{vwvw} = -\lambda \sum_{\substack{0\leqslant i<j\leqslant 8\\ 0\leqslant k\leqslant 7}} \big\langle (u_0, 0), I_{ij}\big(\mu_k(u_k, 0) + \nu_k(0, u_k)\big)\big\rangle^2 = -\lambda\Big(3\sum_{k=0}^{7}\mu_k^2 + 1\Big).$$

In fact, the operator $I_{ij}$ acts on the basis $\{(u_k, 0), (0, u_k), k = 0, \ldots, 7\}$ as a permutation (up to sign) and for each vector $(u_k, 0)$, $k \geqslant 1$, there exist precisely four different pairs $\{u_i, u_j\}$ for which $u_i(u_ju_k) \overset{\mp}{=} (u_iu_j)u_k \overset{\mp}{=} u_0$ and for each vector $(0, u_k)$, $k \geqslant 0$ there exists a unique pair $\{u_i, u_8\}$ for which $u_iu_k \overset{\mp}{=} u_0$ ($i = k$ in this case), where $\overset{\mp}{=}$ means "equal up to sign."

Taking $\lambda = -\frac{c}{4}$, we see that the absolute value of the sectional curvature belongs to $[|c|/4, |c|]$. □

Brown and Gray give in [13, (6.12)] an explicit expression for the curvature tensor $R_{XY}Z$ of $\mathbb{O}$P(2).

Letting $\mathbb{R}^{16} \equiv \mathbb{O}^2$, according to Lemma 3.1 and formulas (4.1), (4.2), and (6.2) in their paper, and only changing some notations, Brown and Gray's formula for the curvature tensor can be written as $R_{XY}Z = S_{XY}Z - S_{YX}Z$, where

$$\begin{aligned} S_{XY}Z = -\frac{c}{4}\big(&4\langle y_1, z_1\rangle x_1 + (z_1y_2)\bar{x}_2 + (x_1y_2)\bar{z}_2, \\ &4\langle y_2, z_2\rangle x_2 + \bar{x}_1(y_1z_2) + \bar{z}_1(y_1x_2)\big), \end{aligned} \tag{4.2}$$

for $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $Z = (z_1, z_2) \in \mathbb{O}^2$.

They also comment that an expression 'similar' to the well-known ones for the spaces of constant either holomorphic or quaternionic sectional curvature cannot be given, because, differently to U($n$) and Sp($n$)Sp(1), the group Spin(9) has not proper normal subgroups.

However, in [26, Prop. 4] a simple expression for either $R^{\mathbb{O}\mathrm{P}(2)}$ or $R^{\mathbb{O}\mathrm{H}(2)}$ has been given in terms of the nine local symmetric operators. We can write it as $R_{XY}Z = S'_{XY}Z - S'_{YX}Z$, where

$$S'_{XY}Z = -\frac{c}{4}\Big(3g(Y, Z)X + \sum_{0\leqslant i\leqslant 8} g(I_iY, Z)I_iX\Big), \tag{4.3}$$

respectively.

This expression, in terms of the octonion algebra has the following form (see [26, Prop. 4,(15)]) for $X = (x_1, x_2)$, $Y = (y_1, y_2)$ and $Z = (z_1, z_2)$,

$$\begin{aligned} S'_{XY}Z = -\frac{c}{4}\big(&(x_1\bar{y}_1)z_1 + (x_1y_2)\bar{z}_2 + (z_1\bar{y}_1)x_1 + (z_1y_2)\bar{x}_2, \\ &\bar{z}_1(y_1x_2) + z_2(\bar{y}_2x_2) + x_2(\bar{y}_2z_2) + \bar{x}_1(y_1z_2)\big). \end{aligned} \tag{4.4}$$

Using the well-known octonion identities $\langle x, y\rangle = \langle \bar{x}, \bar{y}\rangle$ and $2\langle x, y\rangle a = (ax)\bar{y} + (ay)\bar{x}$ and their conjugated $2\langle x, y\rangle a = y(\bar{x}a) + x(\bar{y}a)$ for arbitrary $x, y, a \in \mathbb{O}$ (see [27, Lect. 15, (1)]), we obtain that

$$\begin{aligned} 4\langle y_1, z_1\rangle x_1 - 4\langle x_1, z_1\rangle y_1 &= (z_1\bar{y}_1 + y_1\bar{z}_1)x_1 + (x_1\bar{y}_1)z_1 + (x_1\bar{z}_1)y_1 \\ &\quad - (z_1\bar{x}_1 + x_1\bar{z}_1)y_1 - (y_1\bar{x}_1)z_1 - (y_1\bar{z}_1)x_1 \\ &= (x_1\bar{y}_1 - y_1\bar{x}_1)z_1 + (z_1\bar{y}_1)x_1 - (z_1\bar{x}_1)y_1, \end{aligned}$$

and

$$\begin{aligned} 4\langle y_2, z_2\rangle x_2 - 4\langle x_2, z_2\rangle y_2 &= x_2(\bar{y}_2 z_2 + \bar{z}_2 y_2) + z_2(\bar{y}_2 x_2) + y_2(\bar{z}_2 x_2) \\ &\quad - y_2(\bar{x}_2 z_2 + \bar{z}_2 x_2) - z_2(\bar{x}_2 y_2) - x_2(\bar{z}_2 y_2) \\ &= z_2(\bar{y}_2 x_2 - \bar{x}_2 y_2) + x_2(\bar{y}_2 z_2) - y_2(\bar{x}_2 z_2), \end{aligned}$$

i.e. the expressions $S_{XY}Z - S_{YX}Z$ and $S'_{XY}Z - S'_{YX}Z$ coincide.

Brada and Pécaut-Tison's [12] expression for the curvature tensor in terms of a "cross product," coincides with Brown and Gray's expression up to a factor $-c/4$ (cf. Remark in [12, p. 145], given without proof). To prove that both expressions coincide it suffices to use the property $(a, b, c) = -(\bar{a}, b, c)$ of the associator of $a, b, c \in \mathbb{O}$, and four different expressions for $2\langle x, y\rangle a$ given above.

Then we have

**Proposition 4.2.** *The curvature tensor of the Cayley planes is given by either the expression* (4.1) *or any of those obtained using* (4.2), (4.3), *or* (4.4).

*Moreover, one has*

$$R_{XY}Z = \frac{1}{5}\sum_{0\leqslant j\leqslant 8} I_j R_{XY} I_j Z. \tag{4.5}$$

*Proof.* Since the equivalence of the expressions (4.3) and (4.4) was proved in [26] and the equivalence of the curvature tensors defined by (4.2) and (4.4) was established above, only the equivalence of the curvature tensor defined by either (4.3) or (4.4) with the curvature tensor (4.1) remains to be proved. We now prove this in two ways.

We know that the operator $R_{XY}$ is a linear combination of the operators $I_{kl}$, $0 \leqslant k < l \leqslant 8$, as the isotropy representation $\mathfrak{spin}(9) \to \mathrm{End}(T_pM)$ is the 16-dimensional spin representation of $\mathfrak{spin}(9)$. Since for any fixed pair $kl$, $0 \leqslant k \neq l \leqslant 8$, by (2.1) one has $\sum_{0\leqslant j\leqslant 8} I_j I_k I_l I_j = 5\, I_k I_l$, we get the formula (4.5).

On account of (4.3) and (4.5) we then obtain that

$$\begin{aligned} 5R_{XY}Z &= \sum_{0\leqslant j\leqslant 8} I_j R_{XY} I_j Z \\ &= -\frac{c}{4}\Big(3\sum_{0\leqslant j\leqslant 8}\big\{g(I_jY, Z)I_jX - g(I_jX, Z)I_jY\big\} + 9\big\{g(Y, Z)X - g(X, Z)Y\big\} \end{aligned}$$

$$+2 \sum_{0\leqslant i<j\leqslant 8} \{g(X, I_{ij}Z)I_{ij}Y - g(Y, I_{ij}Z)I_{ij}X\}\Big).$$

Again using (4.3) we then have

$$R_{XY}Z = \frac{c}{4} \sum_{0\leqslant i<j\leqslant 8} \big(g(Y, I_{ij}Z)I_{ij}X - g(X, I_{ij}Z)I_{ij}Y\big),$$

hence by virtue of Corollary 2.6 we deduce that

$$R_{XY}Z = -\frac{c}{4} \sum_{0\leqslant i<j\leqslant 8} g(X, I_{ij}Y)I_{ij}Z,$$

i.e. formula (4.1).

We can also prove the equivalence of the curvature tensor (4.1) and that defined by (4.3) considering for any vector fields $X, Y$ and the basis of 2-forms $\{\omega_{ij}, \sigma_{ijk}\}$ being as in Section 1, Friedrich's expression [15, Lemma 3.2]

$$8\, X^\flat \wedge Y^\flat = \sum_{0\leqslant i<j\leqslant 8} \omega_{ij}(X, Y)\omega_{ij} + \sum_{0\leqslant i<j<k\leqslant 8} \sigma_{ijk}(X, Y)\sigma_{ijk}, \tag{4.6}$$

where $X^\flat$ and $Y^\flat$ denote the differential 1-forms metrically dual to $X$ and $Y$, respectively. From (4.6), as a simple computation shows, we obtain the formula

$$8 \sum_{0\leqslant l\leqslant 8} (I_l X)^\flat \wedge (I_l Y)^\flat = 5 \sum_{0\leqslant i<j\leqslant 8} \omega_{ij}(X, Y)\omega_{ij} - 3 \sum_{0\leqslant i<j<k\leqslant 8} \sigma_{ijk}(X, Y)\sigma_{ijk}. \tag{4.7}$$

From equations (4.6) and (4.7) one easily concludes. □

We omit for the sake of brevity the discussions corresponding to the three next questions.

*Remark* 4.3. Another (longer but equivalent) expression in terms of the operators $I_j$ for the curvature tensor of the Cayley planes has been given in [25, (4.18)].

*Remark* 4.4. Hangan gave in [18, pp. 68–69] another expression for the curvature tensor of $\mathbb{O}\mathrm{P}(2)$, this space viewed as a differentiable manifold with three charts as in Besse [9, p. 91]. The relation of his expression with those given above remains as an open problem.

*Remark* 4.5. The canonical metric on the open unit ball model of $\mathbb{O}\mathrm{P}(2)$, with

$$B^2 = \{(u, v) \in \mathbb{O}^2 : |u|^2 + |v|^2 < 1\}$$

has been recently found by Held, Stavrov and Van Koten in [19, Sect. 8]. It is given by

$$g = \frac{c}{4} \frac{|\mathrm{d}u|^2(1 - |v|^2) + |\mathrm{d}v|^2(1 - |u|^2) + 2\,\mathrm{Re}\big(u\bar{v}(\mathrm{d}v\, \mathrm{d}\bar{u})\big)}{\big(1 - |u|^2 - |v|^2\big)^2}, \quad c < 0.$$

It would be interesting to relate their expression to the results of the present paper. This also remains as an open problem.

# Appendix A

We now give some comments on Brada and Pécaut-Tison's expression of the canonical 8-form, showing that their 8-form $\omega$ (see [11, Def. 5.2] or [12, Def. 5.2]) is not Spin(9)-invariant, and describing some crucial gaps in their proof.

To define this form $\omega$ they identify the space $\mathbb{R}^{16}$ with the space $\mathbb{O}^2$ and consider the cross product $u \times v = \mathrm{Im}(\bar{v}u) = \frac{1}{2}(\bar{v}u - \bar{u}v)$ of two elements $u, v \in \mathbb{O}$ and the "cross product" of two vectors $U, V \in \mathbb{O}^2$ as

$$U \times V = \bar{u}_1 \times \bar{v}_1 + u_2 \times v_2, \quad \text{where} \quad U = (u_1, u_2), V = (v_1, v_2). \tag{4.8}$$

So the octonion $U \times V = \mathrm{Im}(v_1\bar{u}_1) + \mathrm{Im}(\bar{v}_2 u_2)$ is pure imaginary. By Definition 5.2 in [11, 12] the 8-form $\omega$ is given (up to a non-zero factor) by

$$\begin{aligned}\omega(U_1, U_2, \ldots, U_8) = 2^{-7} \sum_{\sigma \in S_8} \varepsilon(\sigma)&[(U_{\sigma(1)} \times U_{\sigma(2)})(U_{\sigma(3)} \times U_{\sigma(4)})] \\ &\cdot[(U_{\sigma(5)} \times U_{\sigma(6)})(U_{\sigma(7)} \times U_{\sigma(8)})].\end{aligned} \tag{4.9}$$

Putting

$$a = U_1 \times U_2, \quad b = U_3 \times U_4, \quad c = U_5 \times U_6, \quad d = U_7 \times U_8,$$

as in [12] we obtain that

$$\begin{aligned}&2^4[(ab)(cd) + (ab)(dc) + (ba)(cd) + (ba)(dc) \\ &\quad + (cd)(ab) + (cd)(ba) + (dc)(ab) + (dc)(ba)] \\ &= 2^5 \mathrm{Re}[(ab)(cd) + (ab)(dc) + (ba)(cd) + (ba)(dc)] \\ &= 2^5 \mathrm{Re}[(ab + ba)(cd + dc)] \\ &= 2^7 \mathrm{Re}(ab)\,\mathrm{Re}(cd),\end{aligned}$$

because all the elements $a, b, c, d$ are pure imaginary, so that, for example, $\overline{(ab)(cd)} = (dc)(ba)$ and $\overline{ab} = ba$. Remark also that in [12, p. 150] there is a misprint in this formula (i.e. the last expression in [12] is said to be equal to $2^7 \mathrm{Re}[(ab)(cd)]$). Taking into account that by definition the cross product in $\mathbb{O}$ and, consequently, the "cross product" (4.8) in $\mathbb{O}^2$ is skew-symmetric, we obtain that

$$\begin{aligned}\omega(U_1, U_2, \ldots, U_8) = \sum_{\sigma \in S_8^*} \varepsilon(\sigma)\,&\mathrm{Re}[(U_{\sigma(1)} \times U_{\sigma(2)})(U_{\sigma(3)} \times U_{\sigma(4)})] \\ &\cdot \mathrm{Re}[(U_{\sigma(5)} \times U_{\sigma(6)})(U_{\sigma(7)} \times U_{\sigma(8)})].\end{aligned} \tag{4.10}$$

where $S_8^* = \{\sigma \in S_8 : \sigma(2i-1) < \sigma(2i), \sigma(1) < \sigma(3), \sigma(5) < \sigma(7), \sigma(1) < \sigma(5)\}$. It is easy to verify that $\#(S_8^*) = 8!/2^7 = 35 \cdot 9$ and $\sigma(1) = 1$ (its lowest number) for arbitrary $\sigma \in S_8^*$.

To prove that this form $\omega$ is not Spin(9)-invariant it is sufficient to show that for the operator $I_{78}$ (which is an element of the Lie algebra $\rho_*(\mathfrak{spin}(9)) \subset \mathfrak{so}(16)$) and some vectors $U_1, \ldots U_8 \in \mathbb{O}^2$ the following expression

$$\omega(I_{78}U_1, U_2, \ldots, U_8) + \omega(U_1, I_{78}U_2, \ldots, U_8) + \cdots + \omega(U_1, U_2, \ldots, I_{78}U_8) \tag{4.11}$$

does not vanish.

Put $U_1 = (0, u_0)$ and $U_2 = (u_0, 0), \dots, U_8 = (u_6, 0)$. We will show that in this case the first term $T_1$ in (4.11) equals 63 and that $|T_i| \leqslant 9$ for each other term $T_i$, $i = 2, \dots, 8$. Since we have exactly 7 terms $T_i$ with $|T_i| \leqslant 9$, the sum of all these eight terms is necessarily non-zero if, for example, the eighth term $T_8 \geqslant -8$.

Consider the first term $T_1 = \omega(I_{78}U_1, U_2, \dots, U_8)$ in the sum (4.11). By (2.3) $I_{78}(0, u_0) = (u_7, 0)$. Since the product of any pair of elements of the basis $B = \{u_0, \dots, u_7\}$ is an imaginary unit (up to a sign), then each of the $35 \cdot 9$ terms in the expression (4.10) for $\omega(I_{78}U_1, U_2, \dots, U_8)$ is given by

$$\varepsilon(f)\varepsilon(\sigma')\operatorname{Re}[(u_{\sigma'(1)}\bar{u}_{\sigma'(0)})(u_{\sigma'(3)}\bar{u}_{\sigma'(2)})]\operatorname{Re}[(u_{\sigma'(5)}\bar{u}_{\sigma'(4)})(u_{\sigma'(7)}\bar{u}_{\sigma'(6)})], \tag{4.12}$$

where $f$ is the unique bijection such that $\sigma' = f \circ \sigma \circ f^{-1}$ and $\sigma'$ is a permutation of the set $\{0, \dots, 7\}$ with its natural ordering. Here $\varepsilon(f)\varepsilon(\sigma') = \varepsilon(\sigma)$ and $\varepsilon(f) = -1$ because $f(1) = 7$ and $f(i) = i - 2$ for $i \geqslant 2$. Since the product of all the elements of the basis $B$ is a real number $\pm 1$ (see (4.13) below), then the term (4.12) is non-zero iff its first factor of the form $\operatorname{Re}[\,\cdot\,]$ is non-zero. That is, we have 7 possibilities for a choice of the first pair $\{\sigma'(0), \sigma'(1)\}$ because $\sigma'(0) = 7$ ($\sigma(1) = 1$) and 3 possibilities for a choice of the second pair $\{\sigma'(2), \sigma'(3)\}$ such that $u_{\sigma'(2)}u_{\sigma'(3)} = \pm u_{\sigma'(0)}u_{\sigma'(1)}$. Thus the number of non-zero terms (4.12) equals 63 because $\sigma(5)$ is the lowest number of the set $\{\sigma(5), \dots, \sigma(8)\}$ and then for a choice of $\sigma(6)$ one has 3 possibilities. Remark that each such a term equals $\pm 1$ and that at least one of them is positive. This positive term corresponds to the even permutation $\sigma = (1, 2, 3, 8, 4, 5, 6, 7)$ with $\sigma' = (7, 0, 1, 6, 2, 3, 4, 5)$ because by (2.13)

$$\begin{aligned}\operatorname{Re}[(u_0\bar{u}_7)(u_6\bar{u}_1)]\operatorname{Re}[(u_3\bar{u}_2)(u_5\bar{u}_4)] &= (-1)^4\operatorname{Re}[(1 \cdot \mathbf{ke})(\mathbf{je} \cdot \mathbf{i})]\operatorname{Re}[(\mathbf{k} \cdot \mathbf{j})(\mathbf{ie} \cdot \mathbf{e})] \\ &= \operatorname{Re}[(\mathbf{ke})(\mathbf{ke})]\operatorname{Re}[(-\mathbf{i})(-\mathbf{i})] = 1.\end{aligned} \tag{4.13}$$

Now we will prove that all the non-zero terms (4.12) coincide for any $\sigma' \in S_8$. Taking into account the symmetries of the expression (4.12) we can suppose that $\sigma'(0) = 0$. Since all the elements of the imaginary units set $B^0 = B \setminus u_0$ anticommute and $\bar{u} = -u$ for such a unit, we can rewrite the expression (4.12) in the following form (up to a factor $\varepsilon(f)$)

$$\phi(\sigma') = \varepsilon(\sigma')\operatorname{Re}[(u_0 u_{\sigma'(1)})(u_{\sigma'(2)}u_{\sigma'(3)})]\operatorname{Re}[(u_{\sigma'(4)}u_{\sigma'(5)})(u_{\sigma'(6)}u_{\sigma'(7)})],$$

where $\sigma' \in S_8$, $\sigma'(0) = 0$. As we remarked above, this expression is not zero iff its first factor $\operatorname{Re}[\,\cdot\,]$ is not zero. In this case the algebra generated by the three imaginary units $u_{\sigma'(1)}, u_{\sigma'(2)}, u_{\sigma'(3)}$ is isomorphic to the quaternion algebra $\mathbb{H}$. In particular, the imaginary unit $u_{\sigma'(4)}$ is orthogonal to these three vectors and $u_{\sigma'(3)} = \varepsilon_{12}u_{\sigma'(1)}u_{\sigma'(2)}$. Therefore ([27, Lect. 15, Lemma 1]) there exists an automorphism $\Phi$ of $\mathbb{O}$ such that $\Phi(u_{\sigma'(1)}) = u_1$, $\Phi(u_{\sigma'(2)}) = u_2$, and $\Phi(u_{\sigma'(4)}) = u_4$. Then $\Phi(u_{\sigma'(3)}) = \varepsilon_{12}u_3$. It is easy to see that $\Phi$ preserves the set $B^0 \cup (-B^0)$ and, consequently, $\Phi(u_k) = \varepsilon^{\Phi}_{u_k}\sigma^{\Phi}(u_k)$, where $\varepsilon^{\Phi}_{u_k} = \pm 1$ and $\sigma^{\Phi}$ is some permutation in $S_8$ preserving $u_0$, and $\prod_{k=0}^{7} \varepsilon_{u_k}^{\Phi} \cdot \varepsilon(\sigma^{\Phi}) = 1$ (see the proof of Lemma 2.2). Thus

$$\begin{aligned}\varepsilon(\sigma')\,[(u_0 u_{\sigma'(1)})(u_{\sigma'(2)} u_{\sigma'(3)})] \cdot [(u_{\sigma'(4)} u_{\sigma'(5)})(u_{\sigma'(6)} u_{\sigma'(7)})] \\ = \varepsilon(\sigma'')[(u_0 u_1)(u_2 u_3)] \cdot [(u_4 u_{\sigma''(5)})(u_{\sigma''(6)} u_{\sigma''(7)})],\end{aligned}$$

where $\sigma'' = \sigma^{\Phi}\sigma' \in S_8$, because $\varepsilon(\sigma'') = \varepsilon(\sigma')\varepsilon(\sigma^{\Phi}) = \varepsilon(\sigma')\prod_{k=0}^{7} \varepsilon_{u_k}^{\Phi}$. Note also that $\sigma''(j) = j$ if $j = 0,1,2,3,4$ and $\sigma''(j) \in \{5,6,7\}$ for $j = 5,6,7$. Since all the expressions in square brackets are real and $\mathbf{i}\cdot\mathbf{j}\cdot\mathbf{k} = -1$, we have

$$\phi(\sigma') = -\varepsilon(\sigma'') \cdot [(u_4 u_{\sigma''(5)})(u_{\sigma''(6)} u_{\sigma''(7)})].$$

But $u_{\sigma''(4+i)} = u_{\tilde\sigma(i)} u_4$, $i = 1,2,3$, where $\tilde\sigma$ is some permutation in $S_3$. It is clear that $\varepsilon(\sigma'') = \varepsilon(\tilde\sigma)$. Since $(q_1\mathbf{e})(q_2\mathbf{e}) = -\bar q_2 q_1$ by (2.13), we obtain that

$$\phi(\sigma') = -\varepsilon(\tilde\sigma)(-\bar u_{\tilde\sigma(1)})(-\bar u_{\tilde\sigma(3)} u_{\tilde\sigma(2)}) = \varepsilon(\tilde\sigma) u_{\tilde\sigma(1)} u_{\tilde\sigma(2)} u_{\tilde\sigma(3)}.$$

Since the imaginary units $u_1, u_2, u_3$ anticommute, then the non-zero value $\phi(\sigma')$ $= \mathbf{i}\cdot\mathbf{j}\cdot\mathbf{k} = -1$ is independent of $\sigma' \in S_8$ and, consequently, $\omega(I_{78}U_1, U_2, \ldots, U_8)$ $= 63$. Remark here that the value $T_1 = \omega(I_{78}U_1, U_2, \ldots, U_8)$ is calculated in [12, p. 150] but with a mistake. By their calculations $T_1 = 35 \cdot 9$ because the calculations are based on the SO(8) $\subset$ Spin(9) invariance of the form $\omega$ given by (4.9). But as we will prove this form is not Spin(9)-invariant.

Consider now the $i$th term $T_i = \omega(U_1, \ldots, I_{78}U_i, \ldots)$, $2 \leqslant i \leqslant 8$, in (4.11). By (2.3) for $0 \leqslant k \leqslant 6$, one has $I_{78}(u_k, 0) = (0, \pm u_{k'})$ with $1 \leqslant k' \leqslant 7$. Since the “cross product” $(x,0)\times(0,y) = 0$ for any $x, y \in \mathbb{O}$ and $\sigma(1) = 1$, $U_1 = (0, u_0)$, then each non-zero term in the expression (4.10) for $\omega(U_1, \ldots, I_{78}U_i, \ldots)$ is determined by $\sigma \in S_8^*$ such that $\sigma(2) = i$. This term is given by the following expression

$$\phi(\sigma) = \varepsilon(\sigma)\operatorname{Re}[(\mp u_{(i-2)'} u_0)(u_{\varphi(2)}\bar u_{\varphi(1)})]\operatorname{Re}[(u_{\varphi(4)}\bar u_{\varphi(3)})(u_{\varphi(6)}\bar u_{\varphi(5)})], \tag{4.14}$$

where the six-point set $\{\varphi(1), \ldots, \varphi(6)\}$ coincides with the set $\{0,1,\ldots,6\}\setminus\{i-2\}$. If the term (4.14) is non-zero then the first factor of the form $\operatorname{Re}[\,\cdot\,]$ in (4.14) is non-zero. That is, we have at most 3 possibilities for a choice of the second pair $\{\varphi(1), \varphi(2)\}$ because if $\phi(\sigma) \neq 0$ then $u_{\varphi(2)}\bar u_{\varphi(1)} = \pm u_{(i-2)'} \subset B^0 \cup (-B^0)$. Thus the number of non-zero terms (4.14) equals at most 9 because $\sigma(5)$ is the lowest number of the set $\{\sigma(5), \ldots, \sigma(8)\}$ and then for a choice of $\sigma(6)$ one has 3 possibilities. Remark that each such a non-zero term equals $\pm 1$.

Now to prove the non-invariance of the form $\omega$ it is sufficient to find one positive term in the expression for $T_8$. This positive term corresponds to the even permutation $\sigma = (1,8,2,3,4,5,6,7)$ because, by (2.13),

$$\begin{aligned}&\operatorname{Re}[(U_1 \times I_{78}U_8)(U_2 \times U_3)] \cdot \operatorname{Re}[(U_4 \times U_5)(U_6 \times U_7)] \\ &= \operatorname{Re}[((0,u_0)\times(0,u_6u_7))((u_0,0)\times(u_1,0))] \\ &\quad \cdot \operatorname{Re}[((u_2,0)\times(u_3,0))((u_4,0)\times(u_5,0))] \\ &= \operatorname{Re}[(u_7u_6 \cdot u_0)(u_1\bar u_0)]\operatorname{Re}[(u_3\bar u_2)(u_5\bar u_4)]\end{aligned}$$

$$= \mathrm{Re}[(\mathbf{ke}\cdot\mathbf{je})\mathbf{i}]\,\mathrm{Re}[(\mathbf{kj})(\mathbf{ie}\cdot\mathbf{e})] = 1.$$

Thus the 8-form $\omega$ proposed in [11] and [12] is not Spin(9)-invariant.

*Remark* 4.6. Using the method described above one can show that only $T_2 = -9$ and that all the other terms $T_i = 9$ for $i = 3, \ldots, 8$. Thus the expression (4.11) equals 108.

Note also that the proof of the invariance of the form $\omega$ in [12] contains some gaps.

First of all this proof is based on the wrong proposition [12, Prop. 5]. The proof of this proposition relies in turn on the fact that the orthogonal transformations $T_a\colon \mathbb{O} \to \mathbb{O}$, $x \mapsto axa$, of the space $\mathbb{O}$, where $a \in \mathrm{Im}\,\mathbb{O}$, $a^2 = -1$, are pure imaginary octonions of length 1, generate a group $G_T$ isomorphic to SO(8) (cf. [12, p.151]). But this is impossible because $T_a(u_0) = -u_0$ so that for any $g \in G_T$ we have $g(u_0) = \pm u_0$. Thus $G_T$ is locally isomorphic to SO(7) so that $G_T \not\cong \mathrm{SO}(8)$.

Moreover, Prop. 5 in [12] asserts that the group $G^*$ generated by certain one-parameter subgroup and by the orthogonal transformations $\tilde{T}_a\colon \mathbb{O}^2 \to \mathbb{O}^2$, $(x_1, x_2) \mapsto (ax_1, x_2a)$, where $a \in \mathrm{Im}\,\mathbb{O}$, $a^2 = -1$, are pure imaginary octonions of length 1, is isomorphic to the group Spin(9). Now remark that by (4.10) their 8-form is $\omega = \omega' \wedge \omega'$, i.e. it is the square of the 4-form $\omega'$ given by

$$\omega'(U_1, U_2, U_3, U_4) = \sum_{\sigma \in S_4} \varepsilon(\sigma)\,\mathrm{Re}[(U_{\sigma(1)} \times U_{\sigma(2)})(U_{\sigma(3)} \times U_{\sigma(4)})].$$

In [12, p. 152] it is proved that this 4-form $\omega'$ is $G^*$-invariant. But we know (Brown and Gray [13, Sect. 4.5]) that such a non-zero Spin(9)-invariant 4-form cannot exist, so that $G^* \not\cong \mathrm{Spin}(9)$.

# Appendix B

We now comment on Abe and Matsubara's expression of $\Omega^8$. Remark first of all that using some computer calculations we can obtain the expression for our Spin(9)-invariant 8-form in some natural basis of $\mathbb{O}^2$. This expression contains 702 terms.

Abe and Matsubara attempted to describe this 702-terms expression for $\Omega^8$ in their paper [2] (see also the short announce by Abe [1]). The form $\Omega^8$ is exhibited there as a sum of eight 8-forms $\Omega^8_1, \ldots, \Omega^8_8$. The combinatorial descriptions of these forms given in [2] are based on certain two $7 \times 8$ integer-valued matrices. But the combinatorial definitions of these eight 8-forms contain some mistakes, for example the definition of the form $\Omega^8_8$ (see [2, p. 8]) is not correct. Moreover, the papers [1] and [2] contain different expressions for the aforementioned form $\Omega^8_8$. The expression given in [1] contains at most $7 \cdot 7 \cdot 4 = 196$ terms (in some canonical basis) though it is asserted in [2, p.12] that $\Omega^8_8$ contains 336 terms. Therefore we can not compare Abe-Matsubara's formula and our formula for the canonical form $\Omega^8$.

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