Land, Technical Progress and the Falling Rate of Profit

Howard Petith[1]

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Abstract;

The paper sets out a one sector growth model with a neoclassical production function in land and a capital-labour aggregate. Capital accumulates through capitalist savings, the labour supply is infinitely elastic at a subsistence wage and all factors may experience factor augmenting technical progress. The main result is that if the elasticity of substitution between land and the capital-labour aggregate is less than one and if the rate of capital augmenting technical progress is strictly positive, then the rate of profit will fall to zero. The surprise is that this result holds regardless of the rate of land augmenting technical progress; that is no amount of technical advance in agriculture can stop the fall in the rate of profit. The paper also discusses the relation of this result to the classical and Marxist literature and sets out the path of the relative price of land.

JEL classification; B24,E11, O41.

Key words: Marx, classical economics, falling rate of profit.

[1] Universitat Autònma de Barcelona, Departament d'Economia i d'Història Econòmica, Edifici B, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona) Spain. howard.petith@uab.es. The paper in its present form could hardly have been written without the help of Gerard Duménil, Duncan Foley and Domenic Lévy. It appeared as a UAB working paper in 1992. The following people have made useful comments: Hamid Azari, Jordi Brandts, Roberto Burguet, Ramon Caminal, Simon Emsley, Alan Freeman, John Hamilton and Carmen Matutes. It was also presented at the Bellaterra seminar, the Macro workshop at the UAB, The Atlantic International Economic Conference, El Simposió De Análisi Economica, The IWGVT Conference, Nueva Direccionas en El Pensamiento Economico, The Conference on New and Old Growth Theories and the III Coloquio de Economistas Politicos de America Latina. The author is responsible for all errors. Finally thanks go to my late wife, Deirdre Herrick, for her considerable help. Financial assistance is acknowledged from the Spanish Ministry of Science and FEDER through grant SEC2003-00306 and from the Barcelona Economics Program of CREA.

1. Introduction

This paper is basically about the falling rate of profit. It develops an essentially neoclassical growth model with land, labour and capital as factors of production. Capital accumulates through capitalist savings, the labour supply is infinitely elastic at a subsistence wage and all factors experience factor augmenting technical progress. The main result is that, if the elasticity of substitution between land and a capital-labour aggregate is less than one and if the rate of technical progress experienced by capital is positive, then the model has no steady state and, as a consequence, the capital-labour ratio rises toward infinity, the share of capital approaches one and the rate of profit falls toward zero. This result holds regardless of the speed of technical progress that land experiences. Surprisingly, technical advance in agriculture can not halt the fall of the rate of profit.

This introduction discusses two themes: first, the relation of the main result to the literature and second, a specific characteristic of the model. With regard to the first, it covers the classical and the Marxist literature and then a particular induced innovation mechanism. With regard to the second it sets out a way to resolve a problem caused by the existence of two assets, land and capital, in a model without a steady state.

With respect to the classical literature, the result of this paper supports the conclusions, but not the logic, of the classical authors against those of modern writers. As a bench mark it is convenient to start with a simplified version of the "corn model" with technical progress in agriculture: The production of corn is constant returns to scale in labour and homogenous land. Capitalists rent land from landlords, paying the marginal product of land after the harvest has been collected and hire labor, paying in advance with their accumulated stock of corn. They save a portion of their profits which becomes zero when the rate of profit reaches its minimum level. The labor force grows only when the wage is above subsistence. At at each moment the wage is determined so that the entire stock of corn is used to pay wages. The classical model, when stripped of its frills[1], corresponds to this corn model. One

[1] One of the frills is non-homogenous land and rent. This is important for distribution but not relevant for the falling rate of profit and the approach to the stationary state.

of the main conclusions of the classical school is that the equilibrium of this model will approach a stationary state with the rate of profit at a minimum, the wage at subsistence and no growth. Now add land augmenting technical progress. With the intuition of the neoclassical growth model, one sees that this model has a steady state in which the rate of profit is above the minimum level, the wage is above subsistence, and output, labour and the stock of corn grow at the rate of technical progress[2]. That is, once land augumenting technical progress is added to the classical model, its equilibrium does not approach the stationary state.

In the light of this, consider the positions of David Ricardo and John Steward Mill on the falling rate of profit and the approach to the stationary state in the presence of technical progress Ricardo (1817 , p. 120) stated;

> The natural tendency of profits then is to fall; for...the additional quantity of food required is obtained by the sacrifice of more and more labour. This tendency...is happily checked at repeated intervals by improvements in machinery connected with the production of necessaries, as well as by discoveries in the science of agriculture...which enable us to lower the price of the prime necessaries of the labourer. But the rise ...in the wages is, however, limited; for as soon as wages should be equal...to...the whole receipts of the farmer, there must be an end to accumulation;...

The common interpretation of this has been that technical progress will only slow the fall of the rate of profit. For example Eltis (1988, p. 278), in the New Palgrave, writes of Ricardo that technical progress "...reduces the rate at which profits decline, without affecting the proposition that they must fall eventually to the minimum stationary level." Mill (1965, p. 743) also considered the same issues. He concluded

> All improvements, therefore, in production of almost any commodity, tend to widen the interval which has to be passed before arriving at the stationary state.

Again the common interpretation is as with Ricardo. According to Eltis (1988, p. 279) "...Mill did not envisage that technical progress...would be sufficient to overcome the influence of population growth and agricultural diminishing returns so profits would continually fall towards (the minimum level)". Thus, if one takes the corn model as the basis for classical thinking, it must be concluded that Ricardo and Mill did not understand that, if there was any technical progress at all, the economy would never arrive at the stationary state and the rate of profit and

[2] This is confirmed by the work of modern authors cited below

the wage would be forever above their minimum levels. This lack of understanding, to my knowledge, has not been pointed out before.

Now turn to the modern treatments of technical progress in the context of the classical model: Johansen (1967) and Samuelson (1976). Both of these authors have the classical labour markets and capitalist behavior. Their models differ mainly in production since both have neoclassical capital rather than corn as an arguments in their production functions. This is important because it allows for the possibility of capital augmenting technical progress. Specifically Johansen has a Cobb Douglas production function in a capital-labor aggregate and land and capital augmenting technical progress while Samuelson has a general neoclassical production function in the same arguments and land augmenting technical progress. Both authors show that their models exhibit steady states with the rate of profit above the minimum level, the wage above subsistence and positive growth. They then state that their results corroborate those of the classical authors. This gives rise to two questions; First, how can these results corroborate those of the classical authors when they are exactly the contrary? and second, how can these models, which are similar to the present model, have a steady state with the rate of profit above its minimum level?

In respect to the first question Johansen's justification (p. 21) is that

> In the classical writings one can find some suggestions about technical progress postponing stagnation, perhaps for an infinite future.

Samuelson (p. 1416) notes:

> Mill went on to emphasize that technical innovation, continued in the long-run steady state, would imply rising output forever, we can show on Mill's behalf that, if there is land augmenting technical progress at a steady exponential rate (the above described stationary state will occur).

No references for these justifications are given and, in any case, they do not hold water. With respect to Johansen, it might be the case that there are suggestions, but his results contradict the basic beliefs of the classical economists. The situation is even worse with Samuelson. He says "On Mill's behalf" and then goes on to demonstrate that Mill's understanding of the future was wrong. I think that what happened was that these authors were more interested in drawing the logical consequences of the classical assumptions rather than engaging in a detailed analysis of whether the classical economists correctly understood all the implications of their assumptions.

In respect to the second question, it is certainly true that for general models of this type the rate of profit will fall to its minimum level. What happened in the two cited cases is that the authors chose accidentally, and without justification, the two special cases where this does not happen: Johansen has a unitary elasticity of substitution while Samuelson has no capital augmenting technical progress.

With respect to the classical literature the contribution of this paper is two fold: first it shows that the conclusions of the modern writers are not correct generally; and second, it shows that, if one takes the model with neoclassical capital as the basis for classical thinking, then these authors, although they were unaware of the necessary reasoning, had accidently reached the generally correct conclusion.

I think that the result has implications beyond the characteristics of the classical model. It would seem that most economists, if asked why the rate of profit has not consistently fallen, would point to the rapid technical progress in agriculture. Eltis (1988, p.280) states

> ...technical progress has raised productivity enormously in both industry and agriculture and there has been no tendency for a rising relative cost of food to squeeze profits in the manner that Ricardo and Mill expected.

It may be true that the rate of profit has not fallen consistently, but the result of this paper implies that this can not be attributed to rapid progress in agriculture and is, thus, rather mysterious.

With regard to the Marxist literature, in the first place Marx's central idea is well-known: He thought that capitalism would fall and be replaced by socialism. He further held that capitalist development would be characterized by the following "historical tendencies": A rising capital-labour ratio, a rising share of capital and a falling rate of profit. In addition he seems to have initially thought that the wage would stay at subsistence but later changed his mind about this. These tendencies play important roles in Marx's various theories (never well worked out) of the end of capitalism[3]. However the focus of interest has been on Marx's theory of the falling rate of profit.

Marx thought that, in a temporary fashion, a shortage of labour could cause the wage to rise and the rate of profit to fall; but that the long run fall in the rate of profit would be due to firms choosing progressively more capital intensive means of production[4]. This

[3] The phrase "historical tendencies" was coined by Duménil and Lévy (2003). Their list is slightly longer than the one given above. They emphasize that a constant share of capital is also consistent with Marx's writing. I have abbreviated the list and chosen the rising share of capital because these are the tendencies that carry the weight in Marx's theories of the end of capitalism. See Petith (2002) for a summary of these theories.

[4] Marx explains that the fall in the rate of profit is due to the technical choices of the firms and "Nothing is more absurd, for this reason, than to explain the fall

dichotomy has given rise to two distinct "lines of thought". The first, which is loosely connected with the idea of the profit squeeze, attempts to explain the fall in the rate of profit in industrialized countries that has taken place since 1973 mainly in terms of rising wages[5]. The second, which I will refer to as the technical choice school, attempts to give a general explanation of a long run fall in the rate of profit when this is the result of firms' technical decisions, and not pressure from the labour market. Although a natural background assumption would seem to be that of a constant wage, this has been discarded for two reasons: first, one can find some justification for a rising wage in Marx's writings; and second, Marx's own argument about firms' technical choices has been shown by Okishio (1961) to imply a rising wage. Thus the objective of the technical choice school seems to have become to explain a long run fall in the rate of profit in terms of the technical choices of firms where lack of labour market pressure is evidenced by a non-rising labour share[6] .

There are three distinct contributions to this school. First, Skott (1992) and Michl (1994) use a monopolistic competition setting and a Kalecki type wage determination to show that the rate of profit will fall as firms adjust slowly to an optimal capital-labour ratio. Here, since the models have a steady state, the fall in the rate of profit comes to an end. Second, Skillman (1997) has a matching and bargaining model of the labour market where the individual outcomes depend on economy wide determined outside options. In the presence of capital using labour saving innovations, firms make technical choices which are profit maximizing at the old outside options but change these in a way that the labour share remains constant and the rate of profit falls. Here there is no natural end to the fall in the rate of profit. Finally Duménil and Lévy, in a sequence of papers culminating in (2003) have a model with a steady state in which the share of labour is constant and the rate of profit falls. The model has an endogenous labour supply, a link between the rates of growth of employment and the wage, and an induced innovation mechanism in which the factor shares determine the rates of change of the input coefficients. Their contributions are notable because they span the two lines of thought: In (1993, chapter

in the rate of profit by a rise in the rate of wages, although this may be the case by way of an exception."Marx (1984, p.240). A bit further along, on p.256, he explains this exception:"...the competitive struggle (among capitalists) is accompanied by a temporary rise in wages and a resultant further temporary fall in the rate of profit."

[5] These writings are surveyed in Howard and King (1992, chapter 16). A recent contribution which contains a critical survey is Brenner (1998).

[6] See Duménil and Lévy (2003, p.206) for a detailed justification of this in terms of Marx's writings.

15) they explain the post 1973 fall in the rate of profit in terms of an early version of the model and, in the latest version, primarily thanks to the induced innovation mechanism, they manage to generate all of the historical tendencies.

The present paper belongs to the technical choice school since the assumption of an infinitely elastic labour supply means that labour market pressure is absent. In this area, its contribution is to show that, by adding land, the historical tendencies can be generated without having a rising wage. This is important both for understanding what causes a falling rate of profit and for the coherence of Marx's overall view. First one can take the labour market assumption as an extreme case where there is no pressure on profits from this quarter. This means that when we observe a falling rate of profit (as in the post 1973 period) it is perfectly possible that it may have little to do with a rising wage. Second, the historical tendencies are important, not for themselves, but because of the roles they played in Marx's various accounts of the end of capitalism. But these accounts are much weakened if they include a rising wage. Thus the importance here is that the present paper shows how the historical tendencies can be generated without, at the same time, calling into doubt their raison d'être.

The induced innovation mechanism has a paradoxical relation to the present paper which can be illustrated by looking at Foley (2003) . Foley has a model which is very similar to the one of this paper with output produced by land and a capital-labour aggregate and an elasticity of substitution less than one. Yet the model converges to a steady state with a constant positive rate of profit. This seems to contradict the main result of the present paper; what happened? Foley incorporates a version of the Duménil-Lévy induced innovation mechanism in his model. This, in turn, implies that the rate of capital augmenting technical progress approaches zero[7] so that one of the two assumptions of the present model is violated. The paradox is that, if one wants to generate the historical tendencies as is done in the present model, one

[7] This is a simplification: there are two distinct mechanisms and Foley has two versions. The first mechanism is associated with Kenedy (1964) , involves the rates of factor augmenting technical progress and is set in the context of a Solow growth model Drandakis and Phelps. (1966) showed that this implies that the rate of capital augmenting technical progress approaches zero. The Duménil-Lévy mechanism involves rates of change of the input coefficients and is set in the context of a classical fixed coefficient model. In one of Duménil and Lévy's cases the change in the capital input coefficient approaches zero. Foley has a classical and a neoclassical version. In both of these, if land is not considered a free good but is priced at its marginal product, then the rate of capital augmenting technical progress or the rate of change of the capital input coefficient approaches zero and the rate of profit approaches a positive constant.

has to deny the validity of just the induced innovation mechanism that was responsible for them in the Duménil-Lévy model.

Turning to the second theme, the fact that the model has two assets, land and capital, and no steady state causes a problem. Since the capitalists hold both assets, the rate of return must be the same on both of them and since capital gains are part of the return to land, this leads to a differential equation in the price of land and the task of choosing an initial value for the price. This is a problem that one generally meets in models with more than one asset, the Hahn problem (Hahn (1966)). The solution is to choose an initial price in such a way that the model approaches the steady state. The justification of this is that any other choice will lead to negative prices in finite time (Shell and Stiglitz (1967)). In the present paper, this solution does not work because there is no steady state and all choices of an initial price lead to a negative price of land. One's first reaction is that there must be something wrong with the model, but since the model is conventional except that capital augmenting technical progress has been added, it is more apt to say that the problem does not usually appear because, with no justification, a special case is chosen. Thus the problem must be faced.

The solution I have chosen is to assume that the world will end at a certain terminal date, which may approach infinity. Then, for any given terminal date, one can find an initial price of land such that the price of land will be positive until the terminal date. Furthermore one can fix the initial price with the following argument: Since capital and the good are equivalent, capital can always be consumed and so should have value at any given moment. But the value of land is that it contributes to future production. Thus it should have no value at the terminal date and its price in terms of the good should be zero. This condition fixes the initial price of land.

The paper is organized as follows. Section 2 presents the model. This generates a single non-autonomous differential equation in the aggregate-land ratio. The asymptotic form of this equation is solved and the main result is deduced from this solution. Section 3 provides an intuitive explanation of the main result. Section 4 describes the path of the price of land . Finally Section 5 shows that the path generated by the asymptotic form approaches that generated by the original differential equation.

2. Model and Result

There are two factors of production, each measured in effective units: Land, $M \equiv \underline{M}e^{\delta t}$ where the quantity of physical units $\underline{M}$ is set equal to 1, $\delta \geq 0$ is the rate of land augmenting technical progress and t is time; and a Cobb-Douglas capital-labour aggregate $X \equiv K^{\beta}L^{1-\beta}e^{\gamma t}$, the where K and L are capital and labor in physical units, β is a constant $0 < \beta < 1$ and $\gamma \geq 0$ is the rate of aggregate augmenting technical progress[8]. $x \equiv X/M$ is the aggregate-land ratio in terms of effective units.

$$x = K^{\beta}L^{1-\beta}e^{(\gamma-\delta)t} \tag{1}$$

There is a single good, output Y is produced by a CES production function in M and X, $Y = [\alpha X^{-\rho} + (1-\alpha)\,M^{-\rho}]^{-\frac{1}{\rho}}$ where $0 < \alpha < 1$ is a constant and the elasticity of substitution between M and X is $\sigma = \frac{1}{1+\rho}, -1 \leq \rho \leq \infty$. Bringing M outside the brackets gives

$$Y = f(x)\underline{M}e^{\delta t},\ f(x) \equiv \frac{c_1 x}{(c_2 + x^{\rho})^{\frac{1}{\rho}}},\ \underline{M} = 1 \tag{2}$$

where $c_1 = 1/(1-\alpha)^{\frac{1}{\rho}}$ and $c_2 = \alpha(c_1)^{\rho}$. $f(x)$ is the ratio of output to land in effective units which depends on x.

The supply of labour is infinitely elastic at the subsistence wage w. The demand, and thus the quantity of labour, is determined so that the marginal product of labour is equal to the wage. The marginal product of the aggregate is $f'(x)$ ($f' \equiv df/dx$) and $xe^{\delta t}$ is its quantity so $xf'(x)e^{\delta t}$ is the payment it receives. Because the aggregate is Cobb-Douglas, labour receives $(1-\beta)$ of this payment. Thus

$$L = \frac{1-\beta}{w}xf'(x)e^{\delta t} = \frac{1-\beta}{w}\frac{c_1 c_2 x}{(c_2 + x^{\rho})^{1+\frac{1}{\rho}}}e^{\delta t}. \tag{3}$$

From (1) and (3)

$$K = x^{\frac{1}{\beta}}L^{-\frac{1-\beta}{\beta}}e^{\frac{\delta-\gamma}{\beta}t} = (\frac{1-\beta}{\beta}c_1 c_2)^{-\frac{1-\beta}{\beta}}\frac{x}{(c_2 + x^{\rho})^{\frac{1}{\rho}(1-\mu)}}e^{(\delta-\phi)t} \tag{4}$$

where $\mu = \frac{1+\rho(1-\beta)}{\beta} > 1$ if $\rho > 0$ and $\phi = \frac{\gamma}{\beta} > 0$ if $\gamma > 0$. Thus L and K are given as functions of x and t .

Capitalists own both the stock of capital and the land. They receive the output, pay the wage to the workers and get capital gains on the

[8] But since the aggregate is Cobb-Douglas, it can be thought of as capital augmenting.

land, $\dot{P}\underline{M}$, where P is the price of land in physical units and $\dot{P} \equiv dP/dt$. It is assumed that they save their entire income [9]. Thus savings are $Y - wL + \dot{P}\underline{M}$. The change in wealth is $\dot{K} + \dot{P}\underline{M}$. Setting the two equal gives

$$\dot{K} = Y - wL \tag{5}$$

Thus the assumption that capitalists save all eliminates capital gains and allows the model to be solved without taking the path of the price of land into account[10].

The model reduces to a non-autonomous differential equation in x. From (1)

$$\hat{x} = \beta\hat{K} + (1-\beta)\hat{L} + \gamma - \delta \tag{6}$$

where $\hat{x} \equiv \dot{x}/x$.From (3)

$$\hat{L} = \delta + \frac{c_2 - \rho x^\rho}{c_2 + x^\rho}\hat{x}. \tag{7}$$

From (2), (3) and (5), $\dot{K}$ is a function of x and t . Dividing this expression by (4) to get $\hat{K}$ gives

$$\hat{K} = c_1^{\frac{1}{\beta}} c_2^{\frac{1-\beta}{\beta}} (\frac{1-\beta}{w})^{\frac{1-\beta}{\beta}} \frac{x^\rho + \beta c_2}{(x^\rho + c_2)^{\frac{1}{\beta}(1+\frac{1}{\rho})}} e^{\phi t} \tag{8}$$

Substituting (7) and (8) into (6) gives

$$\dot{x} = F(x)e^{\phi t} + G(x) \text{ with } x(t_0) = X_o > 0 \tag{9}$$

where

$$F(x) \equiv \frac{a\mu(x^\rho + \beta c_2)}{(c_2 + x^\rho)^{\frac{\mu}{\rho}}(c_2 + \mu x^\rho)} x,$$

$$G(x) \equiv (\phi - \delta)\frac{c_2 + x^\rho}{c_2 + \mu x^\rho} x,$$

$$a \equiv \frac{(c_1 c_2^{1-\beta)})^{\frac{1}{\beta}}}{\mu} (\frac{1-\beta}{w})^{\frac{1-\beta}{\beta}}$$

[9] One might object that the saving behavior of capitalists should be determined by intertemporal optimization. There are two questions: what is the effect of continued saving on the rate of profit? and what is the effect of the temporal pattern of the profit rate on saving? This paper only seeks to answer the first question.

[10] If capitalists only saved a proportion of their income, then capital gains on land would affect the accumulation of capital. In a rough way one can see what the effect of this would be. Below it is shown that land experiences first capital gains and then losses. Equating saving with the change in wealth shows that, although the lower saving would generally reduce capital accumulation, the early capital gains would reduce it further while the later capital losses would raise it. If the price of land falls, I am poorer, consume less and thus accumulate more capital.

and it is supposed that the initial value of x is positive.

Let $x(t)$ be the continuous solution to (9) [11]. Lemma 1 gives some of its characteristics,

LEMMA 1. *For $t_0 \leq t < \infty$, $x(t) > 0$. Let $\gamma > 0$, there exists t' such that $\dot{x} > 0$ for $t \geq t'$. Furthermore $x \to \infty$ as $t \to \infty$.*

Proof. *First note that $x(t) > 0$ for $t_0 \leq t < \infty$. Suppose $x(t) = 0$ for some finite value of t . Let t_* be the first such value. Then $x(t) \geq 0$, $t_0 \leq t \leq t_*$. This means that $F(x)/x$ and $G(x)/x$ are bounded below on this domain and there is an A such that*

$$\frac{F(x(t))}{x(t)}e^{\phi t} + \frac{G(x(t))}{x(t)} > A, \ t_0 \leq t \leq t_*.$$

so that $\dot{x}(t) > Ax(t)$ on this domain. Integrating from t_0 to t_* and using both sides of the inequality as exponents gives

$$x(t_*) > x(t_0)e^{a(t_*-t_0)} > 0,$$

contradiction.

From the continuity of $x(t)$, $x(t) \geq 0$, $t_0 \leq t \leq \infty$. Thus there exists an $\underline{F} > 0$ and a $\underline{G}$ such that $F(x(t)) > \underline{F}$ and $G(x(t)) > \underline{G}$ for $t_0 \leq t \leq \infty$. Thus since $\gamma > 0$ implies $\phi > 0$, there exist t' and a $B > 0$ such that

$$F(x(t))e^{\phi t} + G(x(t)) > B > 0, \ t \geq t'$$

so that $\dot{x}(t) > B > 0$ for $t \geq t'$. This proves the second statement. Integrating this gives

$$x(t) > x(t') + B(t - t')$$

which proves the third statement. □

It is instructive to consider what happens to (9) as $x \to \infty$ when $\rho > 0$. Define $\tilde{F}(x)$ and $\tilde{G}(x)$ by

$$F(x) = \frac{a}{x^{\mu-1}}\tilde{F}(x) \text{ and } G(x) = \frac{\phi - \delta}{\mu}x\tilde{G}(x). \tag{10}$$

When $\rho > 0$ these functions satisfy

$$\lim_{x\to\infty} \tilde{F}(x) = \lim_{x\to\infty} \tilde{G}(x) = 1. \tag{11}$$

Consider the equation that arises when the asymptotic values of these functions are substituted into (9) :

$$\dot{x} = \frac{ae^{\phi t}}{x^{\mu-1}} + \frac{\phi - \delta}{\mu}x, \text{ with } x(T) = X. \tag{12}$$

[11] See Petith (2002) for a proof of the existence and uniqueness of $x(t)$ on $[t_0, \infty)$.

In order to determine the solution to (12), substitute the function $y(t)$ for $x(t)$ with $y(t) = a\mu\frac{e^{\phi t}}{(x(t))^{\mu}}$. The function $y(t)$ satisfies the following differential equation $\dot{y} = y(\delta - y)$ which can be easily integrated, $\bar{y}(t) = \frac{\delta}{1-Ce^{-\delta t}}$. Thus, the solutions of (12) can be determined.

$$\bar{x}(t; a, \delta, T, X) = \left[\frac{a\mu}{\delta}e^{\phi t}(1 - C(a, \delta, T, X)e^{-\delta t})\right]^{\frac{1}{\mu}} \tag{13}$$

in which the constant $C(a, \delta, T, X)$ is determined by the initial condition $X = \bar{x}(T; a, \delta, T, X)$ Also the function $\overline{\overline{x}}(t)$ can be defined whose asymptotic behavior is identical to that of $\bar{x}(t)$ when t tends to infinity:

$$\overline{\overline{x}}(t) \equiv (\frac{a\mu}{\delta}e^{\phi t})^{\frac{1}{\mu}}. \tag{14}$$

It would seem likely that the solution to (9), $x(t)$, would tend to $\overline{\overline{x}}(t)$ as $t \to \infty$. Indeed this is the case as will be shown below. This is stated as Lemma 2.

LEMMA 2. *Let* $\rho > 0$ *and* $\gamma > 0$, *then*

$$\lim_{t\to\infty} |x(t) - \overline{\overline{x}}(t)| = 0$$

The elements of the historical tendencies may now be defined. $k \equiv K/L$ is the capital-labour ratio and $s \equiv 1 - \frac{wL}{Y}$ is the share of income received by the capitalists. Next consider the rate of profit. Since capitalists' share of the income of the aggregate is β, as in the justification of (3),

$$rK = \beta x f'(x)e^{\delta t} \tag{15}$$

where r is the marginal product of capital. The rate of profit R is defined as total capitalist income divided by the value of factors of production owned by the capitalists:

$$R \equiv \frac{Y - wL + \dot{P}\underline{M}}{K + P\underline{M}}.$$

Since capitalists hold both capital and land, the rate of return on both of these assets, when calculated in terms of the good, must be the same[12]:

$$r = \frac{\frac{\partial Y}{\partial \underline{M}} + \dot{P}}{P}. \tag{16}$$

[12] r is the own rate of return on capital in the sense that both the gain from a unit of capital and the unit itself are measured in terms of the good. (16) can be interpreted as saying that the own rate of return on capital must be equal to the rate of return on land when both the gain and the stock are measured in terms of the good. It should be noted that this rate of return is not the same as the own rate of return on land, $(\frac{1}{P})\frac{\partial Y}{\partial \underline{M}}$.

It is easily shown that (16) implies that $R = r$. Thus from this point on r will be taken as the rate of profit. From (15) $r = \beta x f'(x) e^{\delta t}/K$.

The main result can now be proved. As explained in the introduction, it is assumed that the world end at a terminal date t_e.The main result states that as the terminal date approaches infinity, the terminal capital-labour ratio approaches infinity, the terminal capitalist income share approaches one and the terminal rate of profit approaches zero.

THEOREM 3. *Let* $\rho > 0$ *and* $\gamma > 0$. *Then* $k(t_e) \to \infty$, $s(t_e) \to 1$, *and* $r(t_e) \to 0$ *as* $t_e \to \infty$.

Proof. Let $L(t_e)$ *and* $K(t_e)$ *be the values of* K *and* L *at* $t = t_e$ *derived from (3), (4) and the function* $x(t)$. *Then* $k(t_e) = K(t_e)/L(t_e)$. *From 3 and 15*

$$r(t_e) = \frac{w\beta}{1-\beta}\frac{1}{k(t_e)}. \tag{17}$$

From (2) and (3)

$$s(t_e) = \frac{\beta c_2 (x(t_e))^{-\rho} + 1}{c_2 (x(t_e))^{-\rho} + 1}.$$

From (3) and(4)

$$k(t_e) = c_3 \left[c_2 + (x(t_e))^{\rho}\right]^{\frac{\mu+\rho}{\rho}} e^{-\phi t_e}$$

where $c_3 = \frac{w}{\beta}(\frac{1-\beta}{\beta} c_1 c_2)^{-\frac{1}{\beta}}$. *Thus, from Lemma 1,*

$$k(t_e) \to c_3 x(t_e)^{\mu+\rho} e^{-\phi t_e} \text{ as } t_e \to \infty.$$

Let $d(t_e)$ *be the distance between* $x(t_e)$ *and* $\overline{\overline{x}}(t_e)$ *that is*

$$d(t_e) \equiv x(t_e) - (\frac{a\mu}{\delta} e^{\phi t})^{1/\mu}$$

where $d(t_e) \to 0$ *as* $t_e \to \infty$ *from Lemma 2. Then*

$$\begin{aligned} k(t_e) &\to c_3 \left[(\frac{a\mu}{\delta})^{1/\mu} e^{\frac{\phi}{\mu} t_e} + d(t_e) \right]^{\mu+\rho} e^{-\phi t_e} \\ &= c_3 \left[(\frac{a\mu}{\delta})^{1/\mu} e^{\frac{\rho}{\upsilon(\mu+\rho)} \phi t_e} + d(t_e) e^{-\frac{1}{\mu+\rho} \phi t_e} \right]^{\mu+\rho} \\ &\to c_3 (\frac{a\mu}{\delta})^{\frac{\mu+\rho}{\mu}} e^{\frac{\rho}{\mu} \phi t_e} \text{ as } t_e \to \infty. \end{aligned} \tag{18}$$

Thus since $\rho > 0$ *and since* $\gamma > 0$ *implies* $\phi > 0$, $k(t_e) \to \infty$ *as* $t_e \to \infty$. *Thus* $r(t_e) \to 0$ *as* $t_e \to \infty$. *Finally, since by Lemma 1* $x(t_e) \to \infty$ *as* $t_e \to \infty$, $\rho > 0$ *means that* $s(t_e) \to 1$ *as* $t_e \to \infty$. □

A quick reading of the theorem might lead one to think that, for example, the rate of profit will be close to zero only when the world is about to end. But this is not the case. If t_e is large, the slope of the rate of profit as a function of time approaches zero as t approaches t_e. Thus the rate of profit will be close to a number close to zero long before t is close to t_e For example the rate of profit could have fallen to a very low level by 2100 even though this is long before the end of the world.

3. A Heuristic Description of the Result[13]

In this description land and the aggregate are always expressed in effective units while labour and capital are in physical units. The first step is to understand why the capital-labour aggregate in effective units, X, grows faster than land, M, in the same units. It must do this in order to keep the marginal product of labour constant because the aggregate admits technical progress. This can be seen as follows: suppose the rate of growth of the aggregate was less than or equal to that of land, $\widehat{X} \equiv \delta' \leq \delta$.

a. $\hat{Y} \geq \delta'$ by constant returns to scale and, since there is a constant savings ratio, $\widehat{Y} = \widehat{K}$ asymptotically so that $\widehat{Y} = \widehat{K} \geq \delta'$ asymptotically.

b. Since $\widehat{X} = \beta\widehat{K} + (1-\beta)\widehat{L} + \gamma$, $\widehat{L} \leq \delta' - \frac{\gamma}{1-\beta}$. That is, since capital is growing at least as fast as the aggregate, labour must grow more slowly to compensate for the technical progress.

c. But in this case the marginal product of labour rises both because the capital to labour ratio increases and the ratio of the aggregate to land falls. Since this is impossible, the aggregate must grow faster than land.

To put this in a nutshell: If the aggregate does not grow faster than land, then capital will grow at least as fast as the aggregate, labour will grow slower to compensate for the technical progress of the aggregate and the impossible wage growth will occur.

The second step is to understand why this implies that capital must grow faster than labour: Since $\sigma < 1$, asymptotically, the slowest growing factor dominates so that $\widehat{Y} = \delta$. Again asymptotically, because of the constant savings ratio, $\widehat{K} = \widehat{Y}$. Finally from the condition that the wage is constant, $\widehat{L} = \delta - \rho\widehat{x}$ which can be understood as follows: If x was constant then the payment to the aggregate would grow at δ and,

[13] The arguments of this section are only approximate. It is frequently stated that the rates of growth of the variables approach limits. This is convenient for an intuitive discussion and may well be the case, but it has not been formally demonstrated. The problem is that Lemma 2 does not imply that $\widehat{x}(t) \rightarrow \widehat{\widehat{\overline{x}}}(t)$ as $t \rightarrow \infty$.

since labour receives a constant proportion of this, the labour supply would have to grow at δ to keep the wage fixed. Since $\sigma < 1$ the growth of x will reduce the payment to the aggregate at an asymptotic rate of $\rho\widehat{x}$ so that the payment to labour grows at $\delta - \rho\widehat{x}$ and the labour supply must grow at this rate to keep the wage constant. Thus it is a combination of the relative growth of the aggregate, $\sigma < 1$ and the constant wage that forces capital to grow faster than labour.[14] .

Finally, what forces drive the historical tendencies? First, the aggregate grows faster than land because of technical progress in the aggregate. Because of this, with $\sigma < 1$, in order to keep the wage constant labour must grow more slowly than capital. This establishes that $k \to \infty$. Second, since both the share of capital in the aggregate and the reward to labour are constant, the faster growth of capital must be compensated for by a fall in its reward. This establishes that $r \to 0$. (Alternatively, the rise in K/L with the marginal product of labour fixed forces the marginal product of capital to fall.) Finally, since $\sigma < 1$, the faster growth of the aggregate forces its share and that of labour to zero. This establishes that $s \to 1$.

With this detailed heuristic account, the reader may be in danger of not seeing the forest through the trees. A simpler, less accurate explanation is the following. Think in terms of a model with only capital and land. Generally, when one adds factor augmenting technical progress to a factor that can be accumulated, like capital, the result is explosive growth in the sense that the rate of growth increases over time. But in our simplified model, with elasticity less than one, this is counteracted by the slower growth of land in efficiency units. The result is a balance between these two forces in which the growth of capital, in efficiency units, is not explosive but is more rapid than that of land with the consequence of a continually falling rate of profit.

4. The Price of Land

As explained in the introduction it is assumed that the world ends at a terminal date t_e. The path of the price of land is determined from the differential equation (16) that arises from the condition that the rates of return on capital and land must be the same, together with the condition that the price of land must be zero at t_e. If, in equation (16) one substitutes the expressions for r and $\partial Y/\partial \underline{M}$ as $t \to \infty$ for the actual ones, then one has an equation which determines the path of the asymptotic price. This section studies the behavior of the

[14] With slightly more effort one can see why $\hat{x} = \gamma/\left[1 + \rho(1-\beta)\right]$: Put $\hat{K} = \delta$ and then the above expression for $\hat{L}$ into (7).

asymptotic price. This is only of interest if the path of the asymptotic price approaches that of the actual price. This is presumably the case but I have not proved it. From this point on the asymptotic price will be written as P and will be referred to as the price of land.

The result is that, if the terminal date is far enough in the future, the price of land first rises and then, at a certain point, begins to fall and continues to fall until it reaches zero at the terminal date. The intuition is the following: Initially the marginal product of capital is high and that of land low. As explained in the previous section, the marginal product of a physical unit of capital falls; also, because $\sigma < 1$ and $x \to \infty$, the marginal product of an efficiency unit of land approaches a constant and so that of a physical unit rises. Intially the marginal product of a physical unit of capital is high and that of land low but this reverses over time. Thus, in order that the rate of return on both assets be the same, land must first experience capital gains and then, later, losses.[15]

The equation which determines the path of the (asymptotic) price of land is got as follows. From (18) and (17), with t substituted for t_e, $r \to d_1 e^{-\theta t}$ as $t \to \infty$ where $d_1 \equiv (\frac{1-\beta}{w} c_1 c_2)^{\frac{1}{\beta}} (\frac{a\mu}{\sigma})^{\frac{\rho}{\mu}}$ and $\theta \equiv \rho \frac{\phi}{\mu}$. Differentiating (2) with respect to $\underline{M}$ and using Lemma 1 give $\partial Y / \partial \underline{M} \to d_2 e^{\delta t}$ as $t \to \infty$ where $d_2 = c_1$. Substituting these expressions into (16) gives[16]

$$\dot{P} = d_1 e^{-\theta t} P - d_2 e^{\delta t}, \text{ with } P(t_e) = 0 \tag{19}$$

Theorem 4 gives the result of this section.

THEOREM 4. *Let $P(t)$ be the solution to (19) . If t_e is sufficiently large, then there exists $0 < t_* < t_e$ such that $\dot{P} \gtreqless 0$ as $t \lesseqgtr t_*$*

Proof. To analyze (19), first let $z = e^{\delta t}$, z goes from 1 to $z_e = e^{\delta t_e}$ as t goes from 0 to t_e. (19) becomes

$$\frac{dP}{dt} - \frac{d_1}{\delta} z^{-\frac{\theta}{\delta} - 1} P = -\frac{1}{\delta} d_2.$$

[15] One can think of the fall of the rate of profit in terms of the definition of R and $\dot{P}$. Since $R = (1 + \dot{P}\underline{M}/Y - wL/Y)/(K/Y + P\underline{M})$ and since $wL/Y \to 0$, $K/Y \to$ a constant and $P \to 0$, it is the capital loss on land, $\dot{P} < 0$, that is responcible for the falling rate of profit.

[16] In footnote 12, when (16) was introduced, it was noted that the own rates of return on capital and land, r and $(\frac{1}{P})(\frac{\partial Y}{\partial \underline{M}})$, were not the same. There is a curious relation between these two rates as t_e becomes large. Let $t \to t_e$, then $r(t)$ approaches a number close to zero. But since $\partial Y / \partial \underline{M}$ does not approach zero and $1/P$ approaches infinity, the own rate of return on land approaches infinity. To put it paradoxically, just as capitalism is about to fall because the rate of profit is approaching zero, the own rate of return on land approaches infinity.

The integrating factor is $e^{\frac{d_1}{\theta} z^{-\frac{\theta}{\delta}}}$ *so that the solution is given by*

$$Pe^{\frac{d_1}{\theta} z^{-\frac{\theta}{\delta}}} = -\frac{d_2}{\delta} \int_1^z e^{\frac{d_1}{\theta} y^{-\frac{\theta}{\delta}}} dy + C$$

where C *is a constant determined by* $P(t_e) = 0$. *Let* $v = z^{-\frac{\theta}{\delta}}$ *and* $u = y^{-\frac{\theta}{\delta}}$ *so that* $v = e^{-\theta t}$ *and* v *goes from* 1 *to* $v_e = e^{-\theta t_e} < 1$ *as* t *goes from* 0 *to* t_e. *Note that* $v_e \to 0$ *as* $t_e \to \infty$ *and that* $dv/dt < 0$. *The previous equation becomes*

$$Pe^{\frac{d_1}{\theta} v} = -\frac{d_2}{\theta} \int_v^1 e^{\frac{d_1}{\theta} u} u^{-\frac{\theta+\delta}{\theta}} du + C \tag{20}$$

where the value of C *is given by setting the RHS of (20) equal to zero and* $v = v_e$*; thus*

$$C = \frac{d_2}{\theta} \int_{v_e}^1 e^{\frac{d_1}{\theta} u} u^{-\frac{\theta+\delta}{\theta}} du.$$

Since $e^{\frac{d_1}{\theta} u} \geq 1$ *on* $[v_e, 1]$

$$\int_{v_e}^1 e^{\frac{\delta}{\theta} u} u^{-\frac{\theta+\delta}{\theta}} du \geq \int_{v_e}^1 u^{-\frac{\theta+\delta}{\theta}} du = -\frac{\theta}{\delta}(1 - v_e^{-\frac{\delta}{\theta}})$$

so that $C \to \infty$ *as* $t_e \to \infty$.

To prove the theorem, differentiate (20) to get

$$\frac{dP}{dv} e^{\frac{d_1}{\theta} v} = \frac{d_2}{\theta} e^{\frac{d_1}{\theta} v} v^{-\frac{\theta+\delta}{\theta}} - P \frac{d_1}{\theta} e^{\frac{d_1}{\theta} v} \equiv g(v).$$

Setting $v = 1$ *in this equation and noting that, from (20), the value of* P *at* $v = 1$ *approaches infinity as* $t_e \to \infty$ *shows that* $dP/dv < 0$ *at* $v = 1$ *for* t_e *sufficiently large. Since*

$$\frac{dg}{dv} = -\frac{\theta + \delta}{\theta} \frac{d_2}{\theta} e^{\frac{d_1}{\theta} v} v^{-\frac{\theta+\delta}{\theta}} < 0,$$

as v *falls from* 1 *to* v_e, $\frac{dP}{dv}$ *is, at first negative, then at* v_* *zero, and finally becomes positive since* $P = 0$ *at* $v = v_e$. *Defining* t_* *by* $v_* = e^{-\theta t_*}$ *and remembering that* $\frac{dv}{dt} < 0$ *gives the result.* □

Let me attempt to justify the idea that the path of price of land is determined by the terminal date of the world. This idea is forced on us as a way to avoid having the price of land eventually become negative. But it seems absurd that one estimates the date at which the sun will have cooled sufficiently to make the earth uninhabitable and then solves a differential equation to find out what the current price of land should

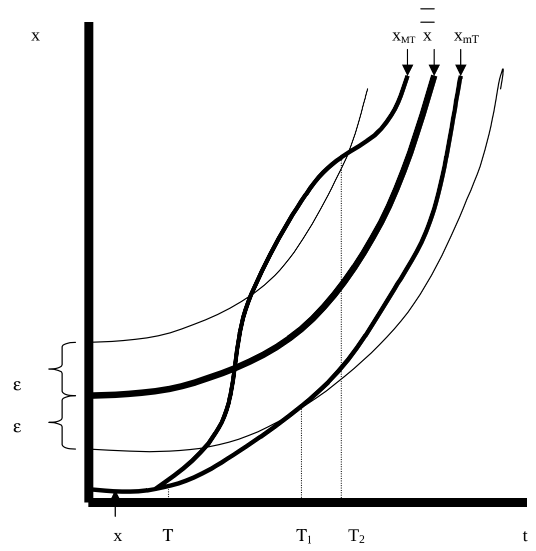

Figure 1. Illustration of lemmas 6 and 7

be. I want to argue that this is reasonable: Under the weak assumptions of the model, as explained above, the marginal product of land in physical units will rise and that of capital fall. It is completely plausible that a similar pattern will occur in the real world. This can easily give rise to the temporal pattern where the price of land first increases and then decreases, driven approximately by the differential equation (19). Now move into the far future, when the date of humanity's departure has (hopefully) has become a factor which affects peoples' calculations. Capital, in one way or another, can be taken along but, it will be supposed, the earth itself must be left behind. If it becomes apparent that, given the current path, land will have a positive price at the last moment, there will be a downward adjustment and visa versa. But the result should be that, without specific calculations now, the price of land will approach zero as humanity prepares to leave the earth.[17].

[17] My late wife grew up in a mansion on the shores of Sidney harbour which later became the Canadian embassy. Her father bought it for a song just after a Japanese pocket submarine had shelled Sydney from the waters of the same harbour.

5. Proof of Lemma 2

This section provides the proof of Lemma 2. It does it only for the case of $\delta - \phi > 0$. This will be called Lemma 7. This is the most difficult and also the most interesting case since it is the one in which the rate of technical progress in land δ can be unboundedly large. The proof of the other case, $\delta - \phi \leq 0$, is set out in Petith (2002). Lemma 7 states that $x(t)$, the solution to (9), approaches $\overline{\overline{x}}(t)$, given by (14), as $t \to \infty$. The proof proceeds in two steps and can be read from figure 1. Lemma 6 uses Chaplygin's theorem to establish that x lies between two bounding functions x_{mt} and x_{MT}. Then Lemma 7 shows that these two bounding functions eventually enter an ϵ tube that surrounds $\overline{\overline{x}}$, thus x is asymptotically equivalent to $\overline{\overline{x}}$.

First Chaplygin's theorem is stated:

THEOREM 5. [18] *Let $x(t)$ be the solution to the equation $\dot{x} = g(x,t)$, $x(T) = X$ and let $x_{mt}(t)$ and $x_{MT}(t)$ be two bounding functions with $x_{mT}(T) = x_{MT}(T) = X$. If the differential inequalities*

$$\dot{x}_{mT}(t) - g(x_{mT}(t), t) < 0$$

$$\dot{x}_{MT}(t) - g(x_{MT}(t), t) > 0$$

hold for $t > T$, then

$$x_{mT}(t) < x(t) < x_{MT}(t)$$

for all $t > T$.

Next the bounding functions are constructed by taking the solutions to modified forms of (12). First modify(12) as

$$\dot{x} = \frac{a}{x^{\mu-1}} e^{\phi t} + m \frac{\phi - \delta}{\mu} x = \frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta_m}{\mu} x, \ x(T) = X \qquad (21)$$

where $\delta_m = \delta m + (1 - m)\phi$. Then take $\bar{x}_{mT}(t)$ as the solution to this equation with δ in (13) replaced by δ_m. Next modify (12) as

$$\dot{x} = M\left(\frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta}{\mu} x\right) = \frac{a_M}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta_M}{\mu} x, \ x(T) = X \qquad (22)$$

where $a_M = Ma$ and $\delta_M = \delta M + (1 - M)\phi$. Then take $\bar{x}_{MT}(t)$ as the solution to this equation with a and δ in (13) replaced by a_M and δ_M.

Now Lemma 6 may be proved.

[18] See Mikhlin and Smolitskiy (1967, pp. 9-12.) or Zwillinger (1989, pp. 388-391.).

LEMMA 6. *Let $x(t)$ be the solution to (9), let $\rho > 0$, and $\gamma > 0$ and take $\phi - \delta < 0$. For any $\tilde{x}$ there exists a T such that $x(T) > \tilde{x}$ and*

$$\bar{x}_{mT}(t) < x(t) < \bar{x}_{MT}(t),\ t > T$$

where

$$\bar{x}_{mT}(t) = \bar{x}(t; a, \delta_m, X(T), T),\ m = \tilde{G}(x(T))/\tilde{F}(x(T))$$

and

$$x_{MT}(t) = \bar{x}(t; a_M, \delta_M, x(T), T),\ M = \tilde{G}(x(T)).$$

Proof. The properties of the bounding functions depend on those of $\tilde{F}(x)$ and $\tilde{G}(x)$. It is clear that

$$\tilde{G}(x)/\tilde{F}(x) > 1, \tag{23}$$

and that there exists and $\bar{x}$ such that

$$\tilde{F}'(x) < 0, \tilde{G}'(x) < 0 \text{ and } (\tilde{G}(x)/\tilde{F}(x))' < 0 \text{ for } x > \bar{x} \tag{24}$$

Next a condition on the derivatives of $\bar{x}_{mT}$ and $\bar{x}_{MT}$ is given. For the given $\tilde{x}$ choose T so that $x(T) > \tilde{x}$, so that $x(T) > \bar{x}$ of (24) and so that $\dot{x}(T) > 0$. This is possible since, by Lemma 1 $x(t)$ is unbounded above. Since $\dot{x}(T) > 0$, writing (9) in terms of $\tilde{F}(x)$ and $\tilde{G}(x)$ gives

$$\frac{a}{x^{\mu-1}} e^{\phi t} \tilde{F}(x) + \frac{\phi - \delta}{\mu} x \tilde{G}(x) = \dot{x}(T) > 0,\ x = x(T), \tag{25}$$

by (25)

$$\dot{\bar{x}}_{mT}(T) = \frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta}{\mu} x \frac{\tilde{G}(x)}{\tilde{F}(x)} > 0,\ x = x_{mT}(T)$$

and finally

$$\dot{\bar{x}}_{mT}(t) > 0,\ t \geq T. \tag{26}$$

The last inequality follows by differentiating (13) with respect to time to get

$$\dot{\bar{x}}(t) = \frac{1}{\mu} \bar{x}^{\frac{1}{\mu}-1} \frac{a\mu}{\delta} e^{\phi t} \left[\phi + (\delta - \phi) C(.) e^{-\delta t}\right]. \tag{27}$$

Replacing δ with δ_m shows that $\dot{\bar{x}}_{mT}$ is either positive (if $(\delta - \phi)C(.) > 0$) or increasing in t.

Also

$$\dot{\bar{x}}_{MT}(T) = \tilde{G}(x)(\frac{a}{x^{\mu-1}} e^{\phi t} + \frac{\phi - \delta}{\mu} x)$$

$$> \tilde{G}(x)(\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x) > 0, \text{ for } x = x_{MT}(T)$$

where the first inequality follows from (23) and the second from (25); and, as above, from (27)

$$\dot{\bar{x}}_{MT}(t) > 0, t \geq T. \tag{28}$$

Finally it is shown that the conditions of Chaplygin's theorem are satisfied.

$$x_{mT}(T) = x(T) = x_{MT}(T)$$

by construction.

$$\bar{x}_{mT}(t) > \bar{x}_{mT}(T),\ \bar{x}_{MT}(t) > \bar{x}_{MT}(T),\ t > T \tag{29}$$

by (26) and (28).

$$\dot{\bar{x}}_{mT}(t) = \frac{a}{x^{\mu-1}}e^{\phi t} + m\frac{\phi-\delta}{\mu}x < (\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\tilde{G}(x)}{\tilde{F}(x)}\frac{\phi-\delta}{\mu}x)\tilde{F}(x)$$

$$= \frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi-\delta}{\mu}x\tilde{G}(x), \text{ for } x = \bar{x}_{mT}(t),\ t > T$$

by (11), (23), (24), and (29).

$$\dot{\bar{x}}_{MT}(t) = M(\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x) > (\frac{\tilde{F}(x)}{\tilde{G}(x)}\frac{a}{x^{\mu-1}}e^{\phi t} + \frac{\phi-\delta}{\mu}x)\tilde{G}(x)$$

$$= \frac{a}{x^{\mu-1}}e^{\phi t}\tilde{F}(x) + \frac{\phi-\delta}{\mu}x\tilde{G}(x) \text{ for } x = \bar{x}_{MT}(t),\ t > T$$

by (23), (24) and (29). This completes the proof. □

Finally Lemma 7 is proved for the case of $\phi - \delta < 0$.

LEMMA 7. *Let $x(t)$ be the solution to (9), let $\rho > 0$ and $\gamma > 0$ and take $\phi - \delta < 0$. Then*

$$\lim_{t\to\infty} |x(t) - \overline{\overline{x}}(t)| = 0$$

Proof. *Choose ϵ arbitrary but with $\epsilon < 1$, $\epsilon < \frac{4}{(\frac{\delta}{\phi}-1)}$. It must be shown that there is a T_ϵ such that*

$$(1-\epsilon)\overline{\overline{x}}(t) < x(t) < (1+\epsilon)\overline{\overline{x}}(t),\ t > T_\epsilon$$

Choose $\tilde{x}$ large enough so that , for the T given by Lemma 6,

$$\frac{\tilde{G}(x)}{\tilde{F}(x)} < 1 + \frac{\epsilon/2}{1-\frac{\phi}{\delta}},\ \tilde{G}(x) < 1 + \frac{1}{\frac{\phi}{\delta}\frac{1+\epsilon/2}{\epsilon/4} - 1},\ \text{for } x \geq x(T). \tag{30}$$

Now apply Lemma 6. The proof is completed by showing that there exists a T_ϵ such that

$$(1-\epsilon)\overline{\overline{x}}(t) < \bar{x}_{mT}(t),\ \bar{x}_{MT}(t) < (1+\epsilon)\overline{\overline{x}}(t),\ t > T_\epsilon.$$

Choose $T_1 > T$ so that

$$(1-\epsilon/2) < 1 - e^{-\delta_m t} C(a, \delta_m, x(T), T),\ t > T_1.$$

(30), and the definitions of m and δ_m imply

$$\frac{1}{1+\epsilon/2} < \frac{1}{m + (1-m)\frac{\phi}{\delta}} = \frac{1}{\delta_m/\delta}.$$

$$(1-\epsilon)^\mu < (1-\epsilon) < \frac{1}{1+\epsilon/2}(1-\epsilon/2) < \frac{1}{\delta_m/\delta}(1-\epsilon/2),$$

$$(1-\epsilon)^\mu \frac{a\mu}{\delta} e^{\phi t} < \frac{a\mu}{\delta_m} e^{\phi t}(1 - e^{-\delta_m t} C(a, \delta_m, x(T), T)),$$

$$(1-\epsilon)\overline{\overline{x}}(t) < \bar{x}_{mT}(t),\ \text{for } t > T_1.$$

Choose $T_2 > T$ such that

$$1 - e^{-\delta_M t} C(a_M, \delta_M, x(T), T) < 1 + \epsilon/4,\ \text{for } t > T_2.$$

(30) and the definition of M imply

$$\frac{M}{M + (1-M)\frac{\phi}{\delta}} < 1 + \epsilon/4.$$

This and the definitions of a_M and δ_M give

$$\frac{a_M}{\delta_M} < \frac{a}{\delta}(1+\epsilon/4).$$

$$\frac{a_M \mu}{\delta_M} e^{\phi t}(1 - e^{-\delta_M t} C(a_M, \delta_M, x(T), T)) < \frac{a\mu}{\delta} e^{\phi t}(1+\epsilon/4)^2$$

$$< \frac{a\mu}{\delta} e^{\phi t}(1+\epsilon) < \frac{a\mu}{\delta} e^{\phi t}(1+\epsilon)^\mu,\ t > T_2$$

$$\bar{x}_{MT}(t) < (1+\epsilon)\overline{\overline{x}}(t),\ t > T_2.$$

The proof is completed by taking $T_\epsilon = Max(T_1, T_2)$. □

References

Brenner, R. 1998. The economics of global turbulence. The New Left Review 29, 1-265.

Drandakis, E. and E. Phelps 1966. A model of induced invention, growth and distribution. The Economic Journal 76, 823-840.

Duménil, G. and D. Lévy 1993. The Economics of the Profit Rate. Aldershot: Edward Elgar Publishing Ltd.

Duménil, G. and D. Lévy 2003. Technology and distribution: historical trajectories a la Marx. Journal of Economic Behavior and Organization 52, 201-233.

Eltis, W. 1988. Falling rate of profit. In J. Eatwell, M Milgate and P. Newman, eds. The New Palgrave. London: Macmillan Press, pp. 276-280.

Foley, D. 2003. Endogenous technical change with externalities in a classical growth model. Journal of Economic Behavior and Organization 52, 167-189.

Hahn, F. 1966. Equilibrium dynamics with heterogeneous capital goods. The Quarterly Journal of Economics 80, 633-646.

Howard, M. C. and J. E. King (1992). A History of Marxian Economics, Volume II. London: Macmillan Education Ltd.

Johansen, L. 1967. A classical model of economic growth. in C. H. Feinstein, ed. Socialism, Capitalism and Economic Growth. Cambridge : Cambridge University Press, 13-29.

Kennedy, C. 1964. Induced bias in innovation and the theory of distribution. The Economic Journal 74, 541-547.

Marx, K. 1984. Capital volume III. London: Lawrrence and Wishart.

Michl, T. 1994. Three models of the falling rate of profit. Review of Radical Political Economics 24, 55-75.

Mikhlin, S. G. and K. L. Smolitskiy 1967. Approximate Methods of Solution of Differential and Integral Equations. New York: American Elsevier Publishing Company.

Mill, J.S. 1965. Principles of Political Economy. V. W. Bladen and J. M. Robson, eds.. Toronto: University of Toronto Press.

Okishio, N. 1961. Technical change and the rate of profit. Kobe University Economic Review 7, 86-99.

Petith, H. 2002. A foundation model for Marxian breakdown theories based on a new falling rate of profit mechanism (long version). UAB Discussion Paper 524.02 available at http://pareto.uab.es/wp.

Ricardo, D. 1817. On the Principles of Political Economy and Taxation. Reprinted as Volume 1 of The Works and Correspondence of David Ricardo. P. Sraffa ed., Cambridge: Cambridge University Press, 1951.

Samuelson, P. 1976. The canonical classical model of political economy. The Journal of Economic Literature 26, 1415-1434

Skott, P. 1992. Imperfect competition and the theory of the falling rate of profit. Review of Radical Political Economy 24, 101-113.

Skillman, G. 1997. Technical change and the equilibrium profit rate in a market with sequential bargaining" Metroeconomica 48, 238-61.

Shell, K and J. Stiglitz 1967. The allocation of investment in a dynamic economy. The Quarterly Journal of Economics 81, 592-609.

Zwillinger, D. 1989. Hand book of Differential Equations. San Diego: Academic Press.