Structures of electromagnetic type on vector bundles

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Abstract

Structures of electromagnetic type on a vector bundle are introduced and studied. The metric case is also defined and studied. The sets of compatible connections are determined and a canonical connection is defined.

1 Introduction

Structures of electromagnetic type (em-structures) and structures of metric electromagnetic type (mem-structures) on a manifold were progressively introduced in [9, 11, 7] (see also [6]) and studied in detail in [5, 7, 8, 13, 14]. In the present paper we define similar structures for the case of a vector bundle $\xi = (E, \pi, M)$, and relate them to product, complex, para-Hermitian, Hermitian, para-Kähler or indefinite Kähler, structures. (In the sequel, by a pseudo-Riemannian metric we shall understand a metric of any signature, and by an indefinite (metric) structure a structure including a pseudo-Riemannian metric.) Then, we determine the set of connections on $\xi$ compatible with those structures and we introduce a canonical connection. Considering an almost para-Hermitian (resp. indefinite Hermitian) structure on the base manifold $M$ and an indefinite Hermitian (resp. para-Hermitian) structure of the bundle $\xi$, we prove that the corresponding diagonal lift of these structures, with respect to a connection on $\xi$, are mem-structures on the total space $E$. Finally, some properties of those mem-structures are established.

We recall the physical origin of the topic ([9, 11]). Let $M^4$ be a spacetime of general relativity, with gravitational tensor $g$ of signature $-+++$. Let $F$ be the electromagnetic field of type $(0, 2)$, which is skewsymmetric, that is a 2-form. Setting $F(X, Y) = g(JX, Y)$, the tensor field $J$ so defined is the electromagnetic tensor field of type $(1, 1)$ associated to $F$. We have $g(JX, Y) + g(X, JY) = 0$. The characteristic equation of $J$ is $\det(J - \lambda I) = 0$, which is satisfied by $J$, and we have

$$J^4 + 2kJ^2 + ll = 0, \quad k = -\frac{1}{4} \text{trace } J^2, \quad l = \det J.$$ 

If $x \in M^4$, it is said that $J_x$ is of $1^{st}$, $2^{nd}$, or $3^{rd}$ class at $x$ if, respectively,

$$l_x \neq 0, \quad l_x = 0, \quad k_x \neq 0, \quad l_x = 0, \quad k_x = 0.$$
It is said that $J$ is of 1st, 2nd, or 3rd class if it is of such class at every $x$. The characteristic polynomial of the second class is $J^2(J^2 + 2k)$, but the minimal polynomial is $J(J^2 + 2k)$, so that the condition $J(J^2 + 2k) = 0$ characterizes the second class. The field of an electromagnetic plane wave is of 3rd class. The field of a moving electron is of 2nd class. More complicated fields belong to the 1st class. The equation one gets from the minimal polynomial in the 1st class is

\[(J^2 - f^2)(J^2 + h^2) = 0.\]

with $f, h$ nowhere-vanishing $C^\infty$ functions on $M^4$. Such a tensor field $J$ on a general manifold $M$ determines a $G$-structure on $M$.

To handle the nonconstant local cross-section situation corresponding to (1.1), one can use the relationships among $G$-structures, related sections of an associated bundle and functions of certain kind on $M$, as follows: Let $(\mathcal{P}, \pi_E, M, H)$ be a principal bundle with group $H$, $H \times W \to W$ a left action of $H$ on a manifold $W$, and $(E = \mathcal{P} \times_H W, \pi_E, M, W)$ the associated bundle. A $J$-subset $S$ of $W$ with corresponding group $G$, a subgroup of $H$, is defined by the conditions: (1) $S \subset$ fixpoint set of $G$, (2) $h \in H, h(S) \cap S \neq \emptyset \Rightarrow h \in G$.

For instance, points are $J$-subsets with $G$ the corresponding isotropy group. A cross-section $K$ of $\pi_E$ is a $J$-section if it can be locally represented as the “product” of a cross-section $\sigma$ of $\pi_P$ and a $S$-valued function $\tilde{K}$, so that

$$K_x = \sigma_x \cdot \tilde{K}_x = \text{equivalence class of } (\sigma_x, \tilde{K}_x) \text{ in } E.$$

Then $\tilde{K}$ is globally defined, and the $\sigma$ generate a principal subbundle of $\mathcal{P}$. $K$ is a constant $J$-section if and only if $\tilde{K}$ is constant. Different sections can generate the same subbundle, and in fact, every principal subbundle can be generated by a constant $J$-section.

Now, let $\mathcal{P}$ be the principal bundle of frames over $M$, so that $H = GL(n, \mathbb{R})$, and let $W$ be a real vector space. If $J \in W$ is given with the conditions stated above, a $J$-section generates a $J$-structure with group $G$, which is a $G$-structure. The tensor $K$ has in principle variable components in adapted frames. This is a slight generalization with respect to the usually considered $G$-structures, given by tensors with constant components, which here correspond to constant $J$-sections. Since every $J$-structure is generated by some constant $J$-section, this generalization is useless for the study of the $J$-structure itself; but if the emphasis shifts to the study of variable $J$-sections, the results are significant, specially with respect to the parallelizability of the tensors.

In the particular case of a $(1, 1)$ tensor field $J$ satisfying $(J^2 - f^2)(J^2 + h^2) = 0$, with characteristic polynomial $(x - p)^{r_1}(x - p)^{r_2}(x^2 + q^2)^s$, $r_1, r_2, s \geq 1$, $r_1 + r_2 + 2s = n = \dim M$, the $J$-subset consists of matrices of the form

$$\begin{pmatrix}
pI_{r_1} & -pI_{r_2} \\
-pI_{r_2} & qI_s
\end{pmatrix}$$

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and the structural group is \( G = GL(r_1, \mathbb{R}) \times GL(r_2, \mathbb{R}) \times GL(s, \mathbb{C}) \). It is proved ([7]) that the \( G \)-structure defined by \( J \) above is also defined by a tensor field, say again \( J \), satisfying \((J^2 - 1)(J^2 + 1) = 0\), that is, the relation \( J^4 = 1 \) considered in the present paper.

Notice that the \( G \)-structure is exactly the same, not an associated or equivalent one. In the 4-dimensional case the group reduces to \( G = GL(1, \mathbb{R}) \times GL(1, \mathbb{R}) \times GL(1, \mathbb{C}) \). It is also proved ([7]) that there exists an adapted Riemannian metric so that the group can be reduced to \( G = O(r_1) \times O(r_2) \times U(s) \), and in the 4-dimensional case to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(1) \), that is, essentially to the unitary group \( U(1) \).

2 Structures of electromagnetic type on a vector bundle

Let \( \xi = (E, \pi, M) \) be a \( C^\infty \) vector bundle with total space \( E \) and projection map \( \pi \) over a connected paracompact base manifold \( M \). The rank of \( E \) is the (common) dimension of the fibres. Let \( C^\infty(M) \) denote the ring of real functions, \( T^q_p(M) \) the \( C^\infty(M) \)-module of \((p, q)\)-tensor fields, and \( T(M) \) the \( C^\infty(M) \)-tensor algebra of \( M \). We respectively denote by \( T^q_p(\xi) \) and \( T(\xi) \) the \( C^\infty(M) \)-module of tensor fields of type \((p, q)\) and the \( C^\infty(M) \)-tensor algebra of the bundle \( \xi \).

We recall that an almost product (resp. almost complex) structure on a manifold \( M \) is defined by a tensor field \( J \) of type \((1, 1)\) satisfying \( J^2 = I \) (resp. \( J^2 = -I \)). An almost para-Hermitian (resp. indefinite almost Hermitian) structure on \( M \) is defined by a couple \((J, g)\), given by an almost product (resp. almost complex) structure \( J \) and a pseudo-Riemannian metric compatible with \( J \) in the sense that \( g(JX, Y) + g(X, JY) = 0, X, Y \in \mathfrak{X}(M) \); that is, as an anti-isometry (resp. isometry). A para-Kähler (resp. indefinite Kähler) manifold is a manifold \( M \) endowed with an almost para-Hermitian (resp. indefinite almost Hermitian) structure such that the Levi-Civita connection of \( g \) parallelizes \( J \).

Definition 2.1. A structure of electromagnetic type on \( \xi = (E, \pi, M) \) is an \( M \)-endomorphism \( J \) of \( \xi \) satisfying

\[
J^4 = I,
\]

with characteristic polynomial \((x - 1)^{r_1}(x + 1)^{r_2}(x^2 + 1)^s\), where \( r_1, r_2, s \) are constants greater than or equal to 1 such that \( r_1 + r_2 + 2s = \text{rank} \ E \).

Setting \( P = J^2 \), we have \( P^2 = I \), so \( P \) is a product structure on \( \xi \), admitting \( J \) as a “square root”. Conversely, if \( P \) is a product structure admitting a “square root” \( J \), then \( J \) is an \( \xi \)-structure on \( \xi \). Denoting by \( \xi_1 \) and \( \xi_2 \) respectively the \(+1\) and \(-1\) eigen-subbundles of \( P \), it is easy to see that \( \xi_1 \) and \( \xi_2 \) are invariant by \( J \) and that \( J_1 = J|_{\xi_1} \) defines a product structure of \( \xi_1 \) and \( J_2 = J|_{\xi_2} \) a complex structure of \( \xi_2 \). So, one has

\[
(2.1) \quad \xi = \xi_1 \oplus \xi_2, \quad J = J_1 \oplus J_2.
\]
Conversely, if $\xi_1$ and $\xi_2$ are two supplementary subbundles of $\xi$, $J_1$ is a product structure of $\xi_1$, and $J_2$ a complex structure of $\xi_2$, then $J = J_1 \oplus J_2$ is an em-structure on $\xi$. Denoting by $P_1$ and $P_2$ the projections of $\xi$ on $\xi_1$ and $\xi_2$ respectively, we obtain

$$P = P_1 - P_2, \quad J = J_1 \circ P_1 + J_2 \circ P_2.$$ 

Summing up we have

Proposition 2.1. An em-structure on the vector bundle $\xi = (E, \pi, M)$ can be defined by each one of the following conditions:

1. An $M$-endomorphism $J$ of $\xi$ satisfying $J^4 = I$,
2. A product structure $P$ of $\xi$ admitting a “square root” $J$,
3. Two supplementary subbundles $\xi_1$ and $\xi_2$ of $\xi$ respectively endowed with a product structure and a complex structure.

Remark 2.1. A product structure $P$ which admits a “square root” is a particular one because rank $\xi_2$ must be even.

Definition 2.2. A structure of metric electromagnetic type (mem-structure) on the vector bundle $\xi$ is a pair $(J, g)$, where $J$ is an em-structure and $g$ a pseudo-Riemannian metric on $\xi$ satisfying the compatibility condition

$$g(JX,Y) + g(X,JY) = 0, \quad X, Y \in \xi.$$ 

Denoting by $\delta_J$ the derivation defined by $J$ in the tensor algebra $T(\xi)$, the relation (2.2) can be written as

$$\delta_J g = 0,$$

from which it follows $g(PX, PY) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. Therefore, the pair $(P, g)$ is a pseudo-Riemannian product structure of $\xi$ and so the subbundles $\xi_1$ and $\xi_2$ are mutually orthogonal with respect to $g$. Denoting respectively by $g_1$ and $g_2$ the restrictions of $g$ to $\xi_1$ and $\xi_2$, from (2.2) we obtain

$$\delta_{J_1} g_1 = 0, \quad \delta_{J_2} g_2 = 0,$$

which may be written

$$g_1(J_1 X, J_1 X) = -g_1(X, Y), \quad g_2(J_2 X, J_2 Y) = g_2(X, Y), \quad X, Y \in \mathfrak{X}(\xi).$$

Hence $(J_1, g_1)$ is a para-Hermitian structure of $\xi_1$ and $(J_2, g_2)$ is an indefinite Hermitian structure of $\xi_2$. Conversely, if $\xi_1$ and $\xi_2$ are two supplementary subbundles of $\xi$ such that $\xi_1$ is endowed with a para-Hermitian structure $(J_1, g_1)$ and $\xi_2$ with an indefinite Hermitian structure $(J_2, g_2)$, then considering $J$ as given by (2.1) and setting

$$g = g_1 \oplus g_2,$$

one obtains a mem-structure on $\xi$. So we have
Proposition 2.2. A mem-structure $(J, g)$ on $\xi$ is equivalent to a pair of supplementary subbundles $\xi_1$ and $\xi_2$ respectively endowed with a para-Hermitian structure $(J_1, g_1)$ and an indefinite Hermitian structure $(J_2, g_2)$.

Remark 2.2. If $(J, g)$ is a mem-structure on $\xi$, then we have: rank $\xi_1$ and rank $\xi_2$ are even; trace $J_1 = \text{trace} J_2 = 0$; sign $g_1 = 0$.

Setting for a mem-structure $(J, g)$ on $\xi$:

$$\Omega(X,Y) = g(JX,Y), \quad \Omega_i(X,Y) = g_i(J_iX,Y), \quad i = 1, 2,$$

it follows that $\Omega$, $\Omega_1$, and $\Omega_2$ are 2-forms which determine almost symplectic structures of $\xi$, $\xi_1$ and $\xi_2$, so that

$$\Omega = \Omega_1 \oplus \Omega_2.$$

These 2-forms satisfy

$$(2.5) \quad \delta J \Omega = 0, \quad \delta J_1 \Omega_1 = 0, \quad \delta J_2 \Omega_2 = 0.$$

Remark 2.3. The meaning of conditions (2.2), (2.3) and (2.5) is the following: The groups of automorphisms of $\mathfrak{X}(\xi_1)$, $\mathfrak{X}(\xi_2)$, and $\mathfrak{X}(\xi)$ given by

$$\alpha_t = I_1 \cosh t + J_1 \sinh t, \quad \beta_t = I_2 \cos t + J_2 \sin t, \quad \gamma_t = \alpha_t \oplus \beta_t,$$

t $\in \mathbb{R}$, determine actions on the tensor algebras $\mathcal{T}(\xi_1)$, $\mathcal{T}(\xi_2)$, and $\mathcal{T}(\xi)$, which respectively preserve the structures $(J_1, g_1, \Omega_1)$, $(J_2, g_2, \Omega_2)$, and $(J, g, \Omega)$.

3 Compatible connections

3.1 The general case

Definition 3.1. A connection $D$ on the vector bundle $\xi$ is said to be compatible with an em-structure $J$ if

$$(3.1) \quad DJ = 0.$$ 

From this it follows $DP = 0$, hence $D$ preserves the subbundles $\xi_1$ and $\xi_2$, i.e., for $X \in \mathfrak{X}(M)$, $Y_1 \in \mathfrak{X}(\xi_1)$, $Y_2 \in \mathfrak{X}(\xi_2)$, one has $D_X Y_1 \in \mathfrak{X}(\xi_1)$, $D_X Y_2 \in \mathfrak{X}(\xi_2)$. Setting then

$$D^1_X Y_1 = D_X Y_1, \quad D^2_X Y_2 = D_X Y_2, \quad X \in \mathfrak{X}(M), \quad Y_1 \in \mathfrak{X}(\xi_1), \quad Y_2 \in \mathfrak{X}(\xi_2),$$

we have that $D^1$ and $D^2$ are respectively connections on $\xi_1$ and $\xi_2$, so that

$$(3.2) \quad D_X = D^1_X \circ P_1 + D^2_X \circ P_2, \quad D^1_X J_1 = 0, \quad D^2_X J_2 = 0, \quad X \in \mathfrak{X}(M).$$

Conversely, if $D^1$ and $D^2$ are respectively connections on $\xi_1$ and $\xi_2$, then $D$ given as in (3.2) is a connection on $\xi$ satisfying $DP = 0$. If $D_1$ and $D_2$ satisfy the respective conditions in (3.2), then $D$ satisfies (3.1) too. Thus, it follows
Proposition 3.1. A connection $D$ on $\xi$ is compatible with the em-structure $J$ if and only if there exist two connections $D^1$ on $\xi_1$ and $D^2$ on $\xi_2$, respectively compatible with the product structure $J_1$ and the complex structure $J_2$, so that

(3.3) \[ D = D^1 \circ P_1 + D^2 \circ P_2. \]

Consider now on the subbundles $\xi_i$ of $\xi$, the operators $\Phi_i$ and $\Psi_i$ given by

(3.4) \[ (\Phi_i, D^i)_X = \frac{1}{2}(D^i_X + J_i^{-1} \circ D_X^i \circ J_i), \quad (\Psi_i, A^i)_X = \frac{1}{2}(A^i_X + J_i^{-1} \circ A_X^i \circ J_i), \]

where $X \in \mathfrak{X}(M)$, $D^i$ is a connection on $\xi_i$, and $A^i \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$ (now and in the sequel we take $i = 1, 2$). From [1, 13] and Proposition 3.1 we obtain

Proposition 3.2. The set of connections on $\xi$ compatible with the em-structure $J$ is given by

\[ D_X = \{(\Phi_i, D^i)_{X} + (\Psi_i, A^i)_{X}\} \circ P_1 + \{(\Phi_j, D^j)_{X} + (\Psi_j, A^j)_{X}\} \circ P_2, \]

where $X \in \mathfrak{X}(M)$ and $D^i$ is an arbitrary fixed connection on $\xi_i$, $A^i$ denotes any element of $\Lambda^1(M) \otimes \mathfrak{X}(\xi_i) \otimes \Lambda^1(\xi_i)$, and $\Phi_i$, $\Psi_i$, $A_i$ are given by (3.4).

Definition 3.2. A connection $D$ on $\xi$ is said to be compatible with the mem-structure $(J, g)$ if

\[ DJ = 0, \quad Dg = 0, \]

From which it follows: $DP = 0$; $D = D^1 \circ P_1 + D^2 \circ P_2$, where $D^i$ are the restrictions of $D$ to $\xi_1$ and $\xi_2$; $D^i J_1 = 0$; and $D^i g_1 = 0$. Conversely, if $D^1$ and $D^2$ are connections on $\xi_1$ and $\xi_2$, compatible with the para-Hermitian structure $(J_1, g_1)$ and the indefinite Hermitian structure $(J_2, g_2)$ respectively, then the connection $D$ given by (3.3) is compatible with the mem-structure $(J, g)$ on $\xi$. So, we have

Proposition 3.3. A connection $D$ on $\xi$ is compatible with the mem-structure $(J, g)$ on $\xi$, if and only if there are two connections $D^1$ and $D^2$ on the subbundles $\xi_1$ and $\xi_2$, respectively compatible with the para-Hermitian structure $(J_1, g_1)$ and the indefinite Hermitian structure $(J_2, g_2)$, so that $D$ is given by (3.3).

Setting then

(3.5) \[ (\Phi_g, D^i)_X = \frac{1}{2}(D^i_X + g_i^{-1} \circ D_X^i \circ g_i), \quad (\Psi_g, A^i)_X = \frac{1}{2}(A^i_X + g_i^{-1} \circ A_X^i \circ g_i), \]

we obtain from [1], Prop. 3.3, and (2.4)

Proposition 3.4. The set of connections on $\xi$ compatible with the mem-structure $(J, g)$ is given by

\[ D_X = \{(\Phi_{g_1} \circ \Phi_J), D^{i_1}\}_{X} + ((\Psi_{g_1} \circ \Psi_J), A^1)_{X}\} \circ P_1 \]
\[ + \{(\Phi_{g_2} \circ \Phi_J), D^{i_2}\}_{X} + ((\Psi_{g_2} \circ \Psi_J), A^2)_{X}\} \circ P_2, \]

where $D^{i_1}$ is an arbitrary fixed connection on $\xi_1$, $A^1 \in \Lambda^1(M) \otimes \mathfrak{X}(\xi_1) \otimes \Lambda^1(\xi_1)$, and $\Phi_J$, $\Phi_g$, $\Psi_J$, $\Psi_g$, are given by (3.4) and (3.5).
3.2 The case of the tangent bundle

We now consider the case of $\xi$ being the tangent bundle of the manifold $M$, i.e., $\xi = (TM, \pi, M)$. In this case, for a mem-structure $(J, g)$ on $M$, the pair $(P, g)$ is a pseudo-Riemannian almost product structure on $M$, and $(J_1, g_1), (J_2, g_2)$, are respectively a para-Hermitian [4] and an indefinite Hermitian structure [10] on $\xi_1$ and $\xi_2$. If $\nabla$ is a linear connection on $M$, compatible with $P$, i.e., $\nabla P = 0$, then its restrictions $\nabla^1$ and $\nabla^2$ to $\xi_1$ and $\xi_2$ are connections on these subbundles.

If $T$ is the torsion tensor of $\nabla$, we shall call torsion tensor of $\nabla^i$ to the tensor fields $T_i$ given by $T_i = P_i \circ T|_{\xi_i}$, or in more detail

$$T_i(X, Y) = \nabla_X Y_i - \nabla_Y X_i - P_i[X, Y], \quad X, Y \in \mathfrak{X}(\xi_i).$$

We call tensors of nonholonomy of the distributions $\xi_1$ and $\xi_2$ to the tensor fields $S^1 = P_2 \circ T|_{\xi_1}$ and $S^2 = P_1 \circ T|_{\xi_2}$, respectively. We obtain

$$S^1(X, Y_1) = -P_2[X, Y_1], \quad S^2(X_2, Y_2) = -P_1[X_2, Y_2].$$

It follows

**Proposition 3.5.** The distribution $\xi_1$ (resp. $\xi_2$) is involutive if and only if $S^1 = 0$ (resp. $S^2 = 0$).

After some computations we obtain from [3, 10, 14]

**Proposition 3.6.** For a mem-structure $(J, g)$ on a manifold $M$, there exists a unique linear connection $\nabla$ with torsion tensor $T$, satisfying the conditions

$$(3.6) \quad \nabla P = 0, \quad T(PX, Y) = T(X, PY),$$

$$(3.7) \quad \nabla_X J_i = 0, \quad \nabla_X g_i = 0, \quad T^i(J_iX, I_iY) = T^i(I_iX, J_iY).$$

**Definition 3.3.** We shall call the canonical connection associated to the mem-structure $(J, g)$ on the manifold $M$ to the connection given by the conditions (3.6) and (3.7).

**Remark 3.1.** Notice that this connection slightly differs from that given in Theorem 5.3 in [14].

For the canonical connection we obtain from (3.6):

$$\nabla^1_{X_2} Y_1 = P_1[X_2, Y_1], \quad \nabla^2_{X_2} Y_2 = P_2[X_2, Y_2].$$

Denoting by $\xi^1_1, \xi^2_1$ the eigen-subbundles of $J_1$ corresponding to $\varepsilon = +1$, $\varepsilon = -1$, by $\pi^1_1, \pi^2_1$ the projection maps of $\xi_1$ on $\xi^1_1$, and $\xi^2_1$ and by $X^1_1, Y^1_1$ any elements of $\mathfrak{X}(\xi^1_1)$, we obtain from the first equation in (3.7)

$$\nabla^1_{X_1^1} Y^1_1 = \pi^1_1 P_1[X^1_1, Y^1_1], \quad \nabla^1_{X_1^1} Y^2_1 = \pi^2_1 P_1[X^1_1, Y^2_1],$$

$$g_1(\nabla^1_{X^1_1} Y^1_1, Z^2_1) = X^1_1g_1(Y^1_1, Z^2_1) - g_1([X^1_1, Z^2_1], Y^1_1),$$

$$g_1(\nabla^1_{X^1_2} Y^2_1, Z^1_1) = X^2_1g_1(Y^2_1, Z^1_1) - g_1([X^2_1, Z^1_1], Y^2_1).$$

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From the second equation in (3.7) above it results, exactly as in [14, Th. 5.1], the expression for $\nabla^2_{X_2} Y_2$

For $J$ and $g$ we obtain

$$(\nabla_{X_2} J) Y_1 = 0, \quad (\nabla_{X_2} J) Y_2 = 0, \quad (\nabla_{X_2} J) Y_2 = (\nabla^2_{X_2} J) Y_2,$$

$$(\nabla_{X_2} J) Y_1 = (\nabla^1_{X_2} J) Y_1, \quad (\nabla_{X_2} g)(Y_1, Z_1) = 0, \quad (\nabla_{X_2} g)(Y_2, Z_2) = 0,$$

$$(\nabla_{X_2} g)(Y_1, Z_1) = (L_{X_2} g)(Y_1, Z_1), \quad (\nabla_{X_2} g)(Y_2, Z_2) = (L_{X_2} g)(Y_2, Z_2),$$

where $L$ stands for the Lie derivative.

4 Structures of electromagnetic type on the total space of a vector bundle

Let $\xi = (E, \pi, M)$ be a vector bundle and $(x^j), (y^a),$ local coordinates in adapted charts on $M, \xi,$ and $E,$ respectively. We denote by $(\partial_j), (e_a), \partial_a\partial_a$ the corresponding local bases, where $\partial_j = \partial/\partial x^j, \partial_a = \partial/\partial y^a, j = 1, 2, \ldots, m,$ $a, b, c = 1, 2, \ldots, n$ (see [2]). Setting for each $x = (x, y) \in E, \nabla_x E = \ker \pi_x,$ we obtain the vertical distribution and so the vertical subbundle of $T E$, denoted by $V E$. Let $C^\infty = \{ f^\nu = f \circ \pi : \nu \in C^\infty(M) \}$ be the subring of $C^\infty(E)$ naturally isomorphic to $C^\infty(M)$. Setting for each $\mu \in \Lambda^1(\xi),$ locally given by $\mu(x) = \mu_a(z) e^a,$

$$\gamma(\mu)(z) = \mu_a(x) y^a,$$

we obtain a class of functions on $E$ enjoying the property that every vector field $A \in \mathcal{X}(E)$ is uniquely determined by its values on those functions. The mapping $\gamma$ may be extended to tensor fields $S \in T^1_1(\xi)$ by

$$(\gamma S)(\gamma(\mu)) = \gamma(\mu \circ S), \quad \mu \in \Lambda^1(\xi).$$

If $S(x) = S^a_b(x) e_a \otimes e^b,$ then $\gamma S(z) = S^a_b(x) y^a \partial_b,$ i.e., $\gamma S$ is a vertical vector field on $E$. Now, let $D$ be a connection on $\xi$ and $X \in \mathcal{X}(M), u \in \mathcal{X}(\xi).$ Setting

$$X^h(\gamma \mu) = \gamma(D X \mu), \quad u^w(\gamma \mu) = \mu(u) \circ \pi, \quad \mu \in \Lambda^1(\xi),$$

we obtain two vector fields $X^h$ and $u^w$ on $E,$ respectively called the horizontal lift of $X$ and the vertical lift of $u.$ We have the useful formulas [2]:

$$(f X)^h = f^w X^h, \quad (f u)^w = f^w u^w, \quad [X^h, Y^h] = [X, Y]^h - \gamma R^D_{X Y}, \quad [u^w, v^w] = 0,$$

$$[X^h, u^w] = (D_X u)^w, \quad f \in C^\infty(M), X, Y \in \mathcal{X}(M), u, w \in \mathcal{X}(\xi).$$

Now, putting

$$Q(X^h) = X^h, \quad Q(u^w) = -u^w, \quad X \in \mathcal{X}(M), u \in \mathcal{X}(\xi),$$

we obtain an almost product $Q$ structure on $E$ whose $+1$ and $-1$ eigendistributions, are respectively called the horizontal distribution $HE$ of the connection $D$ and the vertical distribution $VE$ of the bundle.
For $f \in T^1_1(M)$, $\varphi \in T^1_1(\xi)$, $g \in T_2(M)$, $\psi \in T_2(\xi)$, we define the horizontal lift or the vertical lift $f^h, \varphi^v, g^h, \psi^v$, respectively by

\begin{equation}
(4.1) \quad f^h(X^h) = f(X), \quad f^h(u^v) = 0, \quad \varphi^v(X^h) = 0, \quad \varphi^v(u^v) = \varphi(u), \\
g^h(X^h, Y^h) = g(X, Y)^v, \quad g^h(X^h, u^v) = g^h(u^v, X^h) = g^h(u^v, w^v) = 0, \\
\psi^v(X^h, Y^h) = \psi^v(X^h, u^v) = \psi^v(u^v, Y^h) = 0, \quad \psi^v(u^v, w^v) = \psi(u, w)^v, \\
X, Y \in \mathfrak{X}(M), u, w \in \mathfrak{X}(\xi).
\end{equation}

We then define the diagonal lifts $J$ and $G$ for the pairs $(f, \varphi)$ and $(g, \psi)$ by

\begin{equation}
(4.2) \quad J = f^h + \varphi^v, \quad G = g^h + \psi^v.
\end{equation}

From (4.1) and (4.2) we have

\begin{equation}
J^n(X^h) = (f^n(X))^h, \quad J^n(u^v) = (\varphi^n(u))^v, \quad n \in \mathbb{N}^+.
\end{equation}

So $J^4 = I$, that is $J$ is an em-structure on $E$, if and only if $J^4 = I_1$ and $J^4 = I_2$, that is, either $f$ and $\varphi$ are both em-structures or one is an em-structure and the other an almost product or almost complex structure, or finally $f$ is an almost product (resp. almost complex) and $\varphi$ is a complex (resp. product) structure on $M$ and $\xi$ respectively. In the sequel we only consider the last case.

Hence, let $J$ be an em-structure on the total space $E$ of $\xi$ given by the diagonal lift in the first equation in (4.2) of an almost product (resp. almost complex) structure $f$ on the base manifold $M$ and a complex (resp. product) structure $\varphi$ on the bundle $\xi$, that is, which satisfy

\begin{equation}
f^2 = \varepsilon I_1, \quad \varphi^2 = -\varepsilon I_2, \quad \varepsilon = 1 \text{ (resp. } \varepsilon = -1),
\end{equation}

with respect to a connection $D$ on $\xi$. For the almost product structure $P$ associated to $J$, we obtain $P = \varepsilon Q$, that is, $P$ coincides up to the sign with the almost product structure $Q$ above associated to $D$.

Now, let $G$ be the diagonal lift in the second equation in (4.2), with respect to $D$, for the pair $(g, \psi)$ of metrics on $M$ and $\xi$. From (4.2) we obtain

\begin{equation}
\delta J G = (\delta f g)^h + (\delta \varphi \psi)^v,
\end{equation}

and so $\delta J G = 0$ if and only if $\delta f g = 0$ and $\delta \varphi \psi = 0$. It follows

**Proposition 4.1.** The pair $(J, G)$ of diagonal lifts, with respect to a connection $D$ on $\xi$, of an almost product (resp. almost complex) structure $f$ on $M$ and a complex (resp. product) structure $\varphi$ on $\xi$, and the nondegenerate metrics $g$ on $M$ and $\psi$ on $\xi$, is an em-structure on the total space $E$ of $\xi$ if and only if the pair $(f, g)$ is an almost para-Hermitian (resp. indefinite almost Hermitian) structure on $M$. The pair $(\varphi, \psi)$ is an indefinite Hermitian (resp. para-Hermitian) structure on $\xi$. 

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Denoting by $\omega$ and $\tau$ the 2-forms associated to the structures $(f, g)$ on $M$ and $(\varphi, \psi)$ on $\xi$, and by $\Omega_1, \Omega_2, \Omega$, the 2-forms associated to the structures $(f^h, g^h)$ on $HE$, $(\varphi^v, \psi^v)$ on $VE$ and $(J, G)$ on $TE$, we obtain

$$\Omega_1 = \omega^h, \quad \Omega_2 = \tau^v, \quad \Omega = \omega^h \oplus \tau^v.$$  

From the hypotheses of Prop. 4.1 it follows

$$\delta fg = 0, \quad \delta f\omega = 0, \quad \delta_x \varphi = 0, \quad \delta_x \tau = 0, \quad \delta J G = 0, \quad \delta J \Omega = 0.$$  

Remark 4.1. The groups of automorphisms of $\mathfrak{X}(M), \mathfrak{X}(\xi), \mathfrak{X}(E)$, given respectively for $\varepsilon = 1$ and $\varepsilon = -1$, by

$$\begin{align*}
\alpha t &= I_1 \cosh t + f \sinh t, \quad \beta t = I_2 \cos t + \varphi \sin t, \quad \gamma t = \alpha^h t \oplus \beta^h t, \quad t \in \mathbb{R}, \\
\alpha t &= I_1 \cos t + f \sin t, \quad \beta t = I_2 \cosh t + \varphi \sinh t, \quad \gamma t = \alpha^h t \oplus \beta^h t, \quad t \in \mathbb{R},
\end{align*}$$  

determine on the tensor algebras $T(M), T(\xi), T(E)$, actions which preserve the structures $(f, g, \omega), (\varphi, \psi, \tau)$ and $(J, G, \Omega)$.

For two connections $\nabla$ on $M$ and $D$ on $\xi$, we define the horizontal lift $\nabla^h$ on the subbundle $HE$ and the vertical lift $D^v$ on the subbundle $VE$ (each one with respect to the connection $D$), respectively by

$$\nabla^h_{XY} = (\nabla_X Y)^h, \quad \nabla^h_{u, v} = 0, \quad D^v_{X, w} = (D_X w)^v, \quad D^v_{u, v} w^w = 0.$$  

Putting them

$$D_A X = \nabla^h_A H X + D^v_A V X, \quad A, X \in \mathfrak{X}(E),$$

where $H$ and $V$ denote the horizontal and vertical projectors of $TE$ on $HE$ and $VE$, we obtain a linear connection $D$ on $E$, called the diagonal lift of the pair $(\nabla, D)$ with respect to the connection $D$ (see [2]), whose restrictions to the subbundles $\xi_1 = HE$ and $\xi_2 = VE$ are $D_1 = \nabla^h$ and $D_2 = D^v$. The nonvanishing components of the torsion and curvature tensors of $D$ are given by

$$(4.3) \quad T(X^h, Y^h) = T^\nabla(X, Y)^h + \gamma R^D_{XY},$$  

$$R_{X^h Y^v} Z^h = (R^\nabla_X Z)^h, \quad R_{X^h Y^v} u^w = (R^D_{XY} u)^w,$$

where $T^\nabla, R^\nabla,$ and $R^D$ stand for the torsion tensor of $\nabla$ and the curvature tensors of $\nabla$ and $D$.

For the covariant derivatives, with respect to $D$, of the horizontal lift of $f$ and $g$, and the vertical lift of $\varphi$ and $\psi$ we obtain

$$\begin{align*}
D_X f^h &= (\nabla_X f)^h, \quad D_{u, v} f^h = 0, \quad D_X g^h = (\nabla_X g)^h, \quad D_{u, v} g^h = 0, \\
D_X \varphi^v &= (D_X \varphi)^v, \quad D_{u, v} \varphi^v = 0, \quad D_X \psi^v = (D_X \psi)^v, \quad D_{u, v} \psi^v = 0.
\end{align*}$$

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Proposition 4.2. The diagonal lift
\[ D_P T \circ \] have
\[ (4.4) \]
Hence, \( D_H \) and only if \( \nabla g = 0, D_\psi = 0 \); and \( D_G = 0 \) if and only if \( \nabla g = 0, D_\psi = 0 \). From (4.3) and (4.4) it follows, for \( P = J^2 \), that \( D_P = 0 \) and \( T \circ P \times I = T \circ I \times P \) for any connections \( \nabla \) on \( M \) and \( D \) on \( \xi \). After that we have
\[ \nabla g^h = (\nabla X g)^h, \quad D^v \psi^v = 0, \quad D^v \psi^v = 0, \]
\[ \nabla g^h = (\nabla X f)^h, \quad T^1 (f^h X, I^h Y) = (T^\nabla f X, I^h Y)^h, \quad T^2 (\psi^v X, I^2 Y) = 0, \]
where \( T^1 = HO|_{HE} \) and \( T^2 = V \circ T|_{VE} \). So we obtain
Proposition 4.2. The diagonal lift \( D \) on \( E \), for the connections \( \nabla \) on \( M \) and \( D \) on \( \xi \), is the canonical connection associated to the mem-structure \((J, G)\) if and only if
\[ \nabla f = 0, \quad \nabla g = 0, \quad T^\nabla f X, Y = T^\nabla f X, Y, \]
i.e., the connection \( \nabla \) is the canonical connection [2, 10] associated to the almost para-Hermitian (resp. indefinite almost Hermitian) structure \((f, g)\) on \( M \).

Also from (4.3) and (4.4) we obtain \( D_G = 0 \) and \( T = 0 \) if and only if \( \nabla g = 0, T^\nabla = 0, R^D = 0 \) and \( D_\psi = 0 \). Hence we have
Proposition 4.3. The diagonal lift \( D \) of the pair of connections \((\nabla, D)\) coincides with the Levi-Civita connection of \( G \) if and only if \( \nabla \) is the Levi-Civita connection of \( g, D \) has vanishing curvature and \( \psi \) is covariant constant.

For the Nijenhuis tensor of \( J \),
we obtain
\[ (4.5) \quad N_J (X^h, Y^h) = N_J (X, Y)^h + \gamma (\varepsilon R^D_{XY} - R^D_{fXfY} + \varphi (R^D_{fXfY} + R^D_{XfY})), \]
\[ N_J (X^h, u^v) = (D^f_X u - \varepsilon D_X u - \varphi (D^f_X u + D_X \varphi u))^v, \quad N_J (u^v, w^v) = 0. \]

It follows
Proposition 4.4. The mem-structure \( J \) is integrable (i.e., \( N_J = 0 \), see [8]) if and only if \( f \) is a product (resp. a complex) structure in \( M \), the connection \( D \) has vanishing curvature and the complex (resp. product) structure \( \varphi \) on \( \xi \) is covariant constant.

For the exterior differential of the 2-form \( \Omega \) associated to the mem-structure \((J, G)\) we obtain
\[ d\Omega (X^h, Y^h, Z^h) = d\omega (X, Y, Z)^v, \quad 3d\Omega (X^h, Y^h, w^v) = -\gamma (i_w \varphi \circ R^D_{XY}), \]
\[ 3d\Omega (X^h, u^v, w^v) = D_X \varphi (u, w)^v, \quad d\Omega (u^v, v^v, w^v) = 0. \]

Hence
Proposition 4.5. The almost symplectic structure $\Omega$ associated to the mem-structure $(J,G)$ on $E$ is integrable (i.e., $d\Omega = 0$) if and only if the structure $(f,g)$ is almost para-Kähler (resp. indefinite almost Kähler), the connection $D$ has vanishing curvature, and the 2-form $\tau$ on $\xi$ is covariant constant.

Finally we obtain

Proposition 4.6. For the mem-structure $(J,G)$ on $E$, the structures $J$ and $\Omega$ are simultaneously integrable if and only if the structure $(f,g)$ is a para-Kähler (resp. indefinite Kähler) structure on $M$, $D$ has vanishing curvature and the pair $(\varphi,\psi)$ is covariant constant.

References

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