PRESCRIBING THE NODAL SET OF THE FIRST EIGENFUNCTION IN EACH CONFORMAL CLASS

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Abstract. We consider the problem of prescribing the nodal set of the first nontrivial eigenfunction of the Laplacian in a conformal class. Our main result is that, given a separating closed hypersurface Σ in a compact Riemannian manifold \((M, g_0)\) of dimension \(d \geq 3\), there is a metric \(g\) on \(M\) conformally equivalent to \(g_0\) and with the same volume such that the nodal set of its first nontrivial eigenfunction is a \(C^0\)-small deformation of \(Σ\) (i.e., \(\Phi(Σ)\) with \(\Phi : M \to M\) a diffeomorphism arbitrarily close to the identity in the \(C^0\) norm).

1. Introduction

Let \(M\) denote a closed manifold of dimension \(d \geq 3\) endowed with a fixed Riemannian metric \(g_0\). The eigenfunctions of \(M\) satisfy the equation

\[ \Delta u_k = -\lambda_k u_k, \]

where \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \) are the eigenvalues of \(M\) and \(\Delta\) is the Laplace operator of the manifold. The zero set \(u_k^{-1}(0)\) is called the nodal set of the eigenfunction, and each connected component of \(M \setminus u_k^{-1}(0)\) is known as a nodal domain.

The study of the eigenvalues and eigenfunctions of a manifold is a classic topic in geometric analysis with a number of important open problems [17, 18]. A fundamental fact is that the behavior of \(\lambda_k\) as \(k \to \infty\) is extremely rigid, as captured by Weyl’s law. Major open questions roughly related to this rigidity phenomenon concern the asymptotic behavior as \(k \to \infty\) of, say, the number of nodal domains and the measure of the nodal set of \(u_k\) [7, 13] or the number of critical points of the eigenfunctions [14].

In striking contrast, the low-energy behavior of the eigenvalues is quite flexible. A landmark in this direction is the proof that one can prescribe an arbitrarily high number of eigenvalues of the Laplacian, including multiplicities. More precisely [6], given any finite sequence of positive real numbers \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\), there is a metric \(g\) on \(M\) having this sequence as its first \(N\) nontrivial eigenvalues. This cannot be accomplished if we require the metric \(g\) to be conformally equivalent to the original metric \(g_0\) and of the same volume; in particular, it is known [5, 12, 13] that the supremum of the first nontrivial eigenvalue \(\lambda_1\) corresponding to \(g\) is finite as \(g\) ranges over the set of metrics conformal to \(g_0\) and with the same volume.

The nodal sets of the low-energy eigenfunctions turn out to be surprisingly flexible too. Indeed, it has been recently shown that [11], given a separating hypersurface \(Σ\) in \(M\), there is a metric \(g\) on the manifold for which the nodal set \(u_1^{-1}(0)\) of the first eigenfunction is precisely \(Σ\). Throughout, we will assume that the hypersurfaces are all smooth, connected, compact and without boundary. We recall that
A hypersurface $\Sigma$ is \emph{separating} if its complement $M \setminus \Sigma$ is the union of two disjoint open sets. Similar results for higher eigenfunctions have been established as well. The goal of this paper is to show that the nodal set $u_{-1}(0)$ of the first eigenfunction can be prescribed, up to a small deformation, even if we require the metric $g$ to be conformal to the original metric $g_0$ and of the same volume. More precisely, we will prove the following theorem.

**Theorem 1.1.** Let $M$ be a $d$-manifold ($d \geq 3$) endowed with a Riemannian metric $g_0$ and let $\Sigma$ be a separating hypersurface. Then, given any $\delta > 0$, there is a metric $g$ in $M$ conformally equivalent to $g_0$ and with the same volume such that its first eigenvalue $\lambda_1$ is simple and the nodal set of its first eigenfunction $u_1$ is $\Phi(\Sigma)$, where $\Phi$ is a diffeomorphism of $M$ whose distance to the identity in $C^0(M)$ is at most $\delta$.

The proof of this result is based on the explicit construction of a conformal factor which is of order $\varepsilon$ in a neighborhood $\Omega_\eta$ of $\Sigma$ of width $\eta$. Geometrically, this ensures that the manifold, endowed with the rescaled metric, has the structure of a dumbbell, as depicted in Figure 1. The basic idea behind the proof of the theorem is to exploit this dumbbell structure through a fine analysis of the first eigenfunction. More precisely, we show that as $\varepsilon$ tends to zero the first eigenfunction approximates a harmonic function in $\Omega_\eta$ with constant boundary values. In turn, the zero set of this harmonic function can then be controlled provided that $\eta$ is small. Related results on level sets with prescribed topologies were derived, using completely different methods, for Green’s functions in [9], for harmonic functions in $\mathbb{R}^n$ in [10] and for eigenfunctions of the Laplacian in [11].

An easy application of Theorem 1.1 enables us to prove that, given any Riemannian manifold $(M, g_0)$, there is a metric conformally equivalent to $g_0$ and with the same volume such that the eigenfunction $u_1$ has as many isolated critical points as one wishes.

**Theorem 1.2.** Let $M$ be a $d$-manifold endowed with a Riemannian metric $g_0$, with $d \geq 3$, and let $N$ be a positive integer. Then there is a metric $g$ on $M$, conformally equivalent to $g_0$ and with the same volume, such that its first nontrivial eigenfunction $u_1$ has at least $N$ non-degenerate critical points.

It is worth recalling that, on surfaces, Cheng [4] gave a topological bound for the number of critical points of the $k$th eigenfunction that lie on the nodal line. We do not know if results analogous to Theorems 1.1 and 1.2 hold for surfaces. The
proof of Theorem 1.1 is given in Section 2, although the proofs of several technical lemmas are relegated to Sections 3-7. The proof of Theorem 1.2, which hinges on Theorem 1.1, is then presented in Section 8.

2. Proof of the main theorem

We divide the proof of Theorem 1.1 in five steps. In Step 1 we define a discontinuous metric $g_\varepsilon$ that is conformal to $g_0$ and is of order $\varepsilon$ in a neighborhood $\Omega_\eta$ of $\Sigma$ of width $\eta$; here we have Lemma 2.1 showing that the first nontrivial eigenvalue of $(M, g_\varepsilon)$ is simple and tends to zero as $\varepsilon \to 0$. In Step 2 we exploit the partial regularity of the metric $g_\varepsilon$ to obtain estimates in certain mixed Sobolev norms (cf. Lemma 2.2), which imply that the first eigenfunction $u_\varepsilon$ is Hölder continuous and that its $L^\infty$ norm is bounded uniformly in $\varepsilon$. These estimates are used in Step 3 to show that $u_\varepsilon$ converges in $\Omega_\eta$ to a harmonic function $h$ with constant boundary values on $\partial\Omega_\eta$ (cf. Proposition 2.5). In Step 4 the nodal set of $h$ is analyzed, the main result being Corollary 2.7 where we prove that it is a regular level set diffeomorphic to $\Sigma$ provided that the width $\eta$ is small. Finally, the proof of the theorem is completed in Step 5 taking the two independent parameters $\varepsilon$ and $\eta$ sufficiently small, and using that 0 is a regular value of $h$.

Step 1: Defining a discontinuous metric. For small enough $\eta > 0$, the set

$$\Omega_\eta := \{ x \in M : \text{dist}_{g_0}(x, \Sigma) < \eta \}$$

is diffeomorphic to $(-\eta, \eta) \times \Sigma$. Denoting by $\Omega_\eta^c := M \setminus \Omega_\eta$ the complement of the set $\Omega_\eta$, let us consider the bounded discontinuous function

$$f_\varepsilon(x) := \begin{cases} 
\varepsilon & \text{if } x \in \Omega_\eta, \\
\kappa & \text{if } x \in \Omega_\eta^c.
\end{cases}$$

Here the constant $\kappa \equiv \kappa(\varepsilon, \eta)$ is defined as

$$\kappa := \left(1 + \frac{\varepsilon^2}{\frac{|\Omega_\eta|}{\Omega_\eta^c}}\right)^{2/d},$$

with $|\cdot|$ standing for the volume of a set computed with respect to the metric $g_0$. With this choice of $\kappa$, the volume of $M$ with the metric $g_\varepsilon := f_\varepsilon g_0$ is independent of the choice of $\varepsilon$ and $\eta$. Notice that $\kappa$ is well-behaved since, for $\varepsilon, \eta$ sufficiently small,

$$1 \leq \kappa \leq \left(1 + \frac{|\Omega_\eta|}{|\Omega_\eta^c|}\right)^{2/d} \leq 2.$$

The first (nontrivial) eigenfunction of $M$ with the discontinuous metric $g_\varepsilon$ is a minimizer of the Rayleigh quotient

$$q_\varepsilon(u) := \frac{\int_M |du|^2 \, dV_\varepsilon}{\int_M u^2 \, dV_\varepsilon} = \frac{\varepsilon^{d-1} \int_{\Omega_\eta} |du|^2 + \kappa^{d-1} \int_{\Omega_\eta^c} |du|^2}{\varepsilon^{d} \int_{\Omega_\eta} u^2 + \kappa^{d} \int_{\Omega_\eta^c} u^2}.$$
in the space of nonzero functions \( u \in H^1(M) \) such that

\[
0 = \int_M u \, dV_\varepsilon = \varepsilon^{\frac{d}{2}} \int_{\Omega_\varepsilon} u + \kappa^{\frac{d}{2}} \int_{\Omega^c_\varepsilon} u.
\]

Throughout we are denoting with a subscript \( \varepsilon \) the quantities (norm, Riemannian measure) associated with the metric \( g_\varepsilon \) and we are omitting the measure under the integral sign when it is the one corresponding to \( g_0 \). We will use the notation \( \lambda_\varepsilon \equiv \lambda_{1,\varepsilon} \) for the first nonzero eigenvalue of \((M, g_\varepsilon)\), and call \( u_\varepsilon \equiv u_{1,\varepsilon} \) its corresponding eigenfunction, which we assume to be normalized to have unit norm:

\[
\int_M u_\varepsilon^2 \, dV_\varepsilon = \varepsilon^{\frac{d}{2}} \int_{\Omega_\varepsilon} u_\varepsilon^2 + \kappa^{\frac{d}{2}} \int_{\Omega^c_\varepsilon} u_\varepsilon^2 = 1.
\]

The small-\( \varepsilon \) behavior of the first nontrivial eigenvalue \( \lambda_\varepsilon \) is described in the following lemma, here in particular it is shown to be simple. The proof is presented in Section 3. With a little more work, and using that \( M \setminus \Omega_\varepsilon \) consists of precisely two connected components because \( \Sigma \) separates, the argument in fact implies that, as \( \varepsilon \to 0 \), the second eigenvalue tends to the smallest of the first nonzero Neumann eigenvalue of each connected component.

**Lemma 2.1.** For any small but fixed \( \eta \), the first nontrivial eigenvalue \( \lambda_\varepsilon \) is simple and converges to zero as \( \varepsilon \searrow 0 \) (more precisely, \( \lambda_\varepsilon \leq C\varepsilon^{\frac{d}{2}-1} \)).

**Step 2:** Anisotropic derivative estimates for the eigenfunction. In Step 1 we defined a metric \( g_\varepsilon \) that is discontinuous across the boundary of the set \( \Omega_\varepsilon \). Two redeeming features of this metric are that it is smooth everywhere but on \( \partial \Omega_\eta \) and that \( g_\varepsilon \) is in fact everywhere smooth along the directions that are tangent to \( \partial \Omega_\eta \).

In view of the discontinuity of the metric, a simple, convenient way of exploiting this partial regularity is by considering mixed Sobolev norms. In order to define them, it is convenient to use the following approach. Denote by \( U_0 \) a neighborhood of \( \Omega_\eta \) where the signed distance \( \rho \) to the hypersurface \( \Sigma \) as measured with respect to the metric \( g_0 \) is a smooth function. Set

\[
Y := \chi_0 \nabla \rho
\]

where \( \chi_0 \equiv \bar{\chi}_0(\rho) \) is a fixed nonnegative smooth function that only depends on \( \rho \), is equal to 1 in a neighborhood \( U_1 \subset U_0 \) of \( \Omega_\eta \) and vanishes outside \( U_0 \). With this definition, \( Y \) defines a smooth vector field in \( M \) that does not vanish in \( U_1 \).

Let us now take a finite collection of vector fields \( X_1, \ldots, X_s \) that are everywhere orthogonal to \( Y \) (i.e., \( g_0(X_j, Y) = 0 \)), supported in \( U_0 \) and such that

\[
\text{span}\{Y|_x, X_1|_x, \ldots, X_s|_x\} = T_xM \quad \text{for all } x \in U_1.
\]

A simple transversality argument shows that in fact one can take \( s = d + 1 \). Given a set \( V \subset M \), functions in \( H^1(V) \) that additionally have \( k \) weak derivatives in the directions tangent to \( \Sigma \) that are also in \( H^1 \) will be characterized through the mixed Sobolev norm

\[
\|u\|_{H^1 H^k_\varepsilon(V)} := \sum_{j_1 + \cdots + j_s \leq k} \|X_1^{j_1} \cdots X_s^{j_s} u\|_{H^1(V)} + \|(1 - \chi_0) u\|_{H^k(V)}.
\]

We will write \( \| \cdot \|_{H^1 H^k_\varepsilon} \equiv \| \cdot \|_{H^1 H^k(M)} \). The following lemma, which is proved in Section 4, establishes an upper bound for the mixed Sobolev norm of \( u_\varepsilon \).
Lemma 2.2. For any nonnegative integer \( k \), there are constants \( C_\varepsilon \) and \( C \) depending on \( k \) such that the normalized eigenfunction \( u_\varepsilon \) satisfies
\[
\| u_\varepsilon \|_{H^1 H^k_T} < C_\varepsilon, \quad \| u_\varepsilon \|_{H^1 H^k_T(\Omega^c_\eta)} < C.
\]
Here the constant \( C \) is independent of \( \varepsilon \).

Since \( u_\varepsilon \) is then in the Sobolev space \( H^1 H^k_T \) for any positive \( k \), \( u_\varepsilon \) is Hölder continuous with exponent \( \alpha \) for any \( \alpha < 1/2 \) on account of the Sobolev embedding theorem [3, Theorem 10.2]. Further, from the second inequality in the lemma we infer that the boundary traces
\[
B^\pm_\varepsilon := u_\varepsilon |_{\partial \Omega^\pm_\eta}
\]
are bounded in \( H^k(\partial \Omega^\pm_\eta) \) uniformly in \( \varepsilon \), i.e.,
\[
\| B^\pm_\varepsilon \|_{H^k(\partial \Omega^\pm_\eta)} < C.
\]
Here we are denoting by \( \Omega^\pm_\eta \) the connected components of the set \( \Omega^c_\eta \) (which are known to be precisely two because \( \Sigma \) is a separating hypersurface). Notice, in particular, that \( B^\pm_\varepsilon \) are smooth functions. Furthermore, the eigenfunction \( u_\varepsilon \) is uniformly bounded in \( L^\infty \), as established in the following argument.

Corollary 2.3. There is a constant \( C \) independent of \( \varepsilon \) such that \( \| u_\varepsilon \|_{L^\infty(M)} < C \).

Proof. In the domains \( \Omega^\pm_\eta \) the metric \( g_\varepsilon = \kappa g_0 \) is smooth, so the eigenfunction \( u_\varepsilon \) is smooth as well and satisfies the boundary value problem
\[
\Delta u_\varepsilon = -\kappa \lambda_\varepsilon u_\varepsilon \quad \text{in} \ \Omega^\pm_\eta, \quad u_\varepsilon |_{\partial \Omega^\pm_\eta} = B^\pm_\varepsilon.
\]
Here and in what follows \( \Delta \) stands for the Laplacian corresponding to the metric \( g_0 \). The standard a priori bounds following from the maximum principle then imply that
\[
\| u_\varepsilon \|_{L^\infty(\Omega^\pm_\eta)} < \| B^\pm_\varepsilon \|_{L^\infty(\partial \Omega^\pm_\eta)} + C\kappa \lambda_\varepsilon \| u_\varepsilon \|_{L^\infty(\Omega^\pm_\eta)},
\]
for an \( \varepsilon \)-independent constant \( C \). Since \( \lambda_\varepsilon \to 0 \) as \( \varepsilon \to 0 \) by Lemma 2.1 and \( B^\pm_\varepsilon \) is bounded by Eq. (2.1), we conclude that
\[
\| u_\varepsilon \|_{L^\infty(\Omega^\pm_\eta)} < C.
\]
Similarly, in the domain \( \Omega_\eta \) the eigenfunction \( u_\varepsilon \) is smooth and solves the boundary value problem
\[
\Delta u_\varepsilon = -\varepsilon \lambda_\varepsilon u_\varepsilon \quad \text{in} \ \Omega_\eta, \quad u_\varepsilon |_{\partial \Omega^\pm_\eta} = B^\pm_\varepsilon.
\]
The same argument as before proves the desired estimate in \( \Omega_\eta \), and the corollary follows. \( \square \)

The following result is a corollary of the proof of Lemma 2.2 and will be instrumental in Step 5 to prove the existence of a smooth metric \( g \) with the properties stated in Theorem 1.1. For the ease of notation, throughout the proof we will denote by \( u_n \) the first eigenfunction of the metric \( g_n \) described in this corollary. One should not confuse this for the \( n^{th} \) eigenvalue of a fixed metric; indeed, only first eigenfunctions will be relevant in what follows.

Corollary 2.4. One can take a sequence of smooth metrics \( g_n \), conformal to \( g_0 \) and with the same volume, such that the first eigenfunction \( u_n \) of \( g_n \) has multiplicity one and converges to \( u_\varepsilon \) in \( C^0(M) \) and in \( C^k(S) \) for any compact set \( S \subset M \setminus \partial \Omega_\eta \).
Proof. In the proof of Lemma 2.2 it is shown that there is a sequence of smooth metrics $g_n := f_n g_0$, with $f_n$ converging to $f_\varepsilon$ pointwise in $M$ and in $C^k(S)$ for any compact set $S \subset M \setminus \partial \Omega_\eta$, such that

$$\lim_{n \to \infty} |\lambda_n - \lambda_\varepsilon| = 0, \quad \lim_{n \to \infty} \|u_n - u_\varepsilon\|_{C^0(M)} = 0.$$  

Here, $\lambda_n$ is the first nontrivial eigenvalue of the Laplacian with the metric $g_n$ (which is simple) and $u_n$ is the corresponding (normalized) eigenfunction. By elliptic regularity the $C^0$ convergence of $u_n$ to $u_\varepsilon$ can be promoted to $C^k$ convergence in any compact set $S$ contained in $M \setminus \partial \Omega_\eta$. To fix the volume of the metric $g_n$ we can rescale it with a constant factor $\gamma_n$ such that $\lim_{n \to \infty} \gamma_n = 1$ and

$$|M| = \int_M dV_{\gamma_n g_n} = \int_M \gamma_n^\frac{d}{2} dV_n$$

for all $n$, where $dV_n$ is the volume element associated with the metric $g_n$. The corollary then follows trivially because this rescaling does have any effect on the convergence (2.2). \hfill $\Box$

Step 3: Approximation by a harmonic function in $\Omega_\eta$. We will show that, as $\varepsilon \to 0$, $u_\varepsilon$ tends to constants outside of $\Omega_\eta$ while, inside $\Omega_\eta$, it can be approximated by certain harmonic function $h$. Indeed, let us define $h$ as the function in $\Omega_\eta$ given by the only solution to the boundary value problem

$$\Delta h = 0 \text{ in } \Omega_\eta, \quad h|_{\partial \Omega_\pm} = c_\pm,$$

where the constants $c_\pm$ have been chosen so that

$$c_+^2 |\Omega_\eta^+| + c_-^2 |\Omega_\eta^-| = \kappa_0^{-\frac{d}{2}},$$

$$c_+ |\Omega_\eta^+| + c_- |\Omega_\eta^-| = 0,$$

with

$$\kappa_0 := \kappa|_{\varepsilon=0} = \left(1 + \frac{|\Omega_\eta|}{|\Omega_\eta^-|}\right)^\frac{d}{2}.$$  

These equations simply ensure that if we extend $h$ by constants outside of $\Omega_\eta$, then the corresponding extension has mean value 0 (i.e., it is orthogonal to constants) and its $L^2$ norm is 1. This determines $c_\pm$ up to a global sign, which can be fixed by requiring that

$$\pm c_\pm > 0.$$  

Proposition 2.5. If $k$ is any integer, one can choose the sign of $u_\varepsilon$ so that it converges to the harmonic function $h$ in $\Omega_\eta$ and to constants in the complement of this set:

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - h\|_{C^k(\Omega_\eta)} = 0, \quad \lim_{\varepsilon \to 0} \|u_\varepsilon - c_\pm\|_{C^0(\Omega_\eta^\pm)} = 0.$$

This result is proved in Section 5. Observe that this proposition shows that, for $\varepsilon$ small enough, the nodal set of the eigenfunction $u_\varepsilon$ can be controlled using the harmonic function $h$, a fact that will be exploited in the next step.
Step 4: Analysis of the harmonic function $h$. The following proposition is proved in Section 6. It provides an asymptotic form of the harmonic function $h$ defined in Step 4 as the width $\eta$ tends to 0, thereby allowing us to understand its nodal set for small $\eta$. It is stated in terms of local coordinates $(\rho, y)$, with $\rho$ the signed distance to the hypersurface $\Sigma$ with respect to the metric $g_0$, the sign being chosen so that $\partial \Omega^+_{\eta} = \rho^{-1}(\pm \eta)$, and $y = (y_1, \ldots, y_{d-1})$ are local coordinates in $\Sigma$. For details about the construction of the coordinate system, see the proof of Lemma 2.2 in Section 4.

Proposition 2.6. Consider the function in $\Omega_{\eta}$ given by

$$h := \frac{c_+ + c_-}{2} + \frac{c_+ - c_-}{2\eta}\rho.$$ 

For any nonnegative integers $j, k$, in the above local coordinates $(\rho, y)$ we then have

$$\lim_{\eta \searrow 0} \eta^j \|\partial_y^k D^k_y (h - \bar{h})\|_{L^\infty(\Omega_{\eta})} = 0.$$

Here we have abused the notation to state the proposition in a coordinate-dependent way, which is slightly more convenient for our purposes. An equivalent intrinsic statement, using the vector fields $Y, X_1, \ldots, X_s$ introduced in Step 2, is

$$\lim_{\eta \searrow 0} \eta^j \|Y^j X_1^{k_1} \cdots X_s^{k_s} (h - \bar{h})\|_{L^\infty(\Omega_{\eta})} = 0.$$

An easy consequence of the proposition is that the nodal set $h^{-1}(0)$ of the harmonic function $h$ is diffeomorphic to $\Sigma$ (through a diffeomorphism of $M$ that is close to the identity in the $C^0$ norm) provided that the width $\eta$ is small enough. Moreover, $0$ is a regular value of $h$, so that $h^{-1}(0)$ is robust under $C^1$-small perturbations of the function $h$, a property that will be key in Step 5. The proof of this corollary is given in Section 7.

Corollary 2.7. For any $\delta > 0$ there is some $\eta_0 > 0$ such that if $\eta < \eta_0$ the nodal set $h^{-1}(0)$ is given by $\Psi(\Sigma)$, where $\Psi : M \to M$ is a diffeomorphism with $\|\Psi - \text{id}\|_{C^0(M)} < \delta$. Moreover, $0$ is a regular value of $h$, that is, $\nabla h(x) \neq 0$ at any point $x \in h^{-1}(0)$.

Step 5: The nodal set of the eigenfunction. By Corollary 2.7, the width $\eta$ can be chosen small enough so that the zero set $h^{-1}(0)$ is regular and given by $\Psi(\Sigma)$, where $\Psi$ is a diffeomorphism of $M$ with $\|\Psi - \text{id}\|_{C^0(M)}$ as small as one wishes. Let us fix some $\eta$ for which this property holds.

In turn, Proposition 2.5 ensures that, for any fixed $\delta > 0$, one can then choose a small $\varepsilon$ so that the difference $u_{\varepsilon} - h$ is smaller than $\delta$ in $C^k(\Omega_{\eta})$. For small enough $\delta$, Thom’s isopy theorem [11] Section 20.2 then implies that the first eigenfunction $u_{\varepsilon}$ has a regular connected component of the level set $u_{\varepsilon}^{-1}(0)$ diffeomorphic to $\Sigma$, and that the corresponding diffeomorphism can be chosen arbitrarily close to the identity in $C^0(M)$. Moreover, this component, which is contained in $\Omega^+_{\eta}$, is the whole nodal set of $u_{\varepsilon}$ because the second estimate of Proposition 2.5 shows that $u_{\varepsilon}$ does not vanish in $\Omega^+_{\eta}$ for small $\varepsilon$ as

$$\lim_{\varepsilon \searrow 0} \|u_{\varepsilon} - c_{\pm}\|_{C^0(\Omega^+_{\eta})} = 0.$$ 

By Corollary 2.4 there is a sequence of metrics $g_n$, conformal to $g_0$ and with the same volume, such that their first eigenfunction $u_n$ converges to $u_{\varepsilon}$ uniformly in $M$ and additionally in $C^k(S)$ for any compact set $S \subset M \setminus \partial \Omega^+_{\eta}$, and the corresponding
eigenvalue $\lambda_n$ has multiplicity one and converges to $\lambda_\varepsilon$. Since $u_n^{-1}(0) \subset \Omega_\eta$, the fact that $u_n \to u_\varepsilon$ uniformly in $M$ then ensures that, for any large enough $n$, $u_n^{-1}(0) \cap \Omega_{\eta}^\varepsilon = \emptyset$. Therefore, another application of Thom’s isotopy theorem yields that, $u_n^{-1}(0)$ is a regular level set given by $\Phi_n(\Sigma)$, with $\Phi_n$ a diffeomorphism of $M$ with $\|\Phi_n - \text{id}\|_{C^0(M)}$ arbitrarily small. The theorem then follows by taking $g_n$ as the metric in the statement, for any sufficiently large $n$.

3. Proof of Lemma 2.1

Let us denote by $\rho$ the signed distance to the hypersurface $\Sigma$ with respect to the metric $g_0$, the sign being chosen so that $\partial \Omega_{\eta}^\pm = \rho^{-1}(\pm \eta)$. To prove that $\lambda_\varepsilon$ tends to zero, let us take the Lipschitz function
\[
u := \chi_{\Omega_\eta^+} + [1 - a(\eta + \rho)] \chi_{\Omega_\eta^-} + (1 - 2a\eta) \chi_{\Omega_\eta^+},
\]
where the constant
\[a := \frac{\kappa|\Omega_\eta^+| + \varepsilon^\frac{d}{2}|\Omega_\eta^-|}{2\eta \kappa^\frac{d}{2}|\Omega_\eta^+| + \varepsilon^\frac{d}{2} \int_{\Omega_\eta^+}(\eta + \rho)}
\]
has been chosen so that
\[\int_M \nu \, dV_\varepsilon = 0.
\]
Since $\rho \geq -\eta$ in $\Omega_\eta$, it is clear that
\[0 < a < \frac{C}{\eta}
\]
for some absolute constant $C$.

Since
\[\int_M |du|^2 \, dV_\varepsilon = a^2 \varepsilon^{\frac{d}{2} - 1} \int_{\Omega_\eta^-} |d\rho|^2 = a^2 \varepsilon^{\frac{d}{2} - 1} |\Omega_\eta^-| \leq C a^2 \varepsilon^{\frac{d}{2} - 1} \eta \leq \frac{C \varepsilon^{\frac{d}{2} - 1}}{\eta},
\]
where we have used that $|d\rho|^2 = 1$, and
\[\int_M u^2 \, dV_\varepsilon > \kappa^\frac{d}{2}|\Omega_\eta^-| > C,
\]
we can take $u$ as a test function to show that
\[\lambda_\varepsilon \leq q_\varepsilon(u) \leq \frac{C \varepsilon^{\frac{d}{2} - 1}}{\eta},
\]
which tends to zero as $\varepsilon \searrow 0$.

Calling $\lambda_{2,\varepsilon}$ the second nontrivial eigenvalue of $(M, g_\varepsilon)$, we now prove that there is some positive constant $\Lambda$, independent of $\varepsilon$, such that
\[\lambda_{2,\varepsilon} \geq \Lambda.
\]
By the min-max principle, $\lambda_{2,\varepsilon}$ is
\[\lambda_{2,\varepsilon} = \inf_{W \in \mathcal{W}} \max_{\varphi \in W \setminus \{0\}} q_\varepsilon(\varphi),
\]
where $\mathcal{W}$ stands for the set of 3-dimensional linear subspaces of $H^1(M)$.

Following [11], let us consider the direct sum decomposition
\[H^1(M) = \mathcal{H}_1 \oplus \mathcal{H}_2,
\]
where
\[ H_1 := \{ \varphi \in H^1(M) : \Delta \varphi |_{\Omega} = 0 \}, \]
\[ H_2 := \{ \varphi \in H^1(M) : \varphi |_{\Omega} \in H^1_0(\Omega), \varphi |_{\Omega^c} = 0 \}. \]

Notice that these subspaces are orthogonal in the sense that
\[ \int_M g_0(d\varphi_1, d\varphi_2) = 0 \]
for all \( \varphi_1 \in H_1 \) and \( \varphi_2 \in H_2 \). Accordingly, any function \( \varphi \in H^1(M) \) can be written
\[ \varphi = \varphi_1 + \varphi_2 \]
in a unique way, with \( \varphi_i \in H_i \). To prove that there is a positive lower bound for the eigenvalues \( \lambda_{2, \varepsilon} \), we can assume that \( \lambda_{2, \varepsilon} < C_0 \) where \( C_0 \) is a constant independent of \( \varepsilon \). Consider the set \( W' \) of 3-dimensional subspaces of \( H^1(M) \) such that
\[ q_\varepsilon(\varphi) \leq C_0 + 1 \]
for all nonzero \( \varphi \in W' \). Therefore, by Eq. (3.2) we have
\[ \lambda_{2, \varepsilon} = \inf_{W \in W'} \max_{\varphi \in W \setminus \{0\}} q_\varepsilon(\varphi). \]

The observation now is that if \( \varphi \) is in a subspace belonging to \( W' \), then
\[ \int \varphi_2^2 dV_\varepsilon \leq C \int \varphi^2 dV_\varepsilon, \]
where \( C \) is an \( \varepsilon \)-independent constant. Indeed,
\[ C_0 + 1 \geq q_\varepsilon(\varphi) \geq \frac{\varepsilon^{\frac{d}{2}-1} \int_{\Omega_\varepsilon} |d\varphi_2|^2}{\int \varphi^2 dV_\varepsilon} \]
\[ = \frac{1}{\varepsilon} \frac{\int_{\Omega_\varepsilon} |d\varphi_2|^2}{\int_{\Omega_\varepsilon} \varphi_2^2} \left\| \varphi_2 \right\|_\varepsilon^2 \]
\[ \geq \frac{\lambda_{\Omega_\varepsilon} \left\| \varphi_2 \right\|_\varepsilon^2}{\varepsilon \left\| \varphi \right\|_\varepsilon^2}, \]
where the positive, \( \varepsilon \)-independent constant \( \lambda_{\Omega_\varepsilon} \) is the first Dirichlet eigenvalue of \( \Omega_\varepsilon \) with the metric \( g_0 \). This inequality implies the estimate (3.5). A straightforward computation also shows that (3.5) implies
\[ \| \varphi \|_\varepsilon^2 \leq (1 + C \varepsilon^{\frac{d}{2}}) \| \varphi_1 \|_\varepsilon^2. \]

A second observation is that if \( \varphi \) is in a subspace belonging to \( W' \), since \( \varphi_1 \) is harmonic in \( \Omega_\varepsilon \), standard elliptic estimates, the trace inequality and Eq. (3.6) imply
\[ \int_{\Omega_\varepsilon} \varphi_1^2 \leq C \| \varphi_1 \|_{H^1(\partial \Omega_\varepsilon)}^2 \leq C \| \varphi_1 \|_{H^1(\Omega_\varepsilon)}^2 \leq C \int_{\Omega_\varepsilon} \varphi_1^2. \]

Now we are ready to show that \( \lambda_{2, \varepsilon} \) is lower bounded by a positive \( \varepsilon \)-independent constant. Using the orthogonality relation (3.3) and the estimates (3.6) and (3.7)
we write, for any nonzero \( \varphi \in W \) with \( W \in W' \),

\[
q_\varepsilon(\varphi) = \frac{\int_M |d\varphi_1|^2 \, dV_\varepsilon + \int_M |d\varphi_2|^2 \, dV_\varepsilon}{\int |\varphi|^2 \, dV_\varepsilon} \\
\geq (1 + C \varepsilon^{\frac{1}{2}}) \frac{\int_{\Omega^+_n} |d\varphi_1|^2}{\int_{\Omega^+_n} |\varphi|^2} \\
\geq (1 + C \varepsilon^{\frac{1}{2}}) \frac{\int_{\Omega^-_n} |d\varphi_1|^2}{\int_{\Omega^-_n} |\varphi|^2} \\
\geq (1 + C \varepsilon^{\frac{1}{2}}) q_{\Omega^-}(\varphi_1|_{\Omega^-}) \text{.}
\]

Here

\[
q_{\Omega^-}(\psi) := \frac{\int_{\Omega^-} |d\psi|^2}{\int_{\Omega^-} |\psi|^2}
\]

is the Rayleigh quotient in \( \Omega^- \). From min-max formulation (3.4) we then infer

\[
\lambda_{2, \varepsilon} \geq (1 + C \varepsilon^{\frac{1}{2}}) \inf_{W \in W'} \max_{\varphi \in W \setminus \{0\}} q_{\Omega^-}(\varphi_1|_{\Omega^-}) \\
\geq (1 + C \varepsilon^{\frac{1}{2}}) \mu_2 \text{,}
\]

where \( \mu_2 \) is the third Neumann eigenvalue of the domain \( \Omega^- \). Since \( \Omega^- \) has two connected components \( \Omega^+_n \) and \( \Omega^-_n \), a simple computation shows that the first Neumann eigenvalues \( \mu_0 = \mu_1 = 0 \), and \( \mu_2 \) is the smallest of the first nontrivial Neumann eigenvalues of \( \Omega^+_n \) and \( \Omega^-_n \), thus bounded by a positive \( \varepsilon \)-independent constant. The lower estimate (3.1) then follows, which establishes that \( \lambda_\varepsilon \) is a simple eigenvalue.

4. Proof of Lemma 2.2

Let us begin by noticing that the function \( f_\varepsilon \) introduced in Step 1 can be written as \( f_\varepsilon(x) := F(\rho(x)) \), where \( \rho \) denotes the signed distance to \( \Sigma \), and

\[
F(t) := \kappa + (\varepsilon - \kappa) 1_{[-\eta,0]}(t) \text{,}
\]

with \( 1_{[-\eta,0]} \) the indicator function of the interval. The dependence of \( F \) on \( \varepsilon \) is not explicitly written for the sake of simplicity. Let us now take a sequence of uniformly bounded smooth functions \( F_n(t) \) that coincide with \( F(t) \) for \( |t| \geq \eta \) and \( |t| \leq \eta - \frac{1}{n} \) and set \( f_n(x) := F_n(\rho(x)) \), which is a smooth function because the signed distance is smooth in the region where \( F_n \) is nonconstant.

It is standard (see e.g. [2]) that, as \( n \to \infty \), the \( k \)-th eigenvalue associated with the smooth metric \( g_n := f_n g_0 \) converges to that of the discontinuous metric \( g_\varepsilon \) and that, if its multiplicity is one, the corresponding eigenfunction converges to the eigenfunction of the metric \( g_\varepsilon \) in \( H^1 \), possibly up to a choice of normalizing factor. In particular, by Lemma 2.1 if \( u_n \) denotes the first nontrivial eigenfunction of \( g_n \) with eigenvalue \( \lambda_n \), we have that

\[
\lim_{n \to \infty} |\lambda_n - \lambda_\varepsilon| = 0 \text{,} \quad \lim_{n \to \infty} \|u_n - u_\varepsilon\|_{H^1} = 0 \text{,}
\]
provided that \( u_n \) is normalized so that
\[
\int u_n^2 \, dV_n = 1,
\]
with \( dV_n \) the volume measure associated with the metric \( g_n \). Observe that the volume of \( M \) with the metric \( g_n \) is different from the volume with the metric \( g_\varepsilon \) (and hence \( g_0 \)), but this will not be relevant for our purposes. The eigenfunctions \( u_n \) are smooth as the metric \( g_n \) is smooth too.

First, let us establish the energy estimate
\[
\| X_1^{j_1} \cdots X_s^{j_s} u_n \|_{H^1} < C_\varepsilon,
\]
for \( 0 \leq j_1 + \cdots + j_s \leq k \), where of course the constant \( C_\varepsilon \) depends on the number of derivatives that we consider and on \( \varepsilon \), but not on \( n \). For this, it is convenient to cover any given point in \( U_1 \) by a patch \( V \) of local coordinates \((\rho, y)\), with \( y = (y_1, \ldots, y_{d-1}) \), so that the metric \( g_0 \) reads as
\[
g_0 = d\rho^2 + \gamma_{ij}(\rho, y) \, dy^i \, dy^j.
\]
In order to do this, let \( \phi_t \) denote the local flow of the vector field \( \nabla \rho \), which is well-defined in \( U_0 \). Any point \( x \) in \( U_0 \) can be written in a unique way as \( x = \phi_t p \) with \( t \in I \supset [-\eta, \eta] \) and \( p \in \Sigma \). It we take local coordinates \( \hat{y}_i \) in an open set \( \Sigma' \subset \Sigma \), it is standard that the desired local coordinates \((\rho, y_i)\) on \( M \) can then be defined by setting
\[
\rho(\phi_t x) := t, \quad y_i(\phi_t p) := \hat{y}_i(p).
\]
(Notice that \( \rho \) is indeed the signed distance to \( \Sigma \).) The estimate (4.3) will clearly follow if we prove that the estimate
\[
\| D^k y u_n \|_{H^1(V)} < C_\varepsilon
\]
holds for all \( k \) (the constant \( C_\varepsilon \) is not uniform on \( k \), of course).

To prove (4.5) for \( k = 0 \), we integrate the eigenvalue equation against a function \( \varphi \) to find
\[
\int g_n^{ij} \partial_i u_n \partial_j \varphi \, dV_n = \lambda_n \int u_n \varphi \, dV_n,
\]
which is valid for all \( \varphi \in H^1(M) \). In particular, since the corresponding eigenvalues \( \lambda_n \) tend to \( \lambda_\varepsilon \) as \( n \to \infty \), by Lemma 2.1 we can assume that \( \lambda_n \leq C_\varepsilon^{\frac{d}{2}-1} \), so by taking \( \varphi := u_n \) in (4.6) we obtain
\[
\int f_n^{\frac{d}{2}-1} |du_n|^2 \leq C_\varepsilon^{\frac{d}{2}-1}.
\]
In particular, if \( V \) is a patch covered by local coordinates \((\rho, y)\) as above, an easy computation shows that
\[
\int_V \left[ (\partial_\rho u_n)^2 + |D_y u_n|^2 \right] \, d\rho \, dy \leq C,
\]
with the obvious notation, and then Eq. (4.2) implies
\[
\| u_n \|_{H^1(V)} < C_\varepsilon.
\]
The estimate (4.5) for $k > 0$ follows by taking in Eq. (4.6)
\[ \varphi := [X_{1}^{j_{1}} \cdots X_{s}^{j_{s}}] \cdot u_{n}, \]
and considering all possible $j$’s with
\[ j_{1} + \cdots + j_{s} \leq k. \]
In fact, in the local coordinate patch $V$, an easy partition of unity argument shows that this is equivalent to taking in Eq. (4.6)
\[ \varphi := [(\chi D_{y})^{\beta}][\chi D_{y}]u_{n}, \]
where $\chi$ is a cutoff function supported in $V$ that is equal to 1 in a certain open subset of $V$, and $\beta$ is a multiindex ranging over the set $|\beta| \leq k$.

For simplicity, let us see the details for $k = 1$, which implies estimating two derivatives of $u_{n}$. Denoting by $O(1)$ a smooth, possibly matrix-valued, function bounded by constants that do not depend on $n$ and $\varepsilon$, we start off by writing
\[
\int g_{n}^{ij} \partial_{i}u_{n} \partial_{j}[(\chi \partial_{y_{i}})\chi \partial_{y_{j}}u_{n}] \, dV_{n} =
\]
\[= - \int f_{n}^{\frac{d}{2}+1} \partial_{y_{i}}[|g_{0}|^{1/2}g_{ij}^{0}\chi \partial_{y_{i}}u_{n}] (\chi \partial_{y_{i}}u_{n}) \, d\rho dy \]
\[- \int (f_{n})^{\frac{d}{2}+1} \partial_{y_{i}}[|g_{0}|^{1/2}g_{ij}^{0}\chi \partial_{y_{i}}u_{n}] \partial_{j}(\chi \partial_{y_{i}}u_{n}) \, d\rho dy \]
\[= \int (f_{n})^{\frac{d}{2}+1} O(1)(\partial_{y_{i}}u_{n})(\chi \partial_{y_{i}}u_{n}) + \int f_{n}^{\frac{d}{2}+1} O(1)(\partial_{y_{i}}u_{n}) \partial_{j}(\chi \partial_{y_{i}}u_{n}) \]
\[+ \int f_{n}^{\frac{d}{2}+1} O(1)(\partial_{y_{i}}u_{n})^{2} - \int f_{n}^{\frac{d}{2}+1} g_{ij}^{0} \partial_{i}(\chi \partial_{y_{i}}u_{n}) \partial_{j}(\chi \partial_{y_{i}}u_{n}). \]
In the first equality we have integrated by parts and used that, since the function $f_{n}$ only depends on $\rho$, we do not pick up any derivatives of $f_{n}$ with respect to $y_{i}$. In the second equality we integrate with the volume element $dV_{0}$ (so it is omitted under the integral sign) and make explicit all the terms that are of the form $O(1)$.

Furthermore, with the above choice of $\varphi$, the RHS of Eq. (4.6) reads:
\[\lambda_{n} \int u_{n}(\chi \partial_{y_{i}})\chi \partial_{y_{i}}u_{n} \, dV_{n} = \lambda_{n} \int f_{n}^{\frac{d}{2}} O(1) u_{n}(\chi \partial_{y_{i}}u_{n}) - \lambda_{n} \int f_{n}^{\frac{d}{2}} (\chi \partial_{y_{i}}u_{n})^{2}. \]
Therefore, using that
\[\|\chi \partial_{y_{i}}u_{n}\|_{H^{1}}^{2} \leq \|u_{n}\|_{H^{1}}^{2} + C_{\varepsilon} \int f_{n}^{\frac{d}{2}} g_{ij}^{0} \partial_{i}(\chi \partial_{y_{i}}u_{n}) \partial_{j}(\chi \partial_{y_{i}}u_{n}) \]
and the Schwartz inequality, we readily arrive at:
\[\|\chi \partial_{y_{i}}u_{n}\|_{H^{1}}^{2} \leq C_{\varepsilon}(\|u_{n}\|_{H^{1}}^{2} + \|u_{n}\|_{H^{1}}^{2} \|\chi \partial_{y_{i}}u_{n}\|_{H^{1}}). \]
Then, for $k \leq 1$ Eq. (4.4) yields
\[\|D_{y}^{k} u_{n}\|_{H^{1}(V)} < C_{\varepsilon} \]
after taking the derivatives with respect to $y_{l}$ for all $1 \leq l \leq d - 1$ in the coordinate patch $V$. Covering the set $U_{0}$ with coordinates patches of the above form, we derive the upper bound
\[\|X_{j} u_{n}\|_{H^{1}} < C_{\varepsilon}, \]
for all $1 \leq j \leq s$. Notice that the reason for which we obtain $\varepsilon$-dependent constants is that $f_{n}$ is of order $\varepsilon$ in $\Omega_{n}$, and it appears as a factor both in Eq. (4.2) and the
integral of the RHS of the estimate (4.9). The case of general \( k \) follows from a completely analogous reasoning using the function \( \varphi \) specified in (4.8), so we will omit the details. The desired estimate (4.3) then follows.

Having already established (4.3), the estimate

\[
\|u_n\|_{H^1 H^k_T} < C_\varepsilon
\]

is now almost immediate. Indeed, in any coordinate patch \( W \subset M \) whose closure does not intersect \( \partial \Omega_\eta \) and which is covered by local coordinates \( x = (x^1, \ldots, x^d) \), the components of the metric \( g_n \) are bounded as \( |D_x^k g_{ij}^n| \leq C_\varepsilon \). Hence it is standard (and follows readily from the above computation, with \( \varphi := \left[ (\chi D_x)\beta \right] g_{ij}^n u_n \) for some smooth cut-off function \( \chi \) supported in \( W \)) that

\[
\|D_x^k u_n\|_{L^2(W)} \leq C_\varepsilon
\]

for any \( k \). This establishes (4.10) after covering the compact manifold \( M \) by a finite number of coordinate patches.

To prove the uniform estimate

\[
\|u_n\|_{H^1 H^k_T(\Omega_\eta^\pm)} < C,
\]

we can use essentially the same argument. For \( k = 0 \), we just observe that

\[
\|u_n\|_{H^1(\Omega_\eta^\pm)} < C
\]

where we have used Eqs. (4.2) and (4.6), the bound for the eigenvalue \( \lambda_n \) and the fact that on \( \Omega_\eta^c \) we have

\[
g_{ij}^n \partial_i \psi \partial_j \psi \geq C \left( (\partial_\rho \psi)^2 + |D_y \psi|^2 \right)
\]

with a constant \( C > 0 \) that does not depend on \( \varepsilon \).

Similarly, for \( k = 1 \), we use a coordinate patch \( V \) covered by coordinates \( (\rho, y^1, \ldots, y^{d-1}) \) as above and an adapted cut-off function \( \chi \). Since we have a uniform bound \( |D_y^k g_{ij}^n| \leq C \) in \( V \) by the definition of the coordinates, we can write

\[
\left\|\chi \partial_{y^i} u_n \right\|_{H^1(V \cap \Omega_\eta^c)}^2 \leq C \left\|u_n\right\|_{H^1(\Omega_\eta^c)}^2 + C \int_{\mathbb{R}^{d-1}} f\left[ y^{d-1} g_{ij}^n \partial_i \chi \partial_{y^i} u_n \partial_j \chi \partial_{y^j} u_n \right].
\]

The preceding discussion using Eq. (4.6) with \( \varphi = (\chi \partial_{y^i})(\chi \partial_{y^i}) u_n \) implies that the second term of the RHS in (4.13) is upper bounded by an \( \varepsilon \)-independent constant, thus establishing the uniform estimate for \( k = 1 \) after taking a finite covering of \( M \) with \( V \) patches and summing all the contributions. For general \( k \), the reasoning is completely analogous.

Since the upper bounds (4.10) and (4.11) for \( u_n \) are uniform in \( n \) and \( u_n \rightarrow u_\varepsilon \) in \( H^1 \), the lemma follows.

5. Proof of Proposition 2.5

A first observation is that, for any integer \( k \), \( u_\varepsilon \) tends to the constants \( c_\pm \) in the anisotropic Sobolev norm \( H^1 H^k_T(\Omega_\eta^\pm) \), up to the choice of a global sign, i.e.,

\[
\|u_\varepsilon - c_\pm\|_{H^1 H^k_T(\Omega_\eta^\pm)} = o(1),
\]

where

\[
\Omega_\eta^\pm \text{ denote the connected components of } \Omega_\eta \text{ that lie in } \Omega^\pm.
\]
where $o(1)$ stands for a quantity that tends to zero as $\varepsilon \searrow 0$. In particular, this asymptotics implies that

\[
\lim_{\varepsilon \searrow 0} \|B_{\varepsilon}^\pm - c_\pm\|_{H^k(\partial\Omega_{\varepsilon}^\pm)} = 0
\]

where we recall that $B_{\varepsilon}^\pm := u_\varepsilon|_{\partial\Omega_{\varepsilon}^\pm}$.

In order to prove Eq. (5.1), notice that

\[
\eta_{\varepsilon} = \pm \varepsilon \lambda_\varepsilon u_\varepsilon = \pm \varepsilon \lambda_\varepsilon (\Delta \eta_{\varepsilon} - 1)
\]

Indeed, using the $L^\infty$ upper bound established in Corollary 2.3, we can take the limit $\varepsilon \searrow 0$ in Eqs. (5.3) and (5.4) to obtain that, in this limit,

\[
\begin{aligned}
(c'_+)^2|\Omega_\eta^+| + (c'_-)^2|\Omega_\eta^-| &= \kappa^{-\frac{\alpha}{2}}|_{\varepsilon=0}, \\
(c'_+|\Omega_\eta^+| + c'_-|\Omega_\eta^-| &= 0,
\end{aligned}
\]

so $c'_\pm = c_\pm$.

In the domain $\Omega_\eta$ the eigenfunction $u_\varepsilon$ satisfies the boundary problem

\[
\Delta u_\varepsilon = -\varepsilon \lambda_\varepsilon u_\varepsilon \quad \text{in} \ \Omega_\eta, \quad u_\varepsilon|_{\partial\Omega_{\varepsilon}^\pm} = B_{\varepsilon}^\pm,
\]

and therefore the difference $u_\varepsilon - h$ satisfies

\[
\Delta(u_\varepsilon - h) = -\varepsilon \lambda_\varepsilon u_\varepsilon \quad \text{in} \ \Omega_\eta, \quad (u_\varepsilon - h)|_{\partial\Omega_{\varepsilon}^\pm} = B_{\varepsilon}^\pm - c_\pm .
\]

We recall that the Laplacian $\Delta$ is computed using the reference metric $g_0$. The $L^\infty$ norm of $u_\varepsilon - h$ can be estimated using the maximum principle to yield

\[
\|u_\varepsilon - h\|_{L^\infty(\Omega_\eta)} \leq \|B_{\varepsilon}^+ - c_+\|_{L^\infty(\partial\Omega_{\varepsilon}^+)} + \|B_{\varepsilon}^- - c_-\|_{L^\infty(\partial\Omega_{\varepsilon}^-)} + C\varepsilon \lambda_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega_\eta)}
\]

\[
= o(1),
\]

where we have used the asymptotics (5.2) and Corollary 2.3. Therefore, standard elliptic estimates imply

\[
\|u_\varepsilon - h\|_{C^{k,\alpha}(\Omega_\eta)} < C\|u_\varepsilon - h\|_{L^\infty(\Omega_\eta)} + C\varepsilon \lambda_\varepsilon \|u_\varepsilon\|_{C^{k,\alpha}(\Omega_\eta)}
\]

\[
< o(1) + C\varepsilon \lambda_\varepsilon \|h\|_{C^{k,\alpha}(\Omega_\eta)} + C\varepsilon \lambda_\varepsilon \|u_\varepsilon - h\|_{C^{k,\alpha}(\Omega_\eta)}
\]

\[
< o(1) + C\varepsilon \lambda_\varepsilon \|u_\varepsilon - h\|_{C^{k,\alpha}(\Omega_\eta)} ,
\]
thus showing that 
\[ \|u_\varepsilon - h\|_{C^{k+2,\alpha}(\Omega_\eta)} = o(1). \]
This completes the proof of the proposition.

6. Proof of Proposition 2.6

Let us describe the domain \( \Omega_\eta \) in terms of the local coordinates \( (\sigma, y) \), where
\[ \sigma : = \frac{(\rho + \eta) \pi}{2\eta} \]
has been rescaled so that it ranges over the interval \((0, \pi)\). In these coordinates, the Laplacian can be written as
\[ \Delta =: \frac{\pi^2}{4\eta^2} \partial_\sigma^2 + \Delta_\Sigma - L, \]
where \( \Delta_\Sigma \) is the Laplacian on the hypersurface \( \Sigma \) (computed with respect to the metric \( \gamma_{ij}(0, y) dy^i dy^j \), in the notation of Eq. (4.4)) and \( L \) is a differential operator of the form
\[ L = G_1 \frac{\partial_\sigma}{\eta} + \eta G_2 D_y^2 + \eta G_3 D_y, \]
where the functions \( G_j = G_j(\sigma, y, \eta) \) are (possibly matrix-valued) functions that depend smoothly on all their variables.

The difference \( w := h - \bar{h} \) then satisfies the equation
\[ (6.1) \quad \frac{\pi^2}{4\eta^2} \partial_\sigma^2 w + \Delta_\Sigma w = Lw + \frac{F}{\eta}, \quad w|_{\sigma=0} = w|_{\sigma=\pi} = 0, \]
with \( F \) a smooth function of \((\sigma, y, \eta)\). In view of the Dirichlet boundary conditions, let us expand \( w \) in a Fourier series in the variable \( \sigma \) of the form
\[ w = \sum_{n=1}^{\infty} w_n(y) \sin n\sigma. \]
It is standard that the estimate presented in the statement of Proposition 2.6 will follow once we prove that, for any integer \( k \),
\[ \|w\|_{H^{2k}} \leq C\eta \]
for a constant that depends on \( k \), where the Sobolev norm is computed by taking derivatives with respect to the variables \( \sigma, y \) and integrating over the set \((0, \pi) \times \Sigma\).

In terms of the Fourier coefficients of \( w \), this is equivalent to showing that
\[ (6.2) \quad \sum_{n=1}^{\infty} \left( n^{2k} \|w_n\|^2 + \|\Delta_\Sigma^k w_n\|^2 \right) < C\eta^2, \]
where we are denoting by \( \| \cdot \| \) the norm in \( L^2(\Sigma) \). Let us denote by \( F_n \) and \( R_n \) the \( n^{th} \) Fourier coefficient of the functions \( F \) and
\[ (6.3) \quad R := Lw, \]
respectively. Writing Eq. (6.1) as
\[ \frac{\pi^2 n^2}{4\eta^2} w_n - \Delta_\Sigma w_n = -\frac{F_n}{\eta} - R_n, \]
we can invert the positive self-adjoint operator \( \frac{n^2}{4\eta^2} - \Delta_{\Sigma} \) in the closed manifold \( \Sigma \) to obtain
\[
w_n = - \left( \frac{\pi^2 n^2}{4\eta^2} - \Delta_{\Sigma} \right)^{-1} \left( \frac{F_n}{\eta} + R_n \right).
\]

Let us fix any integer \( k \geq 1 \). As the \( L^2(\Sigma) \rightarrow L^2(\Sigma) \) norm of the operator \( \left( \frac{n^2}{4\eta^2} - \Delta_{\Sigma} \right)^{-1} \) is at most \( 4\eta^2/n^2 \), we then have
\[
\sum_{n=1}^{\infty} n^{4k} \| w_n \|^2 \leq 2 \sum_{n=1}^{\infty} \left( \eta^2 n^{4k-4} \| F_n \|^2 + \eta^4 n^{4k-4} \| R_n \|^2 \right).
\]
Notice that
\[
\sum_{n=1}^{\infty} n^{4k-4} \| F_n \|^2 \leq C \| \partial_{\sigma}^{2k-2} F \|^2_{L^2},
\]
where \( \| \cdot \|_{L^2} \) refers to the norm computed with respect to the variables \( (\sigma, y) \in (0, \pi) \times \Sigma \) and, by the definition of \( R \) (Eq. (6.3)),
\[
\sum_{n=1}^{\infty} n^{4k-4} \| R_n \|^2 \leq C \| \partial_{\sigma}^{2k-2} R \|^2_{L^2} \leq C \eta^{-2} \| \partial_{\sigma}^{2k-1} w \|^2_{L^2} + C \eta^2 \| \partial_{\sigma}^{2k-2} \Delta_{\Sigma} w \|^2_{L^2}.
\]

Hence
\[
\sum_{n=1}^{\infty} n^{4k} \| w_n \|^2 \leq C \eta^2 \| \partial_{\sigma}^{2k-2} F \|^2_{L^2} + C \eta^2 \| w \|^2_{H^{2k}}.
\]

Likewise,
\[
\sum_{n=1}^{\infty} \| \Delta_{\Sigma}^k w_n \|^2 \leq 2 \sum_{n=1}^{\infty} \left( \frac{\eta^2}{n^2} \| \Delta_{\Sigma}^k F_n \|^2 + \left\| \Delta_{\Sigma}^k \left( \frac{\pi^2 n^2}{4\eta^2} - \Delta_{\Sigma} \right)^{-1} R_n \right\|^2 + 2\eta^2 \sum_{n=1}^{\infty} \left\| \Delta_{\Sigma}^k \left( \frac{\pi^2 n^2}{4\eta^2} - \Delta_{\Sigma} \right)^{-1} \frac{G_1}{\eta} \partial_{\sigma} w \right\|^2 + 2\eta \sum_{n=1}^{\infty} \left\| \Delta_{\Sigma}^k \left( \frac{\pi^2 n^2}{4\eta^2} - \Delta_{\Sigma} \right)^{-1} \left( G_2 D_y^2 w + G_3 D_y w \right) \right\|^2 + 2\eta^2 \sum_{n=1}^{\infty} \left\| \Delta_{\Sigma}^{k-1} \left( G_2 D_y^2 w + G_3 D_y w \right) \right\|^2 \leq C \eta^2 \| w \|^2_{H^{2k}}.
\]

Combining this equation with (6.4) and (6.5) we infer that
\[
\| w \|_{H^{2k}} \leq C \sum_{n=1}^{\infty} \left( n^{4k} \| w_n \|^2 + \| \Delta_{\Sigma}^k w_n \|^2 \right) \leq C \eta^2 \| F \|^2_{H^{2k}} + C \eta^2 \| w \|^2_{H^{2k}},
\]
which implies the estimate (6.2) provided that \( \eta \) is small enough (e.g., if \( C \eta^2 < \frac{1}{2} \)).

The proposition then follows.
7. Proof of Corollary 2.7

Let us work with the rescaled local coordinates \((\sigma, y)\) introduced in Section 6. In these coordinates, the function \(\bar{h}\) reads as

\[
\bar{h} = c_- + \frac{c_+ - c_-}{\pi}\sigma,
\]

so the zero set \(\bar{h}^{-1}(0)\) is \(\{\pi c_-/(c_- - c_+)\} \times \Sigma\). Since the derivative \(\partial_\sigma \bar{h}\) does not vanish and the functions \(h(\sigma, y)\) and \(\bar{h}(\sigma, y)\) are arbitrarily close in \(C^k((0, \pi) \times \Sigma)\) by Proposition 2.6, Thom’s isotopy theorem [1, Section 20.2] shows that \(h^{-1}(0)\) is given by

\[
\Psi\left(\left\{\frac{\pi c_-}{c_- - c_+}\right\} \times \Sigma\right),
\]

where \(\Psi\) is a diffeomorphism that can be taken to be arbitrarily close to the identity in any \(C^k\) norm, computed with respect to the variables \((\sigma, y)\). Therefore, in the unrescaled variables \((\rho, y)\), the diffeomorphism is \(C^k\)-close to the identity. (Observe that the argument does not imply that the diffeomorphism is \(C^k\)-close to the identity because the derivatives with respect to \(\rho\) introduce a large factor \(\eta^{-1}\) in the estimates. In fact, as was to be expected, what one would obtain is again some kind of anisotropic bounds for the derivatives of \(\Phi - \text{id}\).)

8. Proof of Theorem 1.2

Let us fix some ball \(B \subset M\) and take a domain \(D\) whose closure is contained in \(B\). This ensures that \(\Sigma := \partial D\) separates. Theorem 1.1 shows that there is a smooth metric \(g\) conformal to \(g_0\) and of the same volume, such that the nodal set of its first nontrivial eigenfunction \(u\) is diffeomorphic to \(\Sigma\) and the corresponding eigenvalue \(\lambda\) is simple. Furthermore, in Step 5 of Section 2 we showed that the gradient of \(u\) does not vanish on its nodal set.

A theorem of Uhlenbeck [16] ensures that one can take a metric \(\tilde{g}\) that is a \(C^{m+1}\)-small conformal perturbation of the metric \(g\) so that the first eigenfunction is Morse, that is, all their critical points are non-degenerate. It is obvious that one can take \(\tilde{g}\) and \(g\) with the same volume just multiplying by a constant factor, which does not change the eigenfunctions. Standard results from perturbation theory show that the first nontrivial eigenfunction \(\tilde{u}\) of the perturbed metric is close in the \(C^m(M)\) norm to \(u\), so in particular the nodal set \(\Sigma\) of \(\tilde{u}\) is contained in \(B\) and is diffeomorphic to \(\Sigma\). Here we are using the fact that the gradient of \(u\) does not vanish on its nodal set and Thom’s isotopy theorem.

Call \(\tilde{D}\) the domain contained in \(B\) that is bounded by \(\tilde{\Sigma}\) and let us denote by \(\nabla\tilde{\Sigma}\) the covariant derivative associated with the metric \(\tilde{g}\). Since \(\nabla\tilde{u}\) is a nonzero normal vector on \(\tilde{\Sigma}\), which can be assumed to point outwards without loss of generality, we can resort to Morse theory for manifolds with boundary to show that the number of critical points of \(\tilde{u}\) of Morse index \(i\) is at least as large as the \(i^{\text{th}}\) Betti number of the closure of the domain \(\tilde{D}\), for \(0 \leq i \leq d - 1\). Since \(\tilde{D}\) is diffeomorphic to \(D\), the proposition then follows by choosing the domain \(D\) so that the sum of its Betti numbers is at least \(N\) (this can be done, e.g., by taking \(\Sigma\) diffeomorphic to a connected sum of \(N\) copies of any nontrivial product of spheres, such as \(S^1 \times S^{d-2}\), since in this case the first Betti number is \(N\).)
Acknowledgments

The authors are supported by the ERC Starting Grants 633152 (A.E.) and 335079 (D.P.-S.). This work is supported in part by the ICMAT–Severo Ochoa grant SEV-2011-0087 (A.E. and D.P.-S.). S.S. was partially supported by CRC1060 of the DFG. This work was started when S.S. was visiting ICMAT and he is grateful for the enjoyable visit.

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