The Peregrine soliton (PS) concept, as a paradigm of coherent structures in modeling extreme wave events [1,2], is attracting great interest in a variety of scientific areas ranging from hydrodynamics and oceanography to nonlinear optics [3–5]. It was first discovered in 1983 by Peregrine when looking for surface waves in deep water modeled by the nonlinear Schrödinger (NLS) equation [6], hence its current name. Afterwards, its dynamics were seen in a number of physical settings, first in nonlinear fibers [7], then in water wave tanks [8] and plasmas [9], and recently in an irregular oceanic sea state [10]. Contrarily to the usual shape-invariant soliton [11], the PS, basically expressed by a rational function of second degree, presents a double spatiotemporal localization on a finite continuous background. Starting from this continuous background, it develops progressively towards reaching a climax worth 3 times the level of the background field and then vanishes completely as if it had never occurred [12]. This feature explains well the formation of rogue-wave events [13–18], which have a high amplitude and may appear from nowhere and disappear without a trace [19].

Recent studies reveal that the PS is a universal fundamental solution, characteristic of scalar nonlinear integrable systems such as the celebrated NLS equation [20] and also of vector ones such as Manakov systems [21]. In the latter case, due to the energy transfer between different components, the central amplitude of the vectorial PS can be varied from zero to triple that of the background [22–25]. This is markedly different from the PS in scalar systems, which always has a fixed peak amplitude [20,26]. As a special case, the rogue wave that falls to zero in the dip center is usually referred to as a black (or, loosely, a dark) rogue wave [22,24,27], the counterpart of the usual bright PS that has also been observed lately [28]. More interestingly, a frequency-chirped version of the PS, or chirped PS [26], was unequivocally shown to exist in nonlinear media exhibiting a self-steepening effect, which features a doubly localized chirp besides the inherited threefold-amplitude hallmark.

However, there is more and more numerical and experimental evidence that a single rogue-wave event can appear with a peak amplitude higher than 3 (when normalized to the background) [20,29–32]. Such ultrahigh rogue-wave amplitudes are generally attributed to the collisions of several Peregrine breathers [30–32], yielding field profiles that can match well higher-order rational solutions [20]. Now, a fundamental question occurs: Is it possible that a single anomalous PS exists with a peak amplitude more than triple the background height? If so, would such an anomaly possess enough robustness to be observed in a realistic environment, amid the onset of modulation instability (MI)?

In this Letter, we present a first-ever systematic study addressing this question. We clearly show that a PS solution, still expressed by a rational function of second degree, can surprisingly reach an amplitude limit as high as 5 times the background level, in a multicomponent system. Moreover, the interaction of such two PS states can create a spikelike rogue wave of extremely high peak amplitude (nearly 15-fold for certain given parameters). Our result is different from the deterministic colliding events of ordinary solitons [33–35], which could also yield a hump of very high amplitude, yet predictable. We confirm numerically that these high-amplitude PS solutions are robust and can be excited amid the onset of MI. This finding may help shed more light on the anomalous dynamics of rogue waves in other complex multicomponent systems [36].

For our study, we consider an integrable extension of the Manakov system [37], expressed in a normalized form as
where \(u_{1,2}(\xi, \tau)\) are the complex envelopes of the two field components, with \(\xi\) and \(\tau\) the distance and retarded time, respectively. \(D_r = 1 + i\epsilon(\partial/\partial \tau)\) is the differential operator with respect to \(\tau\), and \(\epsilon(>0)\) scales the perturbation to the Manakov system, which corresponds to the \(\epsilon = 0\) case here. The parameter \(\sigma\) (\(= \pm 1\)) denotes the type of dispersion, i.e., +1 for anomalous dispersion and -1 for normal dispersion. As usual, the asterisk denotes the complex conjugate. This coupled form of nonlinear wave equations is a natural extension of the scalar version \cite{38} obtained separately by Fokas on the mathematical side (using a bi-Hamiltonian method) \cite{39} and by Lenells from the standpoint of physics (starting from Maxwell’s equations) \cite{40}; hence it is referred to as the coupled Fokas-Lenells (CFL) equations. As an important generalization, the CFL system includes, besides the group-velocity dispersion and self- and cross-phase modulation terms that constitute the Manakov system, the effects of space-time coupling (through the presence of the differential operator in the first term) \cite{41} and self-steepening (through the presence of the differential operator in the last two terms) \cite{42}. These additional terms arise as corrections to the slowly varying envelope approximation \cite{40}, when considering the propagation of few-cycle pulses, a situation which can be met in ultrafast optics as well as in hydrodynamics. Thus, the CFL system is relevant to model the propagation of ultrashort optical pulses in birefringent optical fibers \cite{40,43,44} or crossing sea waves in the open ocean \cite{3,45}.

By virtue of the standard Darboux transformation procedure \cite{21,46}, a special type of fundamental PS solutions of the CFL Eqs. (1) and (2) can be found to be

\[
u_j = c_\omega \omega_j, \quad \delta = \omega_j - \omega_j, \quad \omega_j = \sqrt{\delta^2 + i\kappa}, \quad \kappa = \sqrt{\delta - i\kappa}, \quad \text{and} \quad \theta = 2\epsilon \tau + \sigma \left[ 1 + \frac{4(3\delta^2 - \kappa^2)}{(3\delta^2 + \kappa^2)^2} \right] \xi.
\]

Here \(u_{j0}\) \((j = 1, 2)\) are the initial plane-wave seeds

\[u_{j0} = a_j \exp(ik_j \xi + i\omega_j \tau),\]

defined by their amplitudes \((a_j)\), frequencies \((\omega_j)\), and wave numbers \((k_j)\) through the dispersion relations

\[k_1 = 2a_1^2 + \frac{a_1^2 \kappa}{\nu_1} + \frac{\sigma(\nu_1 + 1)^2}{2\epsilon \nu_1},\]

and

\[k_2 = 2a_2^2 + \frac{a_2^2 \kappa}{\nu_2} + \frac{\sigma(\nu_2 + 1)^2}{2\epsilon \nu_2},\]

where

\[a_1 = \frac{2\sqrt{-2\sigma \delta \nu_1}}{\epsilon(3\delta^2 + \kappa^2)}, \quad a_2 = \frac{2\sqrt{-2\sigma \delta \nu_2}}{\epsilon(3\delta^2 + \kappa^2)}.
\]

It follows easily from Eq. (11) that the condition of existence of the PS solutions \((3)\) is \(\omega_j < 1/\epsilon\) for the anomalous dispersion \((\sigma > 0)\) case and \(\omega_j > 1/\epsilon\) for the normal dispersion \((\sigma < 0)\) case.

We find from Eqs. (3)–(6) that the above PS solutions, which consist of quadratic polynomials, each have a central amplitude given below:

\[|u_1| = |a_1| \left| \frac{6\nu_2}{\kappa} \right|, \quad |u_2| = |a_2| \left| \frac{6\nu_1}{\kappa} \right|.
\]

These central amplitudes, calculated at the origin, can be varied with the parameter \(\epsilon\) and with the frequencies \(\omega_1\) and \(\omega_2\). For example, as \(\omega_1 = 1/\epsilon - \sigma/(3\epsilon^3)\) and \(\omega_2 = 2\omega_1 - 1/\epsilon\), one can obtain a PS peak amplitude for the \(u_1\) field that triplicates the background height [see Figs. 1(a) and 1(b)]. As \(\omega_1 = 1/\epsilon - \sigma/(21\epsilon^3)\) and \(\omega_2 = 5\omega_1 - 4/\epsilon\), the PS peak amplitude of the \(u_1\) field becomes now 4 times the background height [see Figs. 1(c) and 1(d)]. If we take \(\omega_1 = 1/\epsilon - \sigma/(4033\epsilon^3)\) and \(\omega_2 = 64\omega_1 - 63/\epsilon\), the peak amplitude can reach nearly 5 times the background level, as seen in Figs. 1(e) and 1(f). Correspondingly, in the above three cases, the \(u_2\) field
whereas letting the other one, say, 
would feature an intermediate, a dark, and a near-planar 
solutions in the anomalous dispersion (\(\sigma = 1\)) regime, normalized 
to their respective background heights: (a),(b) \(e = 1\), \(a_1 = 2/3\), 
\(a_2 = 1/3\); (c),(d) \(e = 3/5\), \(a_1 = 820/567\), \(a_2 = 320/567\); (e),(f) 
\(e = 1/5\), \(a_1 = 20040/4033\), \(a_2 = 12165/4033\).

**FIG. 1.** Surface (top) and contour (bottom) plots of the PS 
solutions in the anomalous dispersion regime, normalized 
to their respective background heights: (a),(b) \(e = 1\), \(a_1 = 2/3\), 
\(a_2 = 1/3\); (c),(d) \(e = 3/5\), \(a_1 = 820/567\), \(a_2 = 320/567\); (e),(f) 
\(e = 1/5\), \(a_1 = 20040/4033\), \(a_2 = 12165/4033\).

The fact that this PS may involve a peak amplitude 
higher than threefold is very intriguing, as no such case 
occurs in integrable systems known so far \[12,21\–25,48\].

We attribute this unique amplitude property to the coupling 
between field components and to the coupling between 
spatio-temporal rearrangement such that the upper amplit-
ude limit of the usual PS solution can be breached. It is self-
evident that the specific conditions for which a more 

than threefold PS amplitude appears are absent in the 
Manakov limit \(\epsilon = 0\), where, instead, the peak amplitude 
for both components is just twice the background height \[21\].

Moreover, the rational solutions describing the interaction 
of two such PS states are given by

\[
\begin{align*}
\omega &= \frac{1}{5} \left( \frac{3}{2} + \frac{4}{5} \right) \frac{e^2}{\epsilon^2}, \\
\theta &= \frac{\tau}{e} + \frac{(\epsilon^2 + 4) a_2^2}{2 \epsilon^2}, \quad \phi = \exp(i\pi/3),
\end{align*}
\]

and \(\gamma_1\), \(\gamma_2\), and \(\gamma_3 \neq 0\) being three complex constants. As 

seen, these solutions, compact yet asymmetrical in form, 
are composed of polynomials of degree 4 and can reduce to 
the PS solutions \(3\) when \(\gamma_3 = 0\). They are allowable in 
either the anomalous or the normal dispersion regime, as 

seen in Figs. 2(a)–2(d). It is exhibited that the two PS 
components that constitute a rogue-wave doublet may have 
a different peak amplitude during interaction; e.g., for the 
specific parameters used in Fig. 2(a), if one component 
grows higher than the factor 4, the other one should be 
lowered below 4.

Interestingly, the superposition state of two PS compo-
nents can reach a very high peak amplitude under a certain 
parameter condition. A typical example is shown in 

Figs. 2(e) and 2(f), in which the peak amplitude of the 
superposed two-PS state can reach nearly 15 times the 
background, for which the initial plane-wave parameters 
are the same as in Figs. 1(e) and 1(f), with other parameters 
being given in the caption. This unusual peak-amplitude 
feature, corresponding to an intensity factor of around 225,
is actually consistent with the rogue-wave anomaly recently 
found via a PIC (particle-in-cell) simulations in the 
gyrotron microwave turbulence, where the rogue waves 
generated could have an intensity factor of 100–150 \[36\].

Lastly, one may wonder if these unusual PS solutions 
could be generated by solving numerically the CFL
FIG. 2. PS doublets (normalized to background) in either (a),(b) the anomalous dispersion ($\sigma = 1$) or (c),(d) the normal dispersion ($\sigma = -1$) regime, under the parameter conditions $e = 3/5$, $\omega_1 = 1/e - \sigma/(2e^2)$, $\omega_2 = 5\omega_1 - 4/e$, $\gamma_1 = 0$, $\gamma_2 = -10 + 10i$, and $\gamma_3 = 1$. (e), (f) show the formation of extreme rogue-wave spikes with $\gamma_1 = -1$, $\gamma_2 = 75i$, $\gamma_3 = 1$, and other parameters being the same as in Figs. 1(e) and 1(f).

FIG. 3. (a) Logarithmic MI gain map [$\ln(\gamma_h)$] of the PS of fourfold amplitude and (b) its gain profile $\gamma_h$ at $\omega_1 = 820/567$, under the parameter conditions $\sigma = 1$, $e = 3/5$, and $\omega_2 = 5\omega_1 - 4/e$. The blue cross corresponds to the case seen in Figs. 1(c) and 1(d), showing a maximum gain $\gamma_h^{\text{max}} = 1.026$ at $\Omega \approx 0.2$. Afterwards, other periodic waves manifest because of the inherent spontaneous MI; see those occurring at $\xi = 4$ in Fig. 4.

Getting closer to realistic experimental conditions, we then confirm numerically that PS solutions featuring an unusual peak-amplitude reinforcement can be excited amid a chaotic background field. To do so, we use the plane-wave solutions (8) as initial conditions, perturbed by white noise of a strength of $10^{-2}$ [48]. This noisy background subsequently develops into a “sea” of different waves as seen in Figs. 5(a) and 5(b), among which one such typical PS structure could be singled out. For further comparison, the temporal cross-sectional profiles of the numerical solutions selected by white circles at $\xi = 18$ are plotted in Figs. 5(c) and 5(d), showing an excellent agreement with the analytical solutions, which confirms our observation above. In addition, one can estimate the period of MI waves occurring at $\xi = 5$ to be around $T = 80/3$, corresponding

FIG. 4. Simulations of the PS of fourfold amplitude ($u_1$ field) and the dark PS ($u_2$ field) shown in Figs. 1(c) and 1(d), starting from $\xi = -2$ to a longer distance $\xi = 4$. 

PHYSICAL REVIEW LETTERS 121, 104101 (2018)
to a modulation frequency of $\Omega = 2\pi / T \approx 0.23$, which is also well consistent with our MI analysis shown in Fig. 3.

In conclusion, we reported the discovery of a novel rogue-wave anomaly in the context of first-order PS solutions of integrable multicomponent systems. This contrasts sharply with the generally acknowledged maximum of threefold peak amplitude associated to the usual PS solutions, which was ubiquitous among the various integrable models used to date. The salient original features of our work can be exposed as follows. We have obtained explicit analytical rogue-wave solutions of the CFL equations, comprising the PS solutions as well as their two-soliton interaction solutions. Owing to the spatiotemporal coupling of the two involved waves in the presence of the self-steepening effect, the PS solutions can reach a peak amplitude between 3 and 5 times the background level, according to the parameters used. We have supported these analytical predictions by numerical simulations of the propagation model. This was illustrated by the excitation of a fourfold-amplitude PS in the presence of realistic noise on a continuous background. The latter subsequently developed into a chaotic field among which the PS structure could be singled out, as a robust pattern, therefore anticipating the possibility of experimental observation. Such an anomaly of rogue-wave creation is not an exception but a manifestation of novel physical mechanisms that take place in complex multicomponent nonlinear systems, resulting in an extreme wave event of higher amplitude [36]. We expect that our work may spark significant research interest in generation of rogue waves as well as their anomalies in birefringent fibers [44], crossing sea states [3,45], semiconductor planar waveguides [50], and even microwave turbulence [36].

This work was supported by the National Natural Science Foundation of China (Grant No. 11474051) and by the European Union under the European Unions Horizon 2020 research and innovation program MSCA-RISE-2015 (Grant No. 691051). Ph. G. was supported by the Indo-French Centre for the Promotion of Advanced Research (IFCPAR/CEFIPRA) under Contract No. 5104-2. The work of J. M. S.-C. was supported by MINECO under Contract No. TEC2015-71127-C2-1-R and by the Comunidad Autonoma de Madrid (CAM) under Contract No. S2013/MIT-2790.