“A primordial, mathematical, logical and computable, demonstration (proof) of the family of conjectures known as Goldbach’s”

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For our seven dear direct predecessors and sons; Carlos, Adrián, Álvaro, Pepita, Antonio, Rosalía and Vicente

"Whoever understands Archimedes and Apollonius, will admire less the achievements of later men" 
Gottfried Wilhelm (von) Leibniz 
(1 July 1646, O.S. 21 June – November 14, 1716)

Summary

In this document, by means of a novel system model and first order topological, algebraic and geometrical free-context formal language (NT-FS&L), first, we describe a new signature for a set of the natural numbers that is rooted in an intensional inductive de-embedding process of both, the tensorial identities of the known as “natural numbers”, and the abstract framework of theirs locus-positional based symbolic representations. Additionally, we describe that NT-FS&L is able to: i.- Embed the De Morgan´s Laws and the FOL-Peano´s Arithmetic Axiomatic. ii.- Provide new points of view and perspectives about the succession, precede and addition operations and of their abstract, topological, algebraic, analytic geometrical, computational and cognitive, formal representations. Second, by means of the inductive apparatus of NT-FS&L, we proof that the family of conjectures known as Glodbach’s holds entailment and truth when the reasoning starts from the consistent and finitary axiomatic system herein described.
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1. THE FAMILY OF CONJECTURES KNOWN AS “GOLDBACH’S”

On 7\textsuperscript{th} June 1742, Christian Goldbach wrote of a letter to Leonard Euler in Latin-German (\textit{letter XLIII})\(^3\) in which he proposes the following conjecture about “integers”, which today are recognized by the names “positive integers” and “natural numbers”:\(^4\)

\textit{“Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units”}  
\textit{(Translated to contemporary English)}  
\textit{(GC-1)}

Additionally, he proposed a second conjecture, which appears in the margin of \textit{letter XLIII}:

\textit{“Every integer greater than 2 can be written as the sum of three primes”}  
\textit{(Translated to contemporary English)}  
\textit{(GC-II)}

The two above conjectures are nowadays considered logically correlated and mathematically connected, but this did not seem to be an issue nowadays.\(^5\) A later corollary of Goldbach’s second conjecture is:

\textit{“Every integer greater than 5 can be written as the sum of three primes”}  
\textit{(Translated to contemporary English)}  
\textit{(GC-III)}

It is documented that Euler replied, in a letter dated 30 June 1742, and reminded C. Goldbach of an earlier conversation they had, in which Goldbach remarked that his original conjecture followed from the statement below:

\textit{“Every even integer greater than 2 can be written as the sum of two primes”}  
\textit{(Translated to contemporary English)}  
\textit{(GC-IV)}

This statement, for centuries named and referred to as Goldbach’s Conjecture, is also known as the “\textit{strong}", “\textit{even}”, or “\textit{binary}” Goldbach conjecture, in order to distinguish it from other statements, the “\textit{weak corollaries}”, such the following:

\textit{“All odd numbers greater than 7 are the sum of three odd primes”}  
\textit{(Translated to contemporary English)}  
\textit{(GC-V)}

On the other hand, the statement \textit{GC-III} is today known as the “\textit{weak Goldbach conjecture}”, the “\textit{odd}” Goldbach’s conjecture, or the “\textit{ternary}” Goldbach conjecture.

While the “\textit{weak Goldbach’s conjecture}”, a corollary of the strong conjecture, which is nowadays understood as a corollary of the \textit{GC-IV} statement, appears to have been finally proved in 2013,\(^6\) to the date, at the beginning of the twenty-first century it is globally accepted that:
1. The “strong Goldbach´s conjecture” (GC-IV) remains unsolved.7

2. The “complete family of Conjectures known as Goldbach´s” (set conformed by statements CG I-V) has remained unsolved as a “set of theorems, at least, for more than twenty-three centuries.8 9

Hence, let us highlight that:

1. after 23 centuries from the first systematic definition of a plethora of numerical, calculus, algebraic, analytic, topological and geometrical concepts as unity, part, even number, odd number and prime number, even parity, odd parity, even parity number, odd parity number, composed number, measuring, prime number, magnitude, point, part, figure, equality, proportion, symmetry, etc.,

2. after 275 years from the enunciation a collectivity of statements, herein named as “the family of mathematical conjectures known as Goldbach´s (statements CG 1-4)”, about both; first, elemental numerical concepts and basic properties (even number, odd number and prime number) and, second, the elemental and primordial operation of addition (also, globally called and nicknamed as “the sum of positive integers numbers” and “the sum of natural numbers”), and, three, their networking correlations, combinations;

the collective human effort known as Science & Technology,10 lacks a finite set of statements that can simultaneously structure mathematical, linguistic,11 algebraic,12 analytical,13 computational,14 topological and geometrical (the two last also analytical or algebraic)15 human cognitive concepts with their conventional and globally accepted symbolic representations at any and every of the possible individual and collective levels of diversity and complexity.

As a direct consequence, a plethora of “long-term unsolved problems”16 closely related with the logical-mathematical-computational and cognitive human structure of even, odd, prime and composite numerical concepts, remain as the driving force of the collective philosophical illusion and passion to reach progress and human welfare by means of the attempts to understand, by means of the scientific and technological methodologies, our own living nature and both, our cognitive and our epistemological environments; the so-named “virtual” (also conceptual) and the so-named “real” (also physical).

2. ON SYMBOLIC REPRESENTATIONS, NOMENCLATURE AND FORMULATION OF THE NATURAL NUMBER (ELEMENTS) CONCEPT AND THEIR COLLECTIVITIES (SETS AND SUBSETS), AND THEIR ARITHMETIC

In the beginning of 2017 (Gregorian calendar), after more than a century and a quarter of century since the primordial, colossal, monumental, historic, paradigmatic and fundamental documents “Arithmetices principia, nova method” and “Sur une courbe, qui remplit toute une aire plane”, (Giuseppe Peano, 1889, in Latin and French, respectively),17 both; i.- every scientific-technological formal language18 (Number Theory, included),19 and ii.- every one of their logical, topological, algebraic, analytic geometrical, symbolic and semiotic, representations and interpretations.20
are “ultimately referred in sense and intension”\(^{21}\) to the human numerical concepts named for centuries as “integers” (more recently, math-nicknamed as “positive integers”; hereinafter, “natural numbers”).

Nowadays, the collectivity conformed by “each and every each natural number” human mental concept as element (also, member) is conventionally recognized as the set of the natural numbers concept (herein after, the set of natural number concepts will be represented by the globally accepted symbol \(\mathbb{N}\)).\(^{22}\)

Two equivalent logical-mathematical-computational-epistemological axiomatic formulations\(^{23}\) referred to both, the natural numbers set \(\mathbb{N}\) and their elements, rooted in Peano’s axiomatic description of \(\mathbb{N}\) (briefly \(\mathbb{N}(P)\)) remain valid as the starting point for any systematic reasoning about the concept of natural number:

2.0.0. If “0” is considered as a natural number concept; then, the current global accepted representation of every and each of the numerical natural number rooted in Peano’s concept of “successor of a natural number” is conformed by the following set of five statements that will sustain and/or hold “entailment” and “truth” (herein after, re-named as “five axioms” and/or “2.0.1, 2.0.2 2.03, 2.0.4, 2.0.5, respectively”):

2.0.1.- 0 is a number natural.
2.0.2.- If \(n\) is a natural number, then a successor of \(n\) is also a natural number.
2.0.3.- 0 is not a successor of any natural number.
2.0.4.- If there are two natural numbers \(n\) and \(m\) with the same successors, then \(n\) and \(m\) are the same natural number.
2.0.5. - The statement named “principle of mathematical induction”\(^{24}\) generation of the natural number concepts and of their “locus-based symbolic representations”:\(^{6}\) If 0 belongs to the collectivity of the set of natural numbers, and given any natural number, the successors of that number also belongs to this collectivity, then all natural numbers belong to that set \(\mathbb{N}(P)\).

Hereinafter, the above formulation of the set of the natural numbers will be represented and referred to by the following formula:\(^{25}\)

\[\mathbb{N}[0 \subset \mathbb{P}A]\]

The “natural number “0” concept” identity is “intentionally de-embedded (also, decoupled, descaled the subset of five axioms 2.0.1-2.0.5) as the reference of the
“identity” of every and each “successor of a natural number concept” in the “P’s axiomatic” of set \( \mathcal{N} \), which is conformed by comprehension and extension” by *the five* statements enumerated above –and/or named; also, nicknamed- from 2.0.1 to 2.0.5 of the set of the customary natural numbers “\( \mathcal{N} \)” set, and symbolically represented by \( \mathcal{N}(P) \).

2.1.0 If “0” is not considered as a natural number concept; then, the current global accepted representation of every and each of the numerical natural number rooted in Peano’s concept of “successor of a natural number” is conformed by the following set of *five statements that will sustain and/or hold “entailment” and “truth”* (herein after, re-named as “five axioms” and/or “2.1.1, 2.1.2 2.1.3, 2.1.4, 2.1.5, respectively”):

2.1.1.- 1 is a natural number. 1 belongs to \( \mathcal{N} \), the set of natural numbers.

2.1.2.- Every natural number \( n \) has at least “a successor of \( n \)”.

2.1.3.- 1 is not “successor of any natural number”.

2.1.4.- If there are two natural numbers \( n \) and \( m \) with the same successors, then \( n \) and \( m \) are the same natural number;

2.1.5.- The “principle of mathematical induction” generation of the natural number concepts and of their “locus-based symbolic representations”: If the 1 belongs to a set \( \mathcal{N} \) of natural numbers, and given one element either \( n \), the successors of \( n \) also belongs to the set \( \mathcal{N} \), then all natural numbers belong to the set \( \mathcal{N} \) and number cero concept is not considered a natural number concept belonging to the set of the natural numbers \( \mathcal{N}(P) \).

Hereinafter, this second formulation will be represented by the following formula:

\[
\mathcal{N}[0 \subset PA]
\]

The “natural number “1” concept”-identity is “intentionally de-embedded (also, decoupled descaled; subset of axioms 2.1.1-2.1.5) as the reference of the “identity” of every and each “successor of a natural number concept” in the P’s axiomatic of set \( \mathcal{N} \), which conformed “by comprehension and extension” by “the five” statements enumerated -and named; also, nicknamed- from 2.1.1 to 2.1.5 of the set of the customary natural numbers “\( \mathcal{N} \)” set, and symbolically represented by \( \mathcal{N}(P) \).

In both collectivities (sets) conformed by natural numbers, \( \mathcal{N}[0 \subset PA] \) and \( \mathcal{N}[0 \subset PA] \), natural number 1 and its successors sustain their identities by ensuring an identical (also equal) symbolic representation for every natural number concept that, in turn, in \( \mathcal{N}[0 \subset PA] \), natural number 0 and its successors sustain, ensures and shares with the natural number 1 and theirs successors in \( \mathcal{N}[0 \subset PA] \).
Thus, we are able to state about the natural numbers zero (0) and one (1) the following:

\[
\begin{align*}
\{\{0 \in \mathcal{N}[0 \|$\mathcal{N}]\} & \land 0 \notin \mathcal{N}[0 \|$\mathcal{N}] \land \{1 \in \mathcal{N}[0 \|$\mathcal{N}] \land \mathcal{N}[0 \|$\mathcal{N}]\} \\
\lor
\{0 \notin \mathcal{N}[0 \|$\mathcal{N}] \cap \mathcal{N}[0 \|$\mathcal{N}] \land \{1 \in \mathcal{N}[0 \|$\mathcal{N}] \cap \mathcal{N}[0 \|$\mathcal{N}]\} \\
\rightarrow
\{\mathcal{N}[0 \|$\mathcal{N}] \subset \mathcal{N}[0 \|$\mathcal{N}]\}
\end{align*}
\]

Then, taking in account De Morgan’s Laws: \(^{26,27}\)

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] \cup \mathcal{N}[0 \|$\mathcal{N}] &= \mathcal{N}[0 \|$\mathcal{N}] \cap \mathcal{N}[0 \|$\mathcal{N}] \\
\mathcal{N}[0 \|$\mathcal{N}] \cap \mathcal{N}[0 \|$\mathcal{N}] &= \mathcal{N}[0 \|$\mathcal{N}] \cup \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

We are able to define both, the union and the intersection of both \(\mathcal{N}[0 \|$\mathcal{N}]\) and \(\mathcal{N}[0 \|$\mathcal{N}]\):

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] \cup \mathcal{N}[0 \|$\mathcal{N}] := \{ n \mid 0 \in \mathcal{N}[0 \|$\mathcal{N}] \} \forall n \in \mathcal{N}[0 \|$\mathcal{N}] \land \{0\} 0 \in \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] \cap \mathcal{N}[0 \|$\mathcal{N}] := \{ n \mid 0 \notin \mathcal{N}[0 \|$\mathcal{N}] \} \forall n \in \mathcal{N}[0 \|$\mathcal{N}] \land \{1\} 0 \notin \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

Hence,

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] \subset \mathcal{N}[0 \|$\mathcal{N}] & \rightarrow \{0\} 0 \in \mathcal{N}[0 \|$\mathcal{N}] \cap \{1\} 1 \in \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] = \{0\} \cup \{1\} 0 \notin \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

\[
\begin{align*}
\mathcal{N}[0 \|$\mathcal{N}] = \{0\} 0 \in \mathcal{N}[0 \|$\mathcal{N}] \cap \{1\} 0 \notin \mathcal{N}[0 \|$\mathcal{N}]
\end{align*}
\]

\(\mathcal{N}[0 \|$\mathcal{N}]\) is univocally identified and well symbolically represented by comprehension as the set of natural numbers defined by extension and conformed by the union of both, the subset of \(\mathcal{N}[0 \|$\mathcal{N}]\) which is conformed by the natural number zero concept subset conformed by a unique element (a singleton), and the subset conformed by every natural number with the identity of successor of the natural number when referred to zero (0) natural number concept.

Hence, we are able to establish the next considerations:

1. \(\mathcal{N}[0 \|$\mathcal{N}]\) is defined as the set conformed by the union between the singleton\(^{20}\) of the natural number zero (0) and the subset of the natural numbers identified and symbolically represented as successors of natural number zero (0), which is, in turn, the set \(\mathcal{N}[0 \|$\mathcal{N}]\).\(^{29}\)
2. \( \mathcal{N}[0 \cup \mathbb{PA}] \) is univocally identified and symbolically represented and named as the subset of natural numbers \( \mathcal{N}[0 \cup \mathbb{PA}] \) defined by comprehension and conformed by every natural number sustaining the identity of successor of the natural number zero (0) concept.

3. \( \mathcal{N}[0 \cup \mathbb{CPA}] \) is defined, in turn, as the set conformed by the union between the singleton of the natural number one (1) and the subset of the natural numbers \( \mathcal{N}[0 \cup \mathbb{CPA}] \) identified, defined by comprehension and symbolically represented as successors of natural number one (1), which is named onwards the set \( \mathcal{N}[0 \cup \mathbb{CPA}] \).

4. By means of the principle of mathematical induction of the both above described formulations, \( \mathcal{N}[0 \cup \mathbb{CPA}] \) and \( \mathcal{N}[0 \cup \mathbb{CPA}] \), please, let us generalize above definition:

\( \mathcal{N}[0 \cup \mathbb{CPA}] \) is defined as the set conformed by the union between both; the subset defined by extension and conformed by natural numbers 0, 1, ..., and “\( n \)”, and the subset of the natural numbers identified, defined by comprehension and well symbolically represented as the successors of natural number “\( n \)” (\( n \)) by formula \( \mathcal{N}[0, 1, ..., n \cup \mathbb{CPA}] \).

Thus,

\[
\text{subset-} \mathcal{N}[0 \cup \mathbb{CPA}] := \{0\} \forall n \in \mathcal{N}[0 \cup \mathbb{CPA}]; \\
\text{subset-} \mathcal{N}[0, 1 \cup \mathbb{CPA}] := \{0, 1\} \forall n \in \mathcal{N}[0 \cup \mathbb{CPA}]; \\
\text{subset-} \mathcal{N}[0, 1, ..., n \cup \mathbb{CPA}] := \{0, 1, ..., n\} \forall n \in \mathcal{N}[0 \cup \mathbb{CPA}]
\]

Hence, allow us to point out that:

\[
\text{subset-} \{\mathcal{N}[0, 1, ..., n \cup \mathbb{CPA}]\} \forall n \in \mathcal{N}[0 \cup \mathbb{CPA}] \subseteq \mathcal{N}[0 \cup \mathbb{CPA}]
\]

&

\[
((0, 1, 2, ..., n \in \mathcal{N}[0 \cup \mathbb{CPA}]) \forall n \in \mathcal{N}[0 \cup \mathbb{CPA}] \land \forall (1, 2, ..., n \in \mathcal{N}[0 \cup \mathbb{CPA}] \cap \mathcal{N}[0 \cup \mathbb{CPA}]))
\]

On the other hand, \( \mathcal{N}[0, 1, ..., n \cup \mathbb{CPA}] \) is univocally identified and symbolically represented as the subset of natural numbers \( \mathcal{N}[0 \cup \mathbb{CPA}] \) which is defined by comprehension and conformed by every natural number with the identity of successor of the natural number “\( n \)” (\( n \)) concept.

Please, allow us to point out that for every subset \( \mathcal{N}[0, 1, ..., n \cup \mathbb{CPA}] \), the next statement are true (theorems):

\[
\#- \text{subset-} \mathcal{N}[0 \cup \mathbb{CPA}] = 1, \forall n \ (n \in \mathcal{N}[0 \cup \mathbb{CPA}])
\]

\[
\#- \text{subset-} \mathcal{N}[0, 1 \cup \mathbb{CPA}] = 2, \forall n \ (n \in \mathcal{N}[0 \cup \mathbb{CPA}])
\]

.............

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Concept of natural number (as natural number and as successor of a natural order named if both operation are sustained by (symbolized by customary meaning respectively, which preserve their usual logical, linguistic, scientific recursive PA mathematical induction; set (number concept.

representations concept, their collection (w.f.s) concerning grammar briefly In the following, first order-logic of Peano’s Arithmetic axiomatic (FOL-PA; briefly, also PA)\(^\text{30}\) will be the starting formal background language used (syntax, grammar and semantic) to the formulations of both; i.- well formed statements (w.f.s) concerning nomenclature and formulation referred to the natural number concept, their collections and their arithmetic, and, ii.- well formed symbolic representations (w.f.-sr) statements of new concepts correlated with the natural number concept.

In the following entailment will be sustained by both; the next list of six statements set (2.1.1-2.1.6 axioms) and the induction schema (also named principle of mathematical induction; statement formula 2.1.7):\(^\text{31, 32}\)

\[
\begin{align*}
2.1.1 & \quad \forall x (x \in \mathbb{N}[0 \subset \text{FOL-PA}]) (S(x) \neq 0) \\
2.1.2 & \quad \forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) (S(x) = S(y) \rightarrow x = y) \\
2.1.3 & \quad \forall x (x \in \mathbb{N}[0 \subset \text{FOL-PA}]) (x + 0 = x), \\
2.1.4 & \quad \forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) (x + S(y) = S(x + y)) \\
2.1.5 & \quad \forall x (x \in \mathbb{N}[0 \subset \text{FOL-PA}]) (x \cdot 0 = 0), \\
2.1.6 & \quad \forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) (x \cdot S(y) = x \cdot y + x) \\
2.1.7 & \quad \forall x \forall y (x, y_1 \ldots y_n \in \mathbb{N}[0 \subset \text{FOL-PA}])^{33}
\end{align*}
\]

PA set of axioms incorporates axioms 2.1.3-2.1.4 and 2.1.5-2.1.6 as formal recursive-inductive references of the addition and multiplication operations, respectively, which preserve their usual logical, linguistic, scientific-technological customary meaning.

Please, take into account the following; first, it is allowed to refer the addition (symbolized by “+”) and multiplication (symbolized by “.”) binary operations only if both operation are sustained by the operation “successor of a natural number named x” (above symbolized by “S(x)”: second, entailment refers only to a first order logic PA w.f. and a w.f.s-r statements referred only to both identities of the concept of natural number (as natural number and as successor of a natural number).
number.
Finally, allow us to point out that:

\textit{i.-} For every natural number \( x \) (\( x \in \mathbb{N} \setminus \{0\} \)), the statement (\( S(x) = 0 \)) is false. That is, there is no natural number whose successor is 0.

\textit{ii.-} For every natural number \( x \) (\( x \in \mathbb{N} \setminus \{0\} \)), the statement (\( S(x) = x \)) is false. That is, there is no natural number whose successor is itself.

\textit{iii.-} The operation (also function) symbolized by “\( S(x) \)” (meaning “successor of natural number \( x \)”, \( \forall x \ (x \in \mathbb{N} \setminus \{0\} \)) is an injection as a direct consequence of axiom 2.1.2 and the mathematical induction axiom (2.1.7).

**First Highlighted NT-FS&L-Considerations**

From this point, we will adopt intended interpretation of \( \text{FOL-PA} \) that holds for \( \mathbb{N} \cap \mathbb{FOL-PA} \):

\textit{i.-} Its individuals are recognized and named as the customary natural numbers (both the linguistic & mathematical perspectives of natural number concept hold, sense and intension, identical “identities” of the natural number concept).

\textit{ii.-} \textbf{Natural numbers} are the members of the set named and symbolized by \( \mathbb{N} \cap \mathbb{FOL-PA} \), which \textit{is equivalent to} the set \( \mathbb{N} \cap \mathbb{FOL-PA} \) from a logical perspective, if and only if, natural number one (1) is successor of natural number zero (0) in the subset \( \mathbb{N} \cap \mathbb{FOL-PA} \).

\textit{iii.-} The natural number named “zero” (symbolized by 0) is, in turn, the “identity conserving element” (also named neutral element) for the binary operation (also named function) addition on \( \mathbb{N} \cap \mathbb{FOL-PA} \).

\textit{iv.-} The next statement is a theorem (a w.f.s and a w.f.s-r statement which sustains truth).

\[ S(x) = x + n, \forall x \forall n(x \in \mathbb{N} \cap \mathbb{FOL-PA}, n \in \mathbb{N} \cap \mathbb{FOL-PA} \cap \mathbb{N} \cap \mathbb{FOL-PA}) \]

Please, allow us to point out both: first, in the present intended interpretation of \( \text{FOL-PA} \) “equality” holds the customary meaning generally accepted of “identity” in a first order logic, and second, its embedded entailment with first order axiom 2.1.4 of \( \text{FOL-PA} \):.

\[ \forall x \forall y(x, y \in \mathbb{N} \cap \mathbb{FOL-PA})(x + S(y) = S(x + y)) \]

\textit{v.-} The next statement is true.

\[ 1 \cdot S(y) = 1 \cdot x + 1 \rightarrow S(y) = x + 1, \forall x \forall y(x, y \in \mathbb{N} \cap \mathbb{FOL-PA}) \]

Please, take in account both, that: first, the \textbf{axiom 2.2.6 \( \text{FOL-PA} \)’s referred to the multiplication operation of every couple of natural numbers that belong to \( \mathbb{N} \cap \mathbb{FOL-PA} \), and, second, \textit{the above statement iv referred to the binary addition of}}
a couple of natural numbers, in which one of them belongs to \( \mathbb{N}[0 \subset \text{FOL-PA}] \) and the other one belongs to the \( \mathbb{N}[0 \subset \text{FOL-PA}] \cap \mathbb{N}[0 \not\subset \text{FOL-PA}] \) set, which is the case for natural number 1. That said:

\[
\forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) (x \cdot S(y) = x \cdot (y + 1) \land (x \cdot y + x) \rightarrow x \neq y \land S(y))
\]

&

\[
\forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) ((1 \cdot S(x) = 1 \cdot S(y) \rightarrow 1 \cdot x = x \land 1 \cdot y) \rightarrow x = y)
\]

Additionally,

\[
\forall x \forall y (x, y \in \mathbb{N}[0 \subset \text{FOL-PA}]) (((x \cdot y = y \cdot S(x)) \rightarrow \begin{array}{l}
(x = 0 \rightarrow y = 0) \\
\lor \\
((x = 1 \rightarrow (1 \cdot S(y) = y \cdot S(x) \rightarrow y = 1 \land y \cdot S(x) \land S(x) \land S(y) \land x)) \\
\lor \\
((x \neq 1) \rightarrow (x = y \land S(x) \land S(y)) \rightarrow y \neq 0 \land 1)
\end{array})
\]

Hence, the natural number one (1) has to be intended recognized as the identity of natural number “one” (symbolized by constant 1), which belongs to both \( \mathbb{N}[0 \subset \text{FOL-PA}] \) and \( \mathbb{N}[0 \not\subset \text{FOL-PA}] \) and, in turn, with the embedded identity of the neutral element (also identity conserving element) for the binary multiplication operation on the \( \mathbb{N}[0 \subset \text{FOL-PA}] \) set.

**FIRST FOUNDAMENTAL NT-FS&L THEOREM**

\[\forall n (n \in \mathbb{N}[0 \subset \text{FOL-PA}]) \]

\[\{\text{subset-} \mathbb{N}[0 \subset \text{FOL-PA}] \subseteq \text{subset-} \mathbb{N}[0,1 \subset \text{FOL-PA}] \subseteq \text{subset-} \mathbb{N}[0,1,...,n \subset \text{FOL-PA}] \subseteq \mathbb{N}[0 \subset \text{FOL-PA}] \rightarrow \]

\[\text{#-subset-} \mathbb{N}[0 \subset \text{FOL-PA}] = 1 \land \text{#-subset-} \mathbb{N}[0,1 \subset \text{FOL-PA}] = 2 \land \text{#-subset-} \mathbb{N}[0,1,...,n \subset \text{FOL-PA}] = n + 1 \]

\[(\text{NT-FS&L F.Th. 1})\]

**Second Highlighted NT-FS&L-Consideration**

From this point, we will adopt the next two couples of symbolic
representations, (a) and (b), which symbolize their corresponding embedded definitions by comprehension and extension of their identity, of each of the two equivalent logical-intended axiomatic (a.1 and a.2, and b.1 and b.2, respectively) for the above descriptions of the above set of natural numbers \( \mathcal{N}(0 \subset \text{PA}) \): 37

\[ \mathcal{N}(0 \subset \text{PA}) := \{0, 1, 2, 3, 4, ..., x+0, x+1, x+2, ... \} \quad (a.1) \]
\[ \mathcal{N}(0 \subset \text{FOL-PA}) := \{0, S(0), S(S(0)), S(S(S(0))), ... \} \quad (a.2) \]

b) For the case that 0 is not intended-considered a natural number:
\[ \mathcal{N}(0 \not\subset \text{FOL-PA}) := \{1, 2, 3, 4, 5, ..., x+1, x+2, ... \} \quad (b.1) \]
\[ \mathcal{N}(0 \not\subset \text{FOL-PA}) := \{S(0), S(S(0)), S(S(S(0))), ... \} \quad (b.2) \]

FOL-PA axiomatic (2.1.1-2.1.7 axioms set) defines, on both cases a) and b), a unary irreducible representation of the natural numbers; the natural number 0 can be well defined and symbolically represented as 0; the natural number 1 can be well defined and symbolically represented as S(0); the natural number 2 as S(S(0)) and by S(1), and so on.

**Third Highlighted NT-FS&L-Consideration**

Hereinafter, in the NT-FS&L first order language, every natural number \( x \) will be intended-considered as the result of “\( x \)-fold (also, \( x \)-times) application(s)” of “the succession operation” (abbreviated by symbol “S”) to natural number 0 39 which hereinafter will be noted as \( S_x(0) \). 40

Then, entailment from FOL-PA (axioms 2.1.1-2.1.7) sustaining formula a.3 and b.3 and by means of indexing representation, will be considered definitions by comprehension and extension of both, natural numbers as elements, and two logical formalized collections of them, the set \( \mathcal{N}(0 \subset \text{NT-FOL-PA}) \) and the set \( \mathcal{N}(0 \not\subset \text{NT-FOL-PA}) \):

\[ \mathcal{N}(0 \subset \text{NT-FOL-PA}) := \{S_0(0), S_1(0), S_2(0), S_3(0), ..., S_x(0), ... \} \quad (a.3) \]
\[ \mathcal{N}(0 \not\subset \text{NT-FOL-PA}) := \{S_1(0), S_2(0), S_3(0), S_4(0), ..., S_{x+1}(0), ... \} \quad (b.3) \]

Hereinafter, it will be shown that the NT-FS&L allows referring to the natural number concept zero (0), the ordered enumeration and, in turn, the ordered counting of:

- The ordered enumeration and, in turn, the ordered counting of the elements (also members and links) belonging to any subset of natural numbers.
- The amount of elements belonging to any subset of natural numbers.
- The enumeration, counting and ordering of the operations succession, addition and multiplications by the same logic, mathematical and computable formal system and language.

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- The enumeration, counting and ordering of axioms and theorems by a unique logic, mathematical and computable formal system and language.

Hence, $\mathcal{N}(0\subset\text{NT-FOL-PA})$ will be redefined as the set of natural numbers, which is referred to the natural number concept zero $(0)$, by the seven-membered set of axioms named $\text{FOL-PA}$, to the first fundamental theorem $\text{NT-FS&L F.Th. 1.}

Additionally, the ordered counting and enumeration of its elements under its identities as natural number concepts, allows access to their complete ordering structure under their identities of successors, and vice versa.

$$\mathcal{N}(0\subset\text{NT-FOL-PA}) := \{0, 1, 2, 3, ..., S_{x}(0), S_{x+1}(0), S_{x+2}(0), x+3, ..., n\},$$

\text{(c.1)}

2.2. NT-FS&L Considerations, definitions and theorems about natural numbers. On Elements and Collectives of successors and predecessors.

First Highlighted NT-Consideration on natural numbers and collectives thereof

In NT-FS&L FOL-PA, (in the following abbreviated as NT-PA and $\mathcal{N}(0\subset\text{NT-FOL-PA})$ abbreviated as $\mathcal{N}(0\subset\text{NT-PA})$), the next representation meta-operation $A-B$ will be operative: 41, 42

A. Meta-exchange index-variable and variable-index representing elements of $\mathcal{N}(0\subset\text{NT-PA})$.

$\mathcal{N}(0\subset\text{NT-PA}) \{0, 1, 2, 3, 4, ..., x+0, x+1, ...\}$, then $\forall x, \forall y (x, y \in \mathcal{N}(0\subset\text{NT-PA}))$:

i. $\mathcal{N}(0\subset\text{NT-PA}) := \{S_{x}(0) \mid (0 = S_{0}(0)) \land (x = S_{x}(0) \land S_{0}(x))\} \forall x$

ii. $(S_{x}(y) = S_{y}(x)) \rightarrow x = y$

iii. $S_{x}(1) = S_{y+1}(0) \rightarrow x = y$

B. Meta-exchange index-variable and variable-index between $\mathcal{N}(0\subset\text{PA})$ and $\mathcal{N}(0\subset\text{NT-PA})$

$\mathcal{N}(0\subset\text{NT-PA}) = \{1, 2, 3, 4, 5, ..., x+0, x+1, ...\}$, then:

i. $\mathcal{N}(0\subset\text{NT-PA}) = \{S_{x+1}(0) \mid (1 = S_{1}(0)) \land (x = S_{x+1}(0))\} \forall x$

ii. $\forall x, \{x \in \mathcal{N}(0\subset\text{NT-PA}) \land \forall y (y \in \mathcal{N}(0\subset\text{NT-PA})). ((S_{x}(1) = S_{y+1}(0) \rightarrow (x = y+1))$

iii. $\forall x, \{x \in \mathcal{N}(0\subset\text{NT-PA}) \land \forall y (y \in \mathcal{N}(0\subset\text{NT-PA})). ((S_{x+1}(0) = S_{y}(1) \rightarrow (x = y))$
Hence, the next two statements summarize the above NT-considerations:

\[ \mathcal{N}(0 \subset \text{NT-PA}) = \{0, 1, 2, \ldots, x, \ldots\} \land \{S(0), S(S(0)), \ldots\} \land \{S_0(0), S_1(0), S_2(0), \ldots\}, \]

if an only if; \[ \mathcal{N}(0 \subset \text{NT-PA}) = \{S_{x+0}(0)\} \subseteq \{S_x(0)\} \forall x \in \mathcal{N}(0 \subset \text{NT-PA}) \]

\[ \mathcal{N}(0 \not\subset \text{NT-PA}) = \{1, 2, \ldots, x, \ldots\} = \{1, S(1), S(S(1)), \ldots\} = \{S_1(0), S_2(0), S_3(0), \ldots\} \]

if an only if; \[ \mathcal{N}(0 \not\subset \text{NT-PA}) = \{S_{x+1}(0)\} \subseteq \{S_y(1)\}, \forall x, \forall y \left( x \in \mathcal{N}(0 \not\subset \text{NT-PA}) \land y \in \mathcal{N}(0 \not\subset \text{NT-PA}) \right) \]

Let us consider the next NT-conclusion unavoidable: we can use the same well formed format symbols rooted in a common symbolic representation of the operation \( S \) (successor of a natural number) identity to represent any natural number identity, which satisfies as element of any of the two PA axiomatic initially described, and we can use said format in order to define by extension and comprehension any set or collectivity of them. However, said representation does not provide us a common and in turn intended (sense and intension) unique definition by comprehension of such set or collectivity thereof.

On the other hand, it is possible to symbolized and enumerate the elements of \( \mathcal{N}(0 \subset \text{PA}) \) and \( \mathcal{N}(0 \not\subset \text{PA}) \) by using the alphabet of either of them, as well as reference for enumeration and counting of the elements.\(^{43}\)

Second Highlighted NT-Consideration on meta-representations of natural numbers and collectives thereof

\[ \forall x \forall y \left( x \in \mathcal{N}(0 \subset \text{FOL-PA}) = \{0, 1, 2, 3, 4 \ldots\} = \{0, S(0), S(S(0)), \ldots\} \land \forall y \in \mathcal{N}(0 \not\subset \text{FOL-PA}) = \{1, 2, 3, 4 \ldots\} = \{S(0), S(S(0)), \ldots\} = \{1, S(1), S(S(1)), \ldots\} \right). \]

\[ ((S_0(1) = S_1(0)) \land ((S_0(1) = S_2(0)) \land \ldots) \rightarrow x = y \]

That is to say, both set \( \mathcal{N}(0 \subset \text{NT-PA}) \) and \( \mathcal{N}(0 \not\subset \text{NT-PA}) \) require the same amount of natural\(^{44}\) number symbols (alphabet) for complete representation of their constants, variables and expressions without altering the topological property of non-dense order\(^{45}\) common from their axiomatic logic and mathematical construction, and definition. Besides, this is extensible to any of its possible subsets.

Please, take into account that the order theory herein suggested does not correlate with the ubiquitous concept (idea) of “the cardinality” (neither to the “ordinality” concept) used by others authors. Hence, we suggest a deep revision of the central role that this concepts of cardinal and ordinal numbers play in the state of the art of the past and current Number Theory.\(^{46}\)
**Third Highlighted NT-Consideration on meta-representations of natural numbers and collectives thereof**

The first Highlighted NT-Consideration on Meta-representations of natural numbers and collectives thereof is also “true” (which has two possible intentional meanings, excluding those that come from the unintentional models: it is a “well formed expression” and, hence, the statement remains as a theorem), for any positional number system (also numbering system) symbolic representation and for any numerical construction and lexical numbering system. 47

Thus, in NT-FS&L the following statement is invariant (also, independent) of the symbolical number representation system used:

\[
\forall x \forall y \left( x, y \in \mathcal{N}(0 \subseteq \text{PA}) \right) = \{0, 1, 2, 3, 4 \ldots\} \quad \& \quad \forall y \in \mathcal{N}(0 \subseteq \text{PA}) = \{1, 0, 1, 1, 1, \ldots\} = \{S(0), S(S(0)), S(S(S(0))), \ldots\} = \{1, 1, 1, 1, \ldots\}
\]

\[
((S_{o}(1) = S_{t}(0)) \& ((S_{y}(1) = S_{x+1}(0))) \rightarrow x = y
\]

In the example, a customary binary symbolic expressions for the natural number concept are used instead of the initial proposition expressed in decimal number system. The “non-dense order relation” holds from the operation “successor of a natural number” when expressed in any numbering system (an invariant property) and it is compatible, with the current accepted “rules and machinery of construction” of the accepted symbols to represented well-formed sentences (expressions) in any number system.

\[
\forall x \forall y \left( x, y \in \mathcal{N}(0 \subseteq \text{PA}) \right) = \{0, 1, 2, 3, 4 \ldots\} = \{0, S(0), S(S(0)), \ldots\}, \text{then}
\]

\[
S_{x+y}(1) = S_{x+1+y}(0) = S_{y}(x+1) = S_{x+1}(y) = S_{y+1}(x) = S_{x+y+1}(0),
\]

if an only if

\[
((S_{o}(1) = S_{t}(0)) \& ((S_{y}(1) = S_{x+1}(0)) \rightarrow ((x = y+1))) \forall x \forall y \left( x, y \in \mathcal{N}(0 \subseteq \text{NT-PA})\right)
\]

**First Highlighted Set of NT-Definitions and NT-Theorems on natural numbers and collectives thereof**

From this point, we will considerer the next set of definitions and w.f.s as the core of NT-FS&L:

**HL-1-NT-Definition:** A natural number \( y \) \((y \in \mathcal{N}(0\subset \text{NT-PA}))\) is named (also, called, renamed, nicknamed and defined) “direct successor of natural number \( x \)” if an only if, the next statement is a wfs and wfs-r statement:

\[
\forall x \forall y \left( x, y \in \mathcal{N}(0 \subseteq \text{NT-PA})\right) (y = S_{1}(x) \land (S_{x+1}(0) \rightarrow y = x+1) \quad \text{def. 1})^{48}
\]
We will provide a new identity to natural number \( y \), which will be well symbolically represented (w.f.s-r) by formulas: \( S_1(x) \), \( 1\text{-suc-}x \) and \( y\text{-sucd-}x \), by the next three equivalent equalities:

\[
y = S_1(x) ; \quad y = 1\text{-suc-}x ; \quad y = \text{sucd-}x;
\]

Please, allow us to indicate that def.1 is sustained on: i.- Previous concept of identity (also, equality in FOL-PA); ii.- Previously described theorem \( S(x) = x + 1 \), which is coming from PA axiomatic; iii.- Equivalence of the next equalities representing both of the next well formed symbolic-representations (w.f.s-r.) identity of natural numbers \( x \) and \( y \) \( \forall x \forall y \ (x, y \in \mathbb{N}(0\subset PA)) \).

\[
0 = S_0(0) \land y = S_x(1) \land y = S_{x+1}(0) \land x = S_{x+1}(0) \land y = S_y(0)
\]

**HL-2-NT Definition:** A natural number \( n \) \( (n \in \mathbb{N}(0\subset PA)) \) is named (also, called renamed, nicknamed and defined) as “\( y\text{-successor of natural number } x \)”, if and only if the next statement is a wfs and wfs-r statement:

\[
\forall x \forall y \forall n \ (x, y, n) \in \mathbb{N}(0\subset PA) \ (n = S_x(y) \land S_y(x) \land (S_{x+y}(0) \rightarrow n = x+y) \) (def. 2)
\]

We provide a new identity to natural number \( n \), which will be well symbolically represented (w.s.r) by formulas: \( S_y(x) \) and “\( y\text{-suc-}x \)”, by the next set of equivalent well symbolic (nomenclature and formulation) formed (entailment from FOL-PA’s axiomatic): 49

\[
n = y\text{-suc-}x \ (n = S_y(x) \land S_x(y)), \ if \ and \ only \ if:
\]

\[
n = (y+x)\text{-suc-}0 \ (n = S_{x+y}(0) \land S_{y+x}(0) \land S_0(x+y)), \ if \ and \ only \ if:
\]

\[
n = n\text{-suc-}0 \ (n = S_n(0) \land S_0(n)), \ \forall n \forall x \forall y \ (x, y, n \in \mathbb{N}(0\subset PA))
\]

Please, allow us to highlight that in first order logic, “\( y\text{-successor of natural number } x \)” and “direct-successor of a natural number \( x \)”, define two different partitions of the set of natural numbers under a unique and well defined NT-FOL-PA’s axiomatic referred to the concept of natural number zero \( (0) \). Additionally is very import remember that zero \( (0) \) is member and symbol on \( \mathbb{N}(0\subset NT-PA) \) and natural number zero concept only allows to the symbolic of the corresponding first order alphabet of \( \mathbb{N}(0\subset NT-PA) \).

Please, take in account that in NT-FS&L, from the perspective of the symbolic representation of a natural number concept, “\( S_1(x) \)” and “\( 1\text{-suc-}x \)” are obviously indistinguishable for two meaning (semantics) “\( 1\text{-successor of } x \)” and “\( \text{direct } 1\text{-successor of cero} \)”; but this does not impose any “logic limitation”, it is only a direct consequence of the fact that the symbolic representation of \( 0 \) and \( 1 \) is a “convention” (please note that only if we see the symbol \( 0 \) we can not unsure if the statement referrers about \( \mathbb{N}(0\subset NT-PA) \) or about \( \mathbb{N}(0\subset NT-PA) \) but this does not change the truth sustained by the next statement:

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\[(0 \in \mathcal{N}(0\subset\text{PA}) \land 0 \notin \mathcal{N}(0\subset\text{PA})) \land (1 \in \mathcal{N}(0\subset\text{PA}) \land \mathcal{N}(0\subset\text{PA})) \rightarrow 0 \neq 1\]

& “1-successor of 0 is a identity of natural number one (1)”,

& “the direct 1-successor of natural number 0” is one of the identities of natural number one (1)” embedded in PA.

**HL-3-NT Definition:** The set conformed by every natural number \( y \) (\( y \in \mathcal{N}(0\subset\text{PA}) \)), which is successor of a natural number \( x \) (\( x \in \mathcal{N}(0\subset\text{PA}) \)), is named (also, called, renamed, nicknamed and defined) as “set of successors of natural number \( x \)”, if and only if the next statement is a wfs and wfs-r statement:

\[
\text{Suc}_y(x) := \{ y \in \mathcal{N}(0\subset\text{PA}) \mid x+y = S_{x+y}(0) \land [S_{x+y}(0) + y] \land (y = S_x(y)) \text{ (def. 3)}
\]

**HL-4-NT Definition:** The singleton formed by the natural number \( y \) as unique member, which is the direct successor of a natural number \( x \) (\( x \in \mathcal{N}(0\subset\text{PA}) \)), is named (also, called, renamed, nicknamed and defined) “set of direct successors of natural number \( x \)”, if and only if the next statement is a wfs and wfs-r statement:

\[
\text{SUCd}(x) := \{ x+1 \in \mathcal{N}(0\subset\text{PA}) \mid x = S_{x+1}(0) \land S_{x+1}(0) \text{ (def. 4)}
\]

Please, allow us to point out both the next two statements NT-FOL-PA about sets of natural numbers:

\[
\forall x (x \in \mathcal{N}(0\subset\text{PA})) \left[ \text{Suc}_x(x) \subseteq \text{Suc}_x(0) \cap \text{Suc}_{x+1}(0) \right]
\]

\[
\forall x (x \in \mathcal{N}(0\subset\text{PA})) \left[ \text{SUCd}(x) \subseteq \text{SUCd}(x) \right]
\]

both of them are a direct consequence of initial FOL-PA’s axiomatic definition of the set of natural numbers referred to natural number zero:

\[
\mathcal{N}(0\subset\text{NT-PA}) = \text{SUCd}_x(0) \land S_{x+1}(0) \forall x (x \in \mathcal{N}(0\subset\text{NT-PA})
\]

**HL-1-NT Theorem:** The \( \mathcal{N}(0\subset\text{NT-PA}) \) set, which has been renamed as \( \text{SUCd}_x(0) \), is conformed by the union of every singleton \( \text{Suc}_x(x) \) conformed by one defined and symbolized natural number \( x \)

\[
\bigcup \left[ \text{Suc}_x(x) \right]_{x \in \mathcal{N}(0\subset\text{NT-PA})} \subseteq \mathcal{N}(0\subset\text{NT-PA}) \text{ (NT-th. 1)}
\]

\( \text{SUCd}(x) \) customary represent a partition “member to member” (only singletons are involved) of a set, \( \mathcal{N}(0\subset\text{NT-PA}) \) which is a now “an infinite enumerable and countable element by element” set as a direct consequence of its complete order.
**HL-5-NT Definition:** A natural number \( n \) (\( n \in \mathbb{N}(0\subset\text{NT-PA}) \)) is named (also, called renamed, nicknamed and defined) “\( y \)-predecessor of natural number \( x \)”, if and only if the next statement is a wfs and wfs-r statement:\(^{53} \)

\[
\forall x \forall y \forall n \left( x, y, n \in \mathbb{N}(0\subset\text{NT-PA}) \right) \\
(x := P_0 (x) \land y := P_0 (y) \land n := P_0 (n)) \rightarrow \\
(x = S_x (0) \land y = S_y (0) \land n = S_n (0) \rightarrow x = n + y \land n = y) \quad \text{(def. 5)}
\]

We will provide of a new identity to natural number \( n \), which will be well symbolically represented (w.f.s.r) by both formulas: “\( P_y (x) \)” and “\( y\text{-pre-}x \)”, by the next two equivalent equalities:

\[
\begin{align*}
n &= P_y (x) ; \quad n = y\text{-pre-}x
\end{align*}
\]

**HL-6-NT Definition:** A natural number \( y \) is named (also, called renamed, nicknamed and defined) “direct predecessor of natural number \( x \)”, if an only if the next statement is a wfs and wfs-r statement:

\[
\forall x \forall y \left( x, y \in \mathbb{N}(0\subset\text{NT-PA}) \right) (x = P_1 (x) \land (S_y (0) \rightarrow x = y + 1) \quad \text{(def. 6)}
\]

*That is to say, natural number \( y \) is the direct predecessor of natural number \( x \), if an only if, \( x \) is the direct successor of \( y \) when both of them are members of \( \mathbb{N}(0\subset\text{NT-PA}) \).*

We provide of a new identity to natural number \( x \), which will be well symbolically represented (w.f.s-r) by both formulas: “\( P_1 (x) \)” and “\( 1\text{-pre-}x \)”, by the next statement:\(^{54} \)

\[
\forall x \forall y \left( x, y \in \mathbb{N}(0\subset\text{NT-PA}) \right) (y = P_1 (x) \land 1\text{-pre-}x \rightarrow (x = 0\text{-pre-}x) \land (y = 0\text{-pre-}y)
\]

Please, take in account that the fact that symbolic representations “\( P_1 (x) \)” and “\( 1\text{-pre-}x \)” are indistinguishable (also, degenerated symbolic representations) for the two meaning (semantic identities) 1-predecessor of \( x \) and/or direct 1-predecessor of \( x \); this has to be considered as a direct consequence correlated with the fact that the customary symbolic representation of “cero” (0) and “one” (1) is just a convention. Thus, this “degeneration” has never to be considered “non-context free” from a logic perspective. Thus, please allow us to point out that: first, the presence of the symbol 0 by itself cannot ensure whether we are reasoning about \( \mathbb{N}(0\subset\text{NT-PA}) \) and/or about \( \mathbb{N}(0\subset\text{NT-PA}) \) and, second, this is only true if a previous axiom (theorem) ensures entailment for the statement: \( 0 \in \mathbb{N}(0\subset\text{NT-PA}) \).

*Hence, for every natural number concept, the statements “zero is the direct predecessor of one” and “the unique predecessor, direct and non-direct, of 1 is cero” do not transfer any first logic inconsistence; because only translate an “enumerable and countable” ordered symbolic degeneration, which always appeared in “the semantic of symbolic conventions” by means of non-free-*
context logical-based statements.

Please, allow us to present as formal coherent conceptual core in NT-FOL-PS&L and supported by def. 1-4 the next select set of NT-theorems on \( \mathcal{N}(0 \subset NT-PA) \) about the “precede operation” (also function):

1. \( \forall x \neq 0 \left( P_0(x) \land S_0(x) \right) \land S_x(0) \neq 0 \)
2. \( \forall x \left( P_0(x+0) = x + 0 \land S_0(0) \land S_0(x) \rightarrow (P_0(x) = 0 \land S_0(x) \land S_d(0) \rightarrow x = 0) \right) \)
3. \( \forall x \left( P_0(x+1) = x + 1 \land S_0(x+1) \land S_{x+1}(0) \right) \)
4. \( \forall x \left( P_0(x+2) = x + 2 \land (S_0(x+1) + 1) \land S_1(x+1) \land S_2(2) \land S_{x+2}(0) \right) \)
5. \( \forall x \left( P_0(x) + 0 = S_0(x+0) \land S_{x+0}(0) \right) \)
6. \( \forall x \left( P_0(x) + 1 = S_0(x+1) + 0 \land S_0(x+1) \land S_{x+1}(0) \right) \)
7. \( \forall x \left( P_0(x) + 2 = S_0(x+1) + 1 \land S_0(x+2) \land S_{x+2}(0) \land S_0(x+2) + 0 \right) \)
8. \( \forall x \left( P_0(x) + 3 = S_0(x+1) + 2 \land S_0(x+3) \land S_{x+3}(0) \land S_0(x+3) + 1 \right) \)
9. \( \forall x \left( P_0(x) + 4 = S_0(x+1) + 3 \land S_0(x+4) \land S_{x+4}(0) \land S_0(x+4) + 2 \right) \)
10. \( \forall x \forall y \left( P_0(x+y) = x + y \land S_1(y) \land S_2(y) \land S_3(y) \land S_4(y) \land (S_{x+y}(0) + 0) \right) \)
11. \( \forall x \forall y \left( P_0(x+y) = x + y \land S_1(y) \land S_2(y) \land S_3(y) \land (S_{x+y}(0) + 1) \right) \)
12. \( \forall x \forall y \left( P_0(x+y) = x + y \land S_1(y) \land S_2(y) \land S_3(y) \land (S_{x+y}(0) + 2) \right) \)
13. \( \forall x \forall y \left( P_0(x+y) = x + y \land S_1(y) \land S_2(y) \land S_3(y) \land (S_{x+y}(0) + 3) \right) \)
14. \( \forall x \left( P_0(x+y) = x + y \land S_1(y) \land S_2(y) \land S_3(y) \land S_0(y+z+1) \rightarrow x = y \right) \)
15. \( \forall x \forall y \left( P_0(x+y+z) = 2 \cdot x + y \land (P_0(2 \cdot x) + y \land S_1(2 \cdot x + y) \land S_0(2 \cdot x + y + 1)) \rightarrow x = y \right) \)
16. \( \forall x \forall y \forall z \left( P_0(x+y+z) = x + y + z \land 3 \cdot x \land S_1(3 \cdot x + y + z) \land S_0(3 \cdot x + y + z) \rightarrow x = y = z \right) \)
17. \( \forall x \left( P_0(x+y+2) = 2 \cdot x + 2 \cdot y \land 2 \cdot P_0(x+y) \land S_1(2 \cdot x + 2 \cdot y) \land S_2(0(x+y)+1) \rightarrow x = y \right) \)

Hence, allow us to point out that:

i.- Operation \( P_x(y) \) \( (x\)-predecessor of natural number \( y \) \) could be understood (also translated) as a unirical injective operation (function) for which the domain is the set of any and every succesor element \( y \) of \( \mathcal{N}(0 \subset NT-PA) \) and of course of \( Q \) (Robinson sense and intension). \( P_x(y) \) is a well defined and symbolically represented formula for any of the natural number if and only \( S_y(x) \) it is defined element in NT-FOL-PA’s \( (x \in \mathcal{N}(0 \subset NT-PA) \forall x \).

ii.- Operation \( P_x(y) \) identifies each natural number with a unique finite, enumerable and countable set and/or subset of natural numbers \( \mathcal{N}(0 \subset NT-PA) \), and vice versa.

iii.- \( P_x(y) \) has not the significance (sense and intension) of the customary “subtraction operation” nor of the customary currently accepted “subtration logic”.

iv.- \( P_x(y) \) as symbolic representation of the operation “precede” referred to

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zero \((P_s(y) \in \mathcal{N}(0 \subset \text{NT-PA})\), which is the irreducible representation of the meta-exchange index and variable symbolic operation (a tensorial operation) described for the operation “succession” referred to natural number zero, if and only if, \(x=0\)

\[
((S_y(x) = S_z(y) \land P_0(x) \rightarrow x = 0 \land y)
\]

iv.- The natural number symbolically represented by \(P_1(x+1)\), which is a well symbolic represented formula when referred to natural number zero as the natural number symbolically represented by \(P_0(x)\), is both: the “direct successor of \(x^*\) \(S_1(x)\) \((S_{x+1}(0)\) referred to natural number zero)\) and \(x\)-predecessor of natural number \(y\) \((S_x(y))\) Thus, we are capable of stating:

\[
\forall x \forall y \forall n (x, y \in \mathcal{N}(0 \subset \text{NT-PA})) (x = P_1(x+1) \land P_x(y) \land S_0(x) \rightarrow 0 = P_0(0) \land S_0(0))
\]

**HL-2-NT Theorem:** Every natural number \(n\) \((n \in \mathcal{N}(0 \subset \text{NT-PA}))\) properly described by the addition of two natural numbers \((n = x+y)\), both of them belonging to \(\mathcal{N}(0 \subset \text{NT-PA})\), at least accept to be NT-PA represented by the next two NT-FOL-PA identities as: i.- the natural number represented as \((x+y)\)-successor of natural zero, and ii.- the natural number \((x+y)\) predecessor of zero.

\[
\forall x \forall y \forall n (x, y, n \in \mathcal{N}(0 \subset \text{NT-PA}))
\]

\[
((n = P_0(n) \land S_{x+y}(0) \land P_0(x+y) \rightarrow n = x + y) \rightarrow
\]

\[
(n = x + y \Rightarrow n = P_0(n) \land S_{x+y}(0) \land P_0(x+y)) \quad \text{(NT-th. 2)}
\]

**HL-7-NT Definition:** The set conformed by the natural numbers which are the predecessors of a natural number \(x\) \((x \in \mathcal{N}(0 \subset \text{NT-PA}))\) referred to natural number zero, is named (also, called, renamed, nicknamed and defined) “set of predecessors of natural number \(x^*\)” if and only if, the next statement is a wfs and wfs-r statement:

\[
P_{\text{pre} 0}(x) := \{ x \in \mathcal{N}(0 \subset \text{NT-PA}) \mid x = P_0(x) \} \forall x (x \in \mathcal{N}(0 \subset \text{NT-PA})) \quad \text{(def. 7)}
\]

**HL-3-NT Theorem:**

\[
\forall x(x \in \mathcal{N}(0 \subset \text{NT-PA}))(\mathcal{N}(0 \subset \text{NT-PA}) \subseteq P_{\text{pre} 0}(0) \cup \text{Suc}_{x+1}(0)) \quad \text{(NT-th. 3)}
\]

Additionally, the next statement is in NT-FOL-PA an equivalent theorem referred to \(\mathcal{N}(0 \subset \text{PA})\):

\[
\forall x(x \in \mathcal{N}(0 \subset \text{NT-PA}))(\mathcal{N}(0 \subset \text{PA}) \subseteq P_{\text{pre} 0}(0) \cup \text{Suc}_{x}(0)) \quad \text{(NT-th. 3bis)}
\]

**HL-8-NT Definition:** The singleton conformed by the natural number \(y\), which is the
direct predecessor of a natural number $x$ ($x \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})$), is named (also, called, renamed, nicknamed and defined) as "set of direct predecessors of natural number $x$", if and only if the next statement is a wfs and wfs-r statement:

$$\forall x \forall y \ (x, y \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}))$$

$$\text{Pred}(x) := \{ x \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}) \mid x = y+1 \land P_0(x) \land P_0(y) + 1 \} \forall x \forall y \quad \text{(def. 8)}$$

Hence the next statement is a wfs and a wfs-r:

$$\forall x (x \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}))$$

$$\left(\text{Pred}(x) \subset \text{Pre}_0(x) \subset \text{Suc}_x(0) \rightarrow (\text{Suc}_x(0) \subseteq \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})) \forall x \right)$$

**HL-4-NT Theorem:** The $\mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})$ set, which has been renamed (named called, defined, etc) as $\text{Suc}_x(0)$, can also be conformed by the union of every singleton $\text{Pred}_0(x+1)$ which, in turn, is conformed by one defined and symbolized by the natural number $x$ identity referred to natural number zero

$$\bigcup \left[ \text{Pred}_0(x+1) \right]_{y \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})} \subseteq \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}) \quad \text{(NT-th. 4)}$$

$$\bigcup \left[ \text{Pred}_0(x) \right]_{y \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})} \subseteq \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}) \quad \text{(NT-th. 4 bis)}$$

**Hence, taking in account the nomenclature and notation of the FOL-PA inductions axioms and the NT-SF&L tensorial formulation; NT-th. 4 and 4 bis can be rewritten as:**

$$\forall x \forall y \forall n \ (x, y_1 ... y_n, y_{n+1} \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}), n \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}) \cap \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}))$$

$$x+1 = y_{n+1}$$

$$\bigcup \left[ \text{Pred}_0(x) \right]_{y \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})} \subseteq \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}}) \quad \text{(NT-th. 4 tris)}$$

$\text{Pred}_0(x+1)$, as well as $\text{Pred}_0(x)$, represents "a partition member to member" (only singletons are involved; hence, also, "a partition singleton to singleton") of a set which is "enumerable and countable" by this NT-methodology (Cantor’s, von Neumann’s and Gödel’s numeration and counting methodologies imply different sense and intension concepts for enumeration, numeration and countability), $\mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})$.

**HL-9-NT Definition:** The singletons conformed by the natural number $x$ ($x \in \mathbb{N}(0^\mathbb{C}_{\text{NT-PA}})$) as its unique member, which is the element of the intersection of $\text{Suc}_x(0)$ and $\text{Pred}_0(x)$ sets, will be named (called, defined) “**primordial of natural number $x$ primordial set**”, if and only if the next statement is a wfs and wfs-r statement:

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That is say in NT FS&L; Please, take in account next NT-theorem:

$$
\forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA})) \quad \forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{P}_{\text{Pre}_0}(x)) \quad (\text{def. 9})
$$

Hence;

$$
\forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA}))
(\# - \mathcal{P}_{\text{Pre}_0}(x) = 1 \land \# - \mathcal{P}_{\text{Pre}_0}(x+1) \rightarrow (1 \land (x+1) \in \mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{N}(0\subset\text{NT-PA}))
$$

**HL-5-NT Theorem:** $\mathcal{N}(0\subset\text{NT-PA})$ set, which has been renamed (named, called and, defined) as $\text{Suc}_x(0)$ and the set resulting of the union of every direct predecessor of a natural number $x$, is equal (identity), in turns, to the set conformed by the union of $\mathcal{P}_{\text{Pre}_0}(x)$ primordial set of every natural number.

$$
\bigcup_{x \in \mathcal{N}(0\subset\text{NT-PA})} [\mathcal{P}_{\text{Pre}_0}(x)] \subseteq \mathcal{N}(0\subset\text{NT-PA}) \quad \text{(NT-th. 5)}
$$

**HL-10-NT Definition:** The singleton conformed by the natural number cero as unique member, which is the direct predecessor of natural number 1 ($\forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA}), \mathcal{P}_{\text{Pre}_0}(x) = 1 \land (x \in \mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{S}_{\text{Suc}_0}(x))$), is named (also, called, renamed, nicknamed and defined) "set of predecessors of natural number 1", if and only if the next statement is a wfs and wfs-r statement:

$$
\mathcal{P}_{\text{Pre}_0}(1) := \{0 \mid 0 \in \mathcal{N}(0\subset\text{NT-PA}) \} \forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA})) \quad \text{(def. 10)}
$$

And, hence:

$$
\# - \mathcal{P}_{\text{Pre}_0}(1) = 1
$$

Please, take in account next NT-theorem:

$$
\forall x \ (x \in \mathcal{N}(0\subset\text{NT-PA}))(\mathcal{P}_{\text{Pre}_0}(1) = \mathcal{P}_{\text{Pre}_0}(x) \rightarrow x = 0 \land x \in \mathcal{S}_{\text{Suc}_0}(x) \cap \mathcal{P}_{\text{Pre}_0}(x))
$$

That is say in NT FS&L:

$$
\mathcal{P}_{\text{Pre}_0}(1) = \mathcal{P}_{\text{Pre}_0}(0)
$$

**HL-6-NT Theorem:**

$$
\mathcal{N}(0\subset\text{NT-PA}) \cup \mathcal{P}_{\text{Pre}_0}(0) \subseteq \mathcal{N}(0\subset\text{NT-PA}) \quad \text{(NT-th. 6)}
$$

(Entailment: NT-PA axioms and definitions:

$$
\mathcal{N}(0\subset\text{NT-PA}) := \{S_0(1), S_1(S_0(1)), S_1(S_1(S_0(1))), \ldots\} \quad \Rightarrow \quad \mathcal{N}(0\subset\text{NT-PA}) \cap \mathcal{P}_{\text{Pre}_0}(0)
$$

**HL-7-NT Theorem:** $\# - \mathcal{P}_{\text{Pre}_0}(x)$ is a first order logic universal and unary natural
number enumeration and counting operation (remember, also the function in first order typed category language) for every \( \text{Suc}_x(0) \) and for every subset or equivalence class (quotient class) arising from every partition of both \( \mathbb{N}(0 \not\subset NT\text{-}PA) \) and \( \mathbb{N}(0 \subset NT\text{-}PA) \).

\[
\forall x \ (x \in \mathbb{N}(0 \subset NT\text{-}PA)) \ (x \in \mathcal{P}\mathcal{N}_0(x) \rightarrow \ #\cdot\text{Pre}_0(x) = (x+1) \wedge S_1(x) \wedge S_3(1) \wedge P_0(x+1) \wedge P_1(x+2) \wedge P_1(x+1+1) \wedge \cdots \wedge P_x((x+1)+x) \wedge \cdots \wedge P_x((x+x)+1))
\]

(NT-th. 7)

\[
\{ \ #\cdot\text{Pre}_0(x) \} = \{x+1 \in \mathbb{N}(0 \not\subset NT\text{-}PA) \cup \mathcal{P}\mathcal{N}_0(0) \mid x \in \mathcal{P}\mathcal{N}_0(x) \} \forall x \ (x \in \mathbb{N}(0 \not\subset NT\text{-}PA))
\]

(NT-th. 7 bis; entailment from NT-th. 3, 6 & 7)


Please, allow us to consider \( \forall x \forall y \ (x, y \in \mathbb{N}(0 \subset NT\text{-}PA)) \)

A. On natural numbers as elements of \( \mathbb{N}(0 \subset NT\text{-}PA) \) referred to natural number zero \( (0) \) successor identity.

\[
\text{Suc}_{x+y}(0) \wedge \text{Suc}_{y+x}(0) = (x+y) \wedge (y+x)
\]

\[
\text{Suc}_{x+y}(0) = (x+y) \wedge (y+x) \rightarrow \text{Suc}_{x+y}(0) = \text{Suc}_x(0) \wedge \text{Suc}_y(0)
\]

\[
\text{Suc}_x(0) = y \wedge \text{Suc}_y(0) \rightarrow x = y,
\]

\[
\text{Suc}_{x+y}(0) = P_0(x+x) \rightarrow x = y
\]

B. On \( \mathbb{N}(0 \subset NT\text{-}PA) \) natural number set as a collectivity of natural numbers referred to natural number zero \( (0) \).

\[
\text{Suc}_{x+y}(0) \subseteq \text{Suc}_{y+x}(0) \wedge \text{Suc}_y(x) \wedge \text{Suc}_x(x)
\]

\[
\mathcal{P}\mathcal{N}_0(x) \subseteq \mathcal{P}\mathcal{N}_0(y) \rightarrow x = y
\]

\[
\mathcal{P}\mathcal{N}_0(x+x) \subseteq \mathcal{P}\mathcal{N}_0(y+x) \wedge \mathcal{P}\mathcal{N}_0(x+y) \wedge \mathcal{P}\mathcal{N}_0(y+y) \rightarrow x = y
\]
Thus, let us redefine and enunciate the next NT-methodological theorem:

\[ \forall x \forall y \ (x, y \in \mathcal{N}(0 \subset \text{NT-PA})) \]

\[ \text{Suc}_y(x) := \{ (x+y)+0, (x+y)+1, (x+y)+2, \ldots \} \]

\[ \land \]

\[ \text{Suc}_x(y) := \{ (y+x)+0, (y+x)+1, (y+x)+1, \ldots \} \]

\[ \land \]

\[ \text{Suc}_0(x+y) := \{ (x+y)+0, (x+y)+1, (x+y)+2, \ldots \}; \]

\[ \rightarrow \]

\[ \text{Pre}_0(x) = \{ x, ..., 3, 2, 1, 0 \}, \forall x (x \in \mathcal{N}(0 \subset \text{NT-PA})) \]

\[ \# \cdot \text{Pre}_0(x) = x+1 \]

\[ \land \]

\[ \mathcal{P} \mathcal{N}_0(x) = \{ x \mid x \in \mathcal{N}(0 \subset \text{NT-PA}) \} \] & \[ \mathcal{P} \mathcal{N}_0(y) := \{ y \mid y \in \mathcal{N}(0 \subset \text{NT-PA}) \} \]

\[ \land \]

\[ \mathcal{P} \mathcal{N}_0(x+y) = \{ x+y \mid x+y \in \mathcal{N}(0 \subset \text{NT-PA}) \} \]

\[ \land \]

\[ \# \cdot \mathcal{P} \mathcal{N}_0(x+y) = 1 \]

From this point onwards we are able to define in the first order NT-PA language:

**HL-10-NT Definition:** Every "presentation" (also, representation) of:

**A.** The set of successors of a number \( x \) **referred to zero** (\( \forall x, x \in \mathcal{N}(0 \subset \text{NT-PA}) \)) \( \text{Suc}_x(0) \), will be intended named (also defined renamed, called,...) as a **covariant representation of the successors of a number \( x \) referred to zero**,

\[ \text{Suc}^\alpha_x(0) \]

if and only if, we represent such set in the sense and intension that on its representation by extension, each member (natural number) is followed by its direct successor:

\[ \text{Suc}^\alpha_x(0) := \{ S_x(0), (x)+1, (x+1)+1, (x+1+1)+1, \ldots \} \]

\[ \forall x \ (x \in \mathcal{N}(0 \subset \text{NT-PA}) \& \ x = S_x(0)) \]

(def. 10)

**B.** The set of successors of a number \( x \) **referred to zero** (\( \forall x (x \in \mathcal{N}(0 \subset \text{NT-PA}) \)) \( \text{Suc}_x(0) \), will be intended named (also defined renamed, called...) as a **contravariant representation of the successors of a number \( x \) referred to zero**,

\[ \text{Suc}^\beta_x(0) \]

if and only if, we represent such set in the sense and intension that on its representation by extension, each member (natural number) is followed by its direct
predecessor:

\[ \text{Suc}_x^\beta(0) := \{ P_x(x), \ldots, P_0(2), P_0(1), P_0(0) \} \]
\[ \forall x \ (x \in \mathcal{N}_{(0 \subseteq \text{PA})} \land x = P_0(x) \land S_0(x)) \]

(def. 10bis)

Hence, for example, both previous definitions a.2 of \( \mathcal{N}_{(0 \subseteq \text{PA})} \) and b.2 of \( \mathcal{N}(0 \subseteq \text{PA}) \), respectively, have to be considered as covariant presentations of both sets, \( \mathcal{N}^\alpha(0 \subseteq \text{NT}-\text{PA}) \) and \( \mathcal{N}^\beta(0 \subseteq \text{NT}-\text{PA}) \), respectively. The corresponding contravariant representations and definitions of \( \mathcal{N}^\beta(0 \subseteq \text{NT}-\text{PA}) \) and \( \mathcal{N}^\alpha(0 \subseteq \text{NT}-\text{PA}) \), respectively, are: \(^{62}\)

i.- \( \forall x \ (x \in \mathcal{N}(0 \subseteq \text{NT}-\text{PA})) \)

\[ \mathcal{N}^\beta(0 \subseteq \text{NT}-\text{PA}) := \{ P_x(x), \ldots, P_1(x), P_0(x) \} \] (a.2β)

ii.- \( \forall x \ (x \in \mathcal{N}^\beta(0 \subseteq \text{NT}-\text{PA}) \}

\[ \mathcal{N}(0 \subseteq \text{NT}-\text{PA}) := \{ P_x(x+1), \ldots, P_{x+1}(x+1), P_1(x+1), P_0(x+1), \} \] (b.2β)

HL-8-NT Theorem: \(^{63}\)

\[ \forall x \ (x \in \mathcal{N}(0 \subseteq \text{NT}-\text{PA}) \} (\text{Suc}_x^\alpha(0) \subseteq \text{Pre}_0^\alpha(x) \land \text{Suc}_x^\beta(x) \subseteq \text{Pre}_0^\beta(x) \rightarrow \]
\[ x+1 = \#\text{Suc}_x^\alpha(0) \land \#\text{Suc}_x^\beta(x) \land \#\text{Pre}_0^\beta(x) \land \#\text{Pre}_0^\beta(x) \}

[For example, for the case \( x = 11 \) \( (x \in \mathcal{N}^\alpha(0 \subseteq \text{NT}-\text{PA}) \land \mathcal{N}^\beta(0 \subseteq \text{NT}-\text{PA}) \)]

\[ \text{Suc}_x^\alpha(11)(0) := \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \} \]
\[ \text{Suc}_x^\beta(11)(0) := \{ 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \} \]
\[ \text{Pre}_0^\alpha(11)(0) := \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \} \]
\[ \text{Pre}_0^\beta(11)(0) := \{ 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \} \]

Then:

\[ \text{Suc}_{11}(0) = \text{Suc}_{11}(0) = \text{Pre}_{11}(0) \land \text{Suc}_{11}(0) = \text{Pre}_{11}(0) = \text{Pre}_{11}(0) \]
\[ \#\text{Suc}_{11}(0) \land \#\text{Suc}_{11}(0) \land \#\text{Pre}_0^\beta(x) \land \#\text{Pre}_0^\beta(x) = 12 \]

HL-9-NT Theorem (set of lemmas): The next statements sustain entailment in first order NT-FS&L from PA axiomatic for the natural number identity concept referred to natural number zero identity:

\[ \forall x \ (x \in \mathcal{N}(0 \subseteq \text{NT}-\text{PA}) \land (x \in \text{Suc}_x^\alpha(0) \land \text{Suc}_x^\beta(0) \land \text{Pre}_0^\alpha(x) \land \text{Pre}_0^\beta(x)) \]

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Let us consider the next currently accepted definitions in order to adopt Number Theory (Typed Category Theory) nomenclature for the set of even natural numbers (also, the even operation) and the set operation of odd natural numbers.
\( \forall x \ (x \in \mathbb{N}(0 \subset \text{NT-PA})) \)

\( \mathcal{E} \text{ven}(x) := \{ \ 2 \cdot x \ | \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \} \), \( \forall x \) (def. 0 Parity)

\( \mathcal{O}dd(x) := \{ \ 2 \cdot x + 1 \ | \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \} \), \( \forall x \) (def. 0 Parity)

**HLParity-1-NT Even Definition:** The set of successors of a number \( x \) referred to zero (\( \forall x, \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \)) \( \text{Suc}(0) \), will be intended named (also defined renamed, called...) as the set even successors referred to zero of a natural number \( x \).

\( \mathcal{E} \text{ven}(x) \lor \mathcal{E} \text{ven}(\text{Suc}(0)) \rightarrow \mathcal{E} \text{ven}(x) = \mathcal{E} \text{ven}(\text{Suc}(0)) \)

if and only if the next statement is a wfs and wfs-r in firs order NT-FS&L:

\( \mathcal{E} \text{ven}(\text{Suc}(0)) := \{ \ x+x \ \in \ \text{Suc}(0) \ | \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \} \), \( \forall x \) (def-P. 1)

Please allow us to remind the reader that **NT-th. 10**, only holds true for every couple of natural numbers \( x \) and \( y \) (\( \forall x \forall y \ (x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \)), if and only if, \( x = y \).

**HLParity-2-NT Odd Definition:** \( \forall x \ (x \ \in \ \mathbb{N}(0 \subset \text{NT-PA})) \) \( \text{Suc}(0) \), will be (intended) named (also defined renamed, called...) as the set odd of successors referred to zero of a natural number \( x \).

\( \mathcal{O}dd(x) \lor \mathcal{O}dd(\text{Suc}(0)) \rightarrow \mathcal{O}dd(x) \subseteq \mathcal{O}dd(\text{Suc}(0)) \)

if and only if, the next definition is a wfs and wfs-r in firs order NT-FS&L statement:

\( \mathcal{O}dd(\text{Suc}(0)) := \{ \ (x+x)+1 \ \in \ \text{Suc}(0) \ | \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \} \), \( \forall x \) (def-P. 2)

That statement is equivalent to:

\( \mathcal{O}dd(\text{Suc}(0)) := \{ \ x \ \in \ \text{Suc}(0) \ | \ x \ \in \ \mathbb{N}(0 \subset \text{NT-PA}) \} \), \( \forall x \) (def-P 2bis)

Hence, the parity of the set of “predecessors” and “successors” of natural number “\( x \)” will be represented, considered and defined as follows in terms of the “parity” of every element of \( \mathcal{E} \text{ven}(x) \) and \( \mathcal{O}dd(x) \) sets:

i.- “\( e-Px(0) \)” is in turn an **even-successor of \( \mathcal{E} \text{ven}(\text{Suc}(0)) \)**, if and only if, when the natural number represented and identified by \( x \) is an **even natural number**.
ii.- "e-S_{y}(x)" is an even-successor of Even(Suc(x)), when referred to zero is S_{x+y}(0), if and only if, x+y is an even natural number when both, x and y, are each of them even or odd natural numbers.

Additionally, each of the elements of Odd(x) will be represented as:

i.- o-S_{x}(0), is an odd successor of Odd(Suc(x)), when x of symbol o-S_{x}(0), is representing a odd natural number.

ii.- o-S_{y}(x), odd-y-successor of x, when referred to zero is S_{x+y}(0), if and only if: x+y is an even natural number when x is an even natural number and y is an odd natural number, or vice versa.

ii.- o-P_{x}(0) is an odd predecessor of Odd(Suc(x)), when x of symbol o-P_{x}(0) is an odd natural number.

**HLParity-3-NT Definition:**

A primordial \( P_{\mathbb{N}}(x) \) set of a natural number \( x \) \( (x \in \mathbb{N}(0\subset\text{NT-PA})) \) is defined (called, renamed, nicknamed):

\[
\begin{align*}
A. & \quad \text{"Even primordial of } x", \\
& e-P_{\mathbb{N}}(x) := \{ x \in \text{Even}(x) \land P_{\mathbb{N}}(x) \mid x \in \mathbb{N}(0\subset\text{NT-PA}) \}, \forall x \text{ (def-P. 3)} \\
& e-P_{\mathbb{N}}(x) := \{ x \in \text{Even}(x) \cap P_{\mathbb{N}}(x) \mid x \in \mathbb{N}(0\subset\text{NT-PA}) \}, \forall x \text{ (def-P. 3bis)} \\
B. & \quad \text{"Odd primordial of } x" \\
& o-P_{\mathbb{N}}(x) := \{ x \in \text{Odd}(x) \land P_{\mathbb{N}}(0) \mid x \in \mathbb{N}(0\subset\text{NT-PA}) \}, \forall x \text{ (def-P. 3-tris)} \\
& o-P_{\mathbb{N}}(x) := \{ x \in \text{Odd}(x) \cap P_{\mathbb{N}}(0) \mid x \in \mathbb{N}(0\subset\text{NT-PA}) \}, \forall x \text{ (def-P. 3-tetra)}
\end{align*}
\]

Therefore, according to the theorem NT-th. 5, the next wfs and wfs-r statement is a new definition of the set \( \mathbb{N}(0\subset\text{NT-Pa}) \):

\[
\mathbb{N}(0\subset\text{PA}) := \left[ \{ e-P_{\mathbb{N}}(x) \} \cup \{ o-P_{\mathbb{N}}(0) \} \right] \forall x \in \mathbb{N}(0\subset\text{NT-Pa})
\]

Hence,

1. In a covariant presentation of \( \mathbb{N}(0\subset\text{NT-Pa}) \) by means of \( P_{\mathbb{N}}(x) \) referred to zero \( (0 \in e-P_{\mathbb{N}}(0)) \):

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\[ N^{\alpha}(0 \subset \text{NT-PA}) := \{ P_{00}(0), P_{01}(1), P_{02}(2), P_{03}(3), P_{04}(4), \ldots, P_{0x}(x) \} \]
\[ := \{ P_{01}(0), P_{02}(0), P_{03}(0), P_{04}(0), \ldots, P_{0x}(0) \} \]
\[ := \{ 0, 1, 2, 3, 4, \ldots, x \} \]

2. In a contravariant representation of \( N(0 \subset \text{NT-PA}) \) by means of \( P_{00}(x) \) referred to zero \( (0 \in e \cdot P_{00}(0)) \):

\[ N^{\beta}(0 \subset \text{NT-PA}) := \{ P_{00}(x), \ldots, P_{04}(4), P_{03}(3), P_{02}(2), P_{01}(1), P_{00}(0) \} \]
\[ := \{ P_{x1}(x), P_{x2}(0), P_{x3}(0), P_{x4}(0), P_{x5}(0), P_{00}(0) \} \]
\[ := \{ x, \ldots, 4, 3, 2, 1, 0 \} \]

Only,

\[ P_{00}^{\alpha}(0) = P_{00}^{\beta}(0) \land P_{0x}^{\alpha}(0) = P_{0x}^{\beta}(0) \Rightarrow \]
\[ P_{0x}^{\alpha}(x) = P_{0x}^{\beta}(x), \forall x (x \in N(0 \subset \text{NT-PA})) \]

Then we are able to state:

\[ \forall x, (x \in N(0 \subset \text{NT-PA})) \]
\[ P_{0x}^{\alpha}(x) = P_{0x}^{\beta}(x) \] has the parity of \( x (x \in N(0 \subset \text{NT-PA})) \), if and only if,
\[ P_{0x}^{\alpha}(x+1) = P_{0x}^{\beta}(x+1) \] has parity of \( x+1 \), which is the 1-successor of \( x \) (its direct successor), and vice versa

**HLParity-1-NT Theorem:**

\[ \forall x, (x \in N(0 \subset \text{NT-PA})) \]
\[ e \cdot P_{00}(x+x+0) \subseteq \text{Even}(Succ_{x+x}(0)) \land \text{Even}(Succ_{0}(x+x)) \]
\[ \land \]
\[ o \cdot P_{00}(x+x+1) \subseteq \text{Odd}(Succ_{x+x}(1)) \land \text{Odd}(Succ_{0}(x+x+1)) \land \text{Odd}(Succ_{1}(x+x)+0) \]

Additionally:

1. \( \forall x, (x \in N(0 \subset \text{NT-PA})) \)
\[ 1 = \# \cdot P_{00}(x) \land \# \cdot P_{x+1}(0) \land \# \cdot P_{x+2}(0) \land \ldots \land \# \cdot P_{x+x}(0) \land \ldots \]
\[ 2 = \# \cdot P_{00}(x) + \# \cdot P_{00}(x+1) \]
\[ 3 = \# \cdot P_{00}(x) + \# \cdot P_{00}(x+1) + \# \cdot P_{00}(x+2) \]
2. In first order NT-FS&L,
\[ \forall x \in N(0 \subset NT-PA) \]
\[ N^\alpha(0 \subset NT-PA) = \{ 0, 1, 2, 3, 4, ..., x + 0, x + 1, ..., e-(x+x) \} \] \hspace{1cm} (a.1);

\[ \forall x \in N(0 \notin NT-PA) \]
\[ N^\alpha(0 \notin NT-PA) = \{ 1, 2, 3, 4, 5, ..., x + 1, x + 2, ..., o-(x+x+1) \} \] \hspace{1cm} (b.1);

Hence, the next statement is a theorem:

\[ \forall x \in N(0 \subset NT-PA) \]
\[ e^\cdot PN_0((x+x)+0) \text{ is an even primordial} \]
\[ \& \]
\[ o^\cdot PN_0((x+x)+1) \text{ is an odd primordial} \]
(NT-P-th. 0)

The distribution of even and odd natural numbers (e-\(x\) and o-\(x\), respectively) in NLSF&L PA according to the complete partition named as “parity” is: \(^{68}\)

\[ \forall x \in N(0 \subset NT-PA) \]
\[ \begin{array}{c}
N^\alpha(0 \subset NT-PA) = \{ 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, ..., \} \\
N^\alpha(0 \subset NT-PA) = \{ S_0(0), S_1(0), S_2(0), S_3(0), S_4(0), ..., \} \wedge \\
\{ S_0(0), S_0(1), S_0(2), S_0(3), S_0(4), ..., \} \wedge \\
\{ P_0(0), P_0(1), P_0(2), P_0(3), P_0(4), ..., \} \Rightarrow \\
N^\alpha(0 \subset NT-PA) = \{ (0+0), (0+0)+1), (1+1), ((1+1)+1), (2+2), ((2+2)+1), \\
(3+3), ((3+3)+1), (4+4), ((4+4)+1), ..., \} \Rightarrow \\
\end{array} \]

\[ \begin{array}{c}
N^\alpha(0 \subset NT-PA) = \{ ..., ((x)+(x)), (x+x)+1), ..., \\
..., ((x+x)+(x+x)), ((x+x+x+x)+1), ..., \\
..., ((x+x+x+x)+(x+x+x)), ((x+x+x+x+x+x)+1), ..., \} \Rightarrow \\
N^\alpha(0 \subset NT-PA) = \{ ((1\cdot x)+(1\cdot x)), (2\cdot x)+1), ..., \\
..., ((2\cdot x)+(2\cdot x)), (4\cdot x)+1), ..., \\
..., ((3\cdot x)+(3\cdot x)), (6\cdot x)+1), ..., \}
\end{array} \]
\[\{ \text{even}, \text{odd}, \text{even}, \text{odd}, \text{even}, \text{odd}, \text{even}, \text{odd}, \ldots, \ldots \} \]

\(\mathcal{N}^\alpha(0 \subset \text{NT-PA}) = \{\text{e} \cdot (0+0), \text{o} \cdot (0+0)+1,\)
\(\text{e} \cdot (1+1), \text{o} \cdot ((1+1)+1),\)
\(\text{e} \cdot (2+2), \text{o} \cdot ((2+2)+1),\)
\(\text{e} \cdot (3+3), \text{o} \cdot ((3+3)+1), \ldots, \ldots \} \Rightarrow \)

\(\mathcal{N}^\alpha(0 \subset \text{NT-PA}) := \{\text{e} \cdot (0), \text{o} \cdot (1), \text{e} \cdot (2), \text{o} \cdot (3), \text{e} \cdot (4), \text{o} \cdot (5), \text{e} \cdot (6), \text{o} \cdot (7), \ldots, \ldots \} \wedge \)

\(\mathcal{N}^\alpha(0 \subset \text{NT-PA}) = \{\text{e} \cdot S_0(0), \text{o} \cdot S_0(1), \text{e} \cdot S_0(2), \text{o} \cdot S_0(3), \text{e} \cdot S_0(4), \ldots, \ldots \} \wedge \)
\(\{\text{e} \cdot P_0(0), \text{o} \cdot P_0(1), \text{e} \cdot P_0(2), \text{o} \cdot P_0(3), \text{e} \cdot P_0(4), \ldots, \ldots \} \wedge \)
\(\{\text{e} \cdot S_0(0), \text{o} \cdot S_0(1), \text{e} \cdot S_0(2), \text{o} \cdot S_0(3), \text{e} \cdot S_0(4), \ldots, \ldots \} \wedge \)

\(\mathcal{N}^\alpha(0 \subset \text{NT-PA}) = \{\text{even}, \text{odd}, \text{even}, \text{odd}, \text{even}, \text{odd}, \text{even}, \text{odd}, \ldots, \ldots \} \wedge \)

\(\mathcal{N}^\alpha(0 \subset \text{NT-PA}) \cap \mathcal{N}(0 \subset \text{NT-PA}) \cap \mathcal{N}(0 \subset \text{NT-P}) \)

\(\text{HLParity-1-NT Theorem:}\)

\(\forall x \forall y \left( x, y \in \mathcal{N}(0 \subset \text{NT-PA}) \right) \)
\[\mathcal{P}N_0(x) = \mathcal{P}N_0(y) \rightarrow y = x \]
\&

\(\left( x \in \text{Even}(Suc_0(x)) \Rightarrow y \in \text{Odd}(Suc_0(x)) \rightarrow (x \in \text{Odd}(Suc_0(x)) \Rightarrow y \in \text{Even}(Suc_0(x))) \right) \vee \)

\(\left( x \in \text{Odd}(Suc_0(x)) \Rightarrow y \in \text{Even}(Suc_0(x)) \rightarrow (x \in \text{Even}(Suc_0(x)) \Rightarrow y \in \text{Odd}(Suc_0(x))) \right) \)
\&

\[\left( \sum_{x} [ \# \cdot \mathcal{P}N_0(x) ] = y \rightarrow y = x + 1, y \in \mathcal{N}(0 \subset \text{NT-PA}) \cap \mathcal{N}(0 \subset \text{NT-P}) \right) \]

\((\text{NT-P-th. 1})\)

\(\text{HLParity-2-NT Theorem:}\)

\(\forall x \forall y \forall \emptyset \left( x, y \in \mathcal{N}(0 \subset \text{NT-PA}), \emptyset \in \Theta \right) \)
\&

\(\forall x \forall y \left( x, y \in \text{Suc}_0(0) \right) \land \text{Suc}_0(x) \land \text{Pre}_0(x) \)

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\[ \textbf{Even}(\text{Suc}_0(0)) = \{0 \mid 0 \in \mathcal{N}(0 \cap \text{NT-PA}) \} \quad \text{and} \quad \textbf{Odd}(\text{Suc}_0(0)) \subseteq \{\emptyset \mid \emptyset \in \Theta \land \emptyset \notin \mathcal{N}(0 \cap \text{NT-PA})\}, \]

Then:

1. \((\forall x \in \mathcal{N}(0 \cap \text{NT-PA}))\)
   
   \[
   \begin{align*}
   x \in \textbf{Odd}(x) & \rightarrow \# \cdot \textbf{Even}(\text{Suc}_x(0)) = \# \cdot \textbf{Odd}(\text{Suc}_x(0)) + 0 \\
   x \in \textbf{Even}(x) & \rightarrow \# \cdot \textbf{Even}(\text{Suc}_x(0)) = \# \cdot \textbf{Odd}(\text{Suc}_x(0)) + 1
   \end{align*}
   \]

   (NT-P-th. 2)

2. \((\forall x \in \mathcal{N}(0 \cap \text{NT-PA})) \Rightarrow (S_x(x+x) \land S_{x+x}(x) = S_0(x+x+x) \land S_{x+x+x}(0) \in \text{Suc}_0(x))\)

   \[
   \begin{align*}
   S_0((x+x)+x) \land S_{x+x+x}(0) \in \textbf{Odd}(\text{Suc}_0(x+x+x)) & \rightarrow x \in \textbf{Odd}(\text{Suc}_0(x)) \\
   \lor \quad S_0(x+x+x) \land S_{x+x+x}(0) \in \textbf{Even}(\text{Suc}_0(x+x+x)) & \rightarrow x \in \textbf{Even}(\text{Suc}_0(x))
   \end{align*}
   \]

   (NT-P-th. 2 bis)

3. \((\forall x \forall y \ (x, y \in \mathcal{N}(0 \cap \text{NT-PA})))\)

   \[
   \begin{align*}
   (y = x + 1 \in \mathcal{N}(0 \cap \text{NT-PA}) & \Rightarrow e \cdot S_0(y+y) \in \textbf{Even}(\text{Suc}_0(y)) \rightarrow \\
   S_{(x+2)}(x+0) & \land S_{(x+1)}(x+1) \land S_{(x)}(x+2) \in \textbf{Even}(\text{Suc}_0(x)) \\
   & \land \quad \{P_0(x+x+2) = P_0((x+1)+(x+1)) \land S_0((x+1)+(x+1)) \land S_{(x+1)}(x+1) \in \textbf{Even}(\text{Suc}_0(x))\}
   \end{align*}
   \]

   (NT-P-th. 3 tris)

Hence, we are able to state:70

\[(\forall x \forall y \ (x, y \in \mathcal{N}(0 \cap \text{NT-PA})))\]

\[
\begin{align*}
((S_x(0) + S_0(0)) = ((x + x) \land (x \cdot y + x)) & \rightarrow x = S_x(0) \land y = S_0(0) \land 0) \\
& \land \quad ((S_x(0) + S_0(0)) = ((x + x) \land (x \cdot y + x)) \rightarrow x = S_x(0) \land y = S_1(0) \land 1) \\
& \land \quad \left(\left((S_x(0) + S_0(0) \land S_x(0) \cdot S_{y+1}(0) \land S_x(0) \cdot S_0(y+1) \land S_x(0) \cdot S_y(1)) = \right) \\
& \quad \left(\left((x + x) \land (x \cdot y + x) \land (x \cdot (y + 1))\right) \rightarrow \\
& \quad x = S_x(0) \land y + 1 = S_1(0) + S_1(0) \land 1 + 1 \Rightarrow y = S_2(0) \land 2\right) \\
& \quad x = S_0(x) \land y = 1 \land S_1(0) \lor x = S_0(x) \land y + 1 = 2 \land S_1(1)
\end{align*}
\]
Thus,

\[ x \Rightarrow x\cdot y + x\cdot (y+1) \Rightarrow x \cdot S_1(x) \land x + x \Rightarrow S_2(0) \cdot (S_1(0) + S_1(0)) = S_2(0) \cdot S_2(0) = S_2(0) \cdot (2) \]

\[
\begin{align*}
0 & \Rightarrow 0 \cdot 0 + 0 \cdot (y+1) \Rightarrow 0 \cdot S_1(x) \land 0 + 0 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (0) \\
1 & \Rightarrow 1 \cdot y + 1 \cdot (y+1) \Rightarrow 1 \cdot S_1(x) \land 1 + 1 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (1) \\
2 & \Rightarrow 2 \cdot y + 2 \cdot (y+1) \Rightarrow 2 \cdot S_1(x) \land 2 + 2 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (2) \\
3 & \Rightarrow 3 \cdot y + 3 \cdot (y+1) \Rightarrow 3 \cdot S_1(x) \land 3 + 3 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (3) \\
4 & \Rightarrow 4 \cdot y + 4 \cdot (y+1) \Rightarrow 4 \cdot S_1(x) \land 4 + 4 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (4) \\
5 & \Rightarrow 5 \cdot y + 5 \cdot (y+1) \Rightarrow 5 \cdot S_1(x) \land 5 + 5 \Rightarrow S_2(0) \cdot (S_1(1) + S_1(0)) = S_2(0) \cdot S_2(0) = (2) \cdot (5) \\
\end{align*}
\]

\[ n \Rightarrow x \cdot y + x \cdot (y+1) \Rightarrow x \cdot S_1(x) \land x + x \Rightarrow P_0(0) \cdot (P_0(1) + P_0(1)) = P_0(0) \cdot P_0(0) = P_0(0) \cdot (2) \]

and, then in NT-FS&L, next logical substitution set operations are allowed:

\[
\begin{align*}
0 & \Rightarrow 0 \cdot y + 0 \cdot (y+1) \Rightarrow (0 \cdot (x+1)) \land 0 + 0 \Rightarrow (0 \cdot (1+1)) = (0 \cdot 2) = (2 \cdot 0) & \\
1 & \Rightarrow 1 \cdot y + 1 \cdot (y+1) \Rightarrow (1 \cdot (x+1)) \land 1 + 1 \Rightarrow (1 \cdot (1+1)) = (1 \cdot 2) = (2 \cdot 1) & \\
2 & \Rightarrow 2 \cdot y + 2 \cdot (y+1) \Rightarrow (2 \cdot (x+1)) \land 2 + 2 \Rightarrow (2 \cdot (1+1)) = (2 \cdot 2) = (2 \cdot 2) & \\
3 & \Rightarrow 3 \cdot y + 3 \cdot (y+1) \Rightarrow (3 \cdot (x+1)) \land 3 + 3 \Rightarrow (3 \cdot (1+1)) = (3 \cdot 2) = (2 \cdot 3) & \\
4 & \Rightarrow 4 \cdot y + 4 \cdot (y+1) \Rightarrow (4 \cdot (x+1)) \land 4 + 4 \Rightarrow (4 \cdot (1+1)) = (4 \cdot 2) = (2 \cdot 4) & \\
5 & \Rightarrow 5 \cdot y + 5 \cdot (y+1) \Rightarrow (5 \cdot (x+1)) \land 5 + 5 \Rightarrow (5 \cdot (1+1)) = (5 \cdot 2) = (2 \cdot 5) & \\
\end{align*}
\]

\[
\begin{align*}
\sum_{x=0}^{n} \left[ (x) \cdot S_2(0) \cdot (1+1) \right] = (S_2(0) \cdot 2) = (2 \cdot S_2(0)) & \\
\sum_{x=0}^{n} \left[ (x) \cdot P_0(x+1) \cdot (1+1) \right] = (P_0(x) \cdot 2) = (2 \cdot P_0(x))
\end{align*}
\]

Then, the next w.f. formula in NT-FS&L is representing the accumulated addition of \( n+1 \) natural numbers referred to zero, which in turn, are both: the elements successors of natural number zero \( (0) \) and the predecessor of a natural number \( n \):

\[
\begin{align*}
x & = n \\
\sum_{x=0}^{n} \left[ (x) \right] & = \left( S_0(0) + S_0(1) + \ldots + S_0(n) \right) \land \left( P_0(0) + P_0(1) + \ldots + P_0(n) \right) \\
\sum_{x=0}^{n} \left[ (x) \right] & = \left( S_0(0) + S_0(1) + \ldots + S_0(n) \right)
\end{align*}
\]

Please, allow us to point out, that in NT-FS&L:

\[
0 = S_0(0) \land P_0(0) \land 1 = S_0(1) \land P_0(1) \Rightarrow \sum_{x=0}^{(x)=n} \left[ (x) \right] = \sum_{x=1}^{(x)=n} \left[ (x) \right]
\]
SECOND FOUNDAMENTAL NT-FS&L THEOREM

∀n∀x∀y (n, x ∈ N(0⊂NT-PA), y ∈ N(0⊂NT-PA))

\[ ((x\cdot y) = \left( \sum_{x=1}^{x} (x) + \sum_{x=0}^{x} (x) \right) \to y = 2 \land S_0(1) + S_1(0) ) \]

\[ \land \]

\[ ((x\cdot y) = \left( \sum_{x=0}^{x} (x) + \sum_{x=0}^{x} (x) \right) \to y = S_0(0) + S_1(x) ) \]

\[ \land \]

\[ ((x\cdot y) = \left( \sum_{x=1}^{x} (x) + \sum_{x=1}^{x} (x) \right) \to y = 2 \land S_0(1) + P_0(1) ) \]

\[ \land \]

\[ ((x\cdot y) = \left( \sum_{x=1}^{x} (x) + \sum_{x=1}^{x} (x) \right) \to y = 2 \land P_0(1) + P_0(1) ) \]

\[ \land \]

\[ ((x\cdot y) = \left( 2 \cdot \sum_{x=0}^{x} (x) \right) \to y = n+1 \in N(0⊂PA) \land N(0\not\subset NT-PA) \land Subs(n) \land N(0⊂NT-PA) \]

\[ \land \]

\[ \forall y \in N(0⊂PA) \land N(0\not\subset NT-PA) \land Subs(n) \land N(0⊂NT-PA) \]

(NT-FS&L F.Th. 2) \textsuperscript{71}

Please, allow us to point out that:

\[ N(0⊂NT-PA) := \{0,1,2,3,4,...,x+0, x+1, x+2, x+3,...,(x+x)+0, (x+x)+1,...\}x \in N(0⊂NT-PA) \]

(Entailment from the axiom 7 of PA expressed in first order logic; “the substitution operation” is intended embedded in every first order logic statement in NT-FS&L)

Then,

\[ Subsets-N(0⊂NT-PA) := \{0,1,2,3,4,..., x\}x \in N(0⊂NT-PA) \]

\[ x=(n+0) \to \{0,1,2,3,4,...,(n+0)\} \land \#-Subset-N(0⊂NT-PA) = n+1 \]

\[ x=(n+1) \to \{0,1,2,3,4,...,(n+1)\} \land \#-Subset-N(0⊂NT-PA) = n+2 \]

\[ x=(n+2) \to \{0,1,2,3,4,...,(n+2)\} \land \#-Subset-N(0⊂NT-PA) = n+3 \]

\[ \ldots \]

\[ x=(n+n)+0 \to \{0,1,2,3,4,...,(n+n)+0\} \land \#-Subset-N(0⊂NT-PA) = (n+n)+1 = 2 \cdot n+1 \]

\[ x=(n+n)+1 \to \{0,1,2,3,4,...,(n+n)+1\} \land \#-Subset-N(0⊂NT-PA) = (n+n)+2 = 2 \cdot n+2 \]
\[ x = (n + n) + 2 \rightarrow \{0, 1, 2, 3, 4, \ldots, (n + n) + 2\} \setminus \text{subset-\(\mathbb{N}\)}(0 \subset \text{NT-PA}) = (n + n) + 3 = 2 \cdot n + 3 \]

\[ x = (n + n) + n \rightarrow \{0, 1, 2, 3, 4, \ldots, (n + n + n)\} \setminus \text{subset-\(\mathbb{N}\)}(0 \subset \text{NT-PA}) = (n + n + n) + 1 = 3 \cdot n + 1 \]

\[ x = ((n + n) + ((n + n) + 0) + 0) \rightarrow \{0, 1, 2, 3, 4, \ldots, ((n + n + n + n))\} \setminus \text{subset-\(\mathbb{N}\)}(0 \subset \text{NT-PA}) = (n + n + n + n) + 0 = 4 \cdot n + 1 \]

\[ x = ((n + n) + ((n + n) + 0) + 1) \rightarrow \{0, 1, 2, 3, 4, \ldots, ((n + n + n + n))\} \setminus \text{subset-\(\mathbb{N}\)}(0 \subset \text{NT-PA}) = (n + n + n + n) + 1 = 4 \cdot n + 2 \]

\[ x = ((n + n) + ((n + n) + 0) + 2) \rightarrow \{0, 1, 2, 3, 4, \ldots, ((n + n + n + n))\} \setminus \text{subset-\(\mathbb{N}\)}(0 \subset \text{NT-PA}) = (n + n + n + n) + 2 = 4 \cdot n + 3 \]

Additionally, please note the following:

\[
\{0, 1, 2, 3, 4, \ldots, (2 \cdot n + 0)\} = \{0, 1, 2, 3, 4, \ldots, (x)\} \quad \text{and} \quad x = n \in \text{NT-PA} \\
\{0, 1, 2, 3, 4, \ldots, (2 \cdot n + 1)\} = \{0, 1, 2, 3, 4, \ldots, (x + 1)\} \quad \text{and} \quad x = (n + (n + 1)) \\
\{0, 1, 2, 3, 4, \ldots, (2 \cdot n + 2)\} = \{0, 1, 2, 3, 4, \ldots, (x + 2)\} \quad \text{and} \quad x = ((n + (n + 1)) + 1) \\
\{0, 1, 2, 3, 4, \ldots, (2 \cdot n + 3)\} = \{0, 1, 2, 3, 4, \ldots, (x + 3)\} \quad \text{and} \quad x = (((n + (n + 1)) + 1) + 1)
\]

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1. Case: \text{subset-\(\mathbb{N}\)}(0, 1 \subset \text{NT-PA}) \subseteq \mathbb{N}(0 \subset \text{NT-PA}):=
\{0, 1, 2, 3, 4, 5, \ldots, x\} \forall x = n \in \mathbb{N}(0 \subset \text{NT-PA});
\#\text{-subset-\(\mathbb{N}\)}(0, 1 \subset \text{NT-PA}) \subseteq \mathbb{N}(0 \subset \text{NT-PA}) = n + 1

\[
2 \cdot \sum \{x\} = 2 \cdot (0 + 1 + 2 + 3 + 4 + 5 + \ldots + (n)) \quad \text{and} \quad x \cdot (2 \cdot n + 1) \rightarrow x = S_0(2 \cdot n + 0)
\]

2. Case: \text{subset-\(\mathbb{N}\)}(0, 1 \subset \text{NT-PA}) \subseteq \mathbb{N}(0 \subset \text{NT-PA}):=
\{0, 1, 2, 3, 4, 5, \ldots, x\} \forall x = n + (n + 1) \in \mathbb{N}(0 \subset \text{NT-PA});
\#\text{-subset-\(\mathbb{N}\)}(0, 1 \subset \text{NT-PA}) \subseteq \mathbb{N}(0 \subset \text{NT-PA}) = n + (n + 1) + 1 = 2 + 2n

\[
2 \cdot \sum \{x\} = 2 \cdot (0 + 1 + 2 + 3 + 4 + 5 + \ldots + (n + (n + 1))) \quad \text{and} \quad x \cdot (2 \cdot n + 2) \rightarrow x = S_0(2 \cdot n + 1)
\]

3. Case: \text{subset-\(\mathbb{N}\)}(0, 1 \subset \text{NT-PA}) \subseteq \mathbb{N}(0 \subset \text{NT-PA}):=
\{0, 1, 2, 3, 4, 5, \ldots, x\} \forall x = (n + (n + 1)) + 1 \in \mathbb{N}(0 \subset \text{NT-PA});
\[ (P_0(x)) \cdot (S_y(0)) = \sum [(P_0(x))(S_0(y))] + \sum ((P_0(x))(S_0(y))) \rightarrow \]
\[ P_0(y) = P_0(x+1) \in \mathcal{N}(\mathcal{O}_{\mathcal{PA}}) \cap \mathcal{N}(\mathcal{O}_{\mathcal{PA}}) \]
\[ \& \]
\[ 2 \cdot \sum ((P_0(x))(S_0(y)) = (P_0(x)) \cdot (S_0(y)) \rightarrow S_0(y) = P_0(x+1) \in \mathcal{N}(\mathcal{O}_{\mathcal{PA}}) \cap \mathcal{N}(\mathcal{O}_{\mathcal{PA}}) \]

4. Case: \( x = S_0(x) \land y = S_1(x) \land (x+1), \forall x \forall y (x, y \in \mathcal{N}(0 \subset \mathcal{NT}-\mathcal{PA})) \) & Enthainment from First Fundamental NT-FS\&L Theorem

\[ (x+x) = S_0(x) + S_0(x) = S_0(x+x) \land S_x(x) \land S_{x+x}(0) \rightarrow 2 \cdot x = 2 \cdot S_0(x) = S_0(x+x) \]

\[ (x+1) + (x+1) = S_1(x) + S_1(x) = S_0(x+x +1+1) \land S_0(x+1+x+1) \land S_{x+1}(x+1) \]
\[ S_{1+1}(x+x) \land S_1(x+x+1) \land S_{x+x+1}(1) \rightarrow S_1(x+x+1) \land S_1(x+x+1) \land S_0(x+x+2) \land S_0(2x+2) \land 2 \cdot S_0(x+1) \land 2 \cdot S_1(x) \land S_2(x+x) \]

\[ x \cdot S_0(x) = x \cdot (x) \land x \cdot x + 0 \rightarrow x = S_0(x) \land y = S_1(x) \]
\[ x \cdot S_0(x) + x \cdot S_0(x) = (x \cdot x + 0) + (x \cdot x + 0) \land x \cdot x + x \cdot x = S_0(x \cdot x + x \cdot x) = 2 \cdot S_0(x \cdot x) \]
\[ 2 \cdot x \cdot S_0(x) = 2 \cdot S_0(x \cdot x) \rightarrow x \cdot S_0(x) = S_0(x \cdot x) \]
\[ x \cdot S_0(x) + x \cdot S_0(x) = S_0(x) \cdot (x + x) \land x \cdot (S_0(x) + S_0(x)) \land x \cdot S_0(x)(1+1) \land 2 \cdot x \cdot S_0(x) \]

\[ x \cdot (x+1) = x \cdot S_1(x) \land x \cdot x + x \rightarrow S_0(x \cdot x) + x = S_0(x \cdot x) + S_0(x) \]

\[ (1.) x \cdot S_1(x) + x \cdot S_1(x) = (x \cdot x + x) + (x \cdot x + x) \land x \cdot x \cdot (x \cdot x + 1) \rightarrow \]
\[ (x \cdot x) \cdot S_0(x \cdot x + 1) \land (x \cdot x) \cdot (S_0(x \cdot x) + 1) \land (x \cdot x) \cdot S_1(x \cdot x) \]

\[ (2.) x \cdot S_1(x) + x \cdot S_1(x) = (x \cdot x + x) + (x \cdot x + x) \land (2 \cdot x \cdot x + 2 \cdot x) \land 2 \cdot x \cdot (x + 1) \rightarrow \]
\[ 2 \cdot S_0(x) \cdot (x+1) \land 2 \cdot S_0(x) \cdot S_1(x) \land 2 \cdot S_0(x) \cdot S_1(x) \]

\[ (3.) x \cdot S_1(x) + x \cdot S_1(x) = 2 \cdot (x \cdot S_1(x)) \land 2 \cdot (S_0(x) \cdot S_1(x)) \rightarrow x \cdot S_1(x) = S_0(x) \cdot S_1(x) \]

\( (1 \& 2 \& 3) \)

\[ x \cdot S_1(x) + x \cdot S_1(x) = (x \cdot x) \cdot S_1(x \cdot x) \land 2 \cdot S_0(x) \cdot S_1(x) \]
\[(x+1)\cdot(x+1) \land S_1(x) \cdot S_1(x) = (x\cdot x + 1\cdot x + 1\cdot 1) \land (x\cdot x + 1\cdot x + 1) + (x\cdot x + 1\cdot x + 1) = \]
\[x \cdot S_1(x) + x \cdot S_1(x) \land x \cdot (S_1(x) + S_1(x)) \land x \cdot (S_1(x+x)) \]
\[(x+1)\cdot(x+1) \land S_1(x) \cdot S_1(x) = (x\cdot x + 1\cdot x + 1\cdot 1) \land (x\cdot x + x + 1) = \]
\[S_1(x\cdot x + x) \land S_0(x\cdot x + x) + 1 \rightarrow S_0(x\cdot x) + S_0(x) + 1 \land S_0(x\cdot x) + S_1(x) \land S_0(x\cdot x) + S_0(x) \]

Then,

\[((x+x) = (S_0(x) + S_0(x))) \land ((y+y) = (S_0(y) + S_0(y)) \land (S_0(x+1) + S_0(y)) \land (y\cdot y) = (x+1)\cdot(x+1) \land S_1(x\cdot x) + S_0(x) \rightarrow \]

\[S_1(x\cdot x) = S_0(y) \land y = x+1 \]

Based on all of the above, we can establish that:

**Third Fundamental NT-FS&L Theorem**

(NT-PA Pythagorean Successors & Predecessors)

The next statement is true, which is a theorem in NT-FS&L (Entailment from First Fundamental NT-FS&L Theorem):

\[\forall x \forall y \ (x, y \in \mathbb{N}(0 \subset \text{NT-PA}) \mid x = S_0(x) \land y = S_1(x) \land (x+1)) \]

\[\left( S_1(x+x) = S_0(y\cdot y) \rightarrow (x = S_0(x) \land y = S_1(x) \forall x \forall y (x, y \in \mathbb{N}(0 \subset \text{NT-PA})) \right) \land \]

\[\left( S_{y+y}(1) = S_{x\cdot x}(0) \rightarrow (x = S_0(x) \land y = S_1(x) \forall x \forall y (x, y \in \mathbb{N}(0 \subset \text{NT-PA})) \right) \land \]

\[\left( S_0(x+x) = P_1(y\cdot y) \rightarrow (x = P_0(x) \land x = P_1(y) \forall x \forall y (x, y \in \mathbb{N}(0 \subset \text{NT-PA})) \right) \land \]

\[\left( P_0(x+x) = P_1(y\cdot y) \rightarrow (x = P_0(x) \land x = P_1(y) \forall x \forall y (x, y \in \mathbb{N}(0 \subset \text{NT-PA})) \right) \rightarrow \]

\[ (x = P_1(y) \land y = P_0(y) \land (x+1)), \forall x \forall y \]

(NT-FS&L F.Th. 3)
Additionally, please, allow us to point out that:

\[ \forall x \forall y \ (x, y \in \mathcal{N}(0 \subset \text{PA}) \land \ x = S_0(x) \land y = S_1(x) \land (x+1), \ \forall x \forall y \ (x, y \in \mathcal{N}(0 \subset \text{PA}) \land \ N^{\alpha}(0 \subset \text{PA}) = \{ \text{even, odd, even, odd, even, odd, even, odd, ...} \} \rightarrow \]

\[ \left\{ \begin{array}{ll} (x \in \text{Even}(S_0(x))) & \Rightarrow y = S_1(x) \in \text{Odd}(S_0(x)) \\ (x \in \text{Odd}(S_0(x))) & \Rightarrow y = S_1(x) \in \text{Even}(S_0(x)) \end{array} \right\} \lor \]

\[ (x \in \text{Odd}(S_0(x))) \Rightarrow y = S_1(x) \in \text{Even}(S_0(x)) \rightarrow \]

\[ (x \in \text{Even}(S_0(x)) \Rightarrow y = S_1(x) \in \text{Odd}(S_0(x))) \land \]

\[ ((y + y) \land (x + x) \land S_0(x+x) \land P_0(x+x) \in \text{Even}(S_0(x))) \]

\[ x = S_0(x) \land y = S_1(x) \land (x+1), \ \forall x \forall y \ (x, y \in \mathcal{N}(0 \subset \text{PA})) \]

\[ x = S_0(x) \land S_0(x) \in \text{Even-Suc}(S_0(x)) \Rightarrow y = x + 1 \land S_1(x) \in \text{Odd-Suc}(S_0(x)) \]

\[ e \cdot S_0(x) + e \cdot S_0(x) = e \cdot S_0(x+1) \land e \cdot S_0(x+x+0) \Rightarrow S_0(x+x+0) \in \text{Even-Suc}(S_0(x)) \]

\[ e \cdot S_0(x) + o \cdot S_1(x) = o \cdot S_1(x+1) \land o \cdot S_0(x+x+1) \Rightarrow S_0(x+x+1) \in \text{Odd-Suc}(S_0(x)) \]

\[ e \cdot S_0(x) \cdot o \cdot S_1(x) = e \cdot S_0(x \cdot (x+1)) \land e \cdot S_0(x \cdot x + x) \Rightarrow S_0(x \cdot x + x) \in \text{Even-Suc}(S_0(x)) \]

\[ x = S_0(x) \land S_0(x) \in \text{Odd-Suc}(S_0(x)) \Rightarrow y = x + 1 \land S_1(x) \in \text{Even-Suc}(S_0(x)) \]

\[ o \cdot S_0(x) + o \cdot S_0(x) = e \cdot S_0(x+x) \land e \cdot S_0(x+x+0) \Rightarrow S_0(x+x+0) \in \text{Even-Suc}(S_0(x)) \]

\[ o \cdot S_0(x) + e \cdot S_0(x) = o \cdot S_0(x+x+1) \land o \cdot S_1(x+1) \Rightarrow S_0(x+x+1) \in \text{Odd-Suc}(S_0(x)) \]

\[ o \cdot S_0(x) \cdot e \cdot S_1(x) = e \cdot S_0(x \cdot (x+1)) \land e \cdot S_0(x \cdot x + x) \Rightarrow S_0(x \cdot x + x) \in \text{Even-Suc}(S_0(x)) \]

Finally, please, allow us to point out that in \( \mathcal{N}(0 \subset \text{PA}) \):

\[ \forall x \forall y \forall n \ (x, y, n \in \mathcal{N}(0 \subset \text{PA}) \land \ x = S_0(x) = S_1(0) \land P_0(x), y \in \text{Suc}(S_0(x)), n \in \text{Suc}(S_0(y)) \land \text{Even}(x) := \{(x+x) = (2 \cdot x) \mid x \in \mathcal{N}(0 \subset \text{PA})\}, \forall x \land \]

\[ \text{Odd}(x) := \{(x+x+1) = (2 \cdot x + 1) \mid x \in \mathcal{N}(0 \subset \text{PA})\}, \forall x \}

\[ (e \cdot x + e \cdot y) = e \cdot (n) \land e \cdot x \land e \cdot y = e \cdot (n) \land \]

\[ e \cdot x + o \cdot y = o \cdot (n) \land e \cdot x \land o \cdot y = e \cdot (n) \land \]

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o·(x) + e·(y) = o·(n) &  o·(x) · e·(y) = e·(n) ∧  o·(x) + o·(y) = e·(n) &  o·(x) · o·(y) = o·(n) \to

\left\{
\begin{aligned}
\left(\mathbb{N}_{0<\text{NT-PA}}\right) &= \{e·0, o·1, e·2, o·3, \ldots, x+(e·0), x+(o·1), x+(e·2), x+(o·3), \ldots, \\
e·(2·x)+(e·0), o·((2·x)+(o·1)), e·((2·x)+(e·2)), o·((2·x)+(e·3)), \\
e·((2·2·x)+(e·0)), o·((2·2·x)+(o·1)), e·((2·2·x)+(e·2)), o·((2·2·x)+(e·3)), \ldots, \\
e·(2·2·2·x)+(e·0)), o·((2·2·2·x)+(o·1)), e·((8·x)+(e·2)), o·((8·x)+(o·3)), \ldots,\} \forall x
\end{aligned}\right.

As summary of the parity (and non-parity) NTFS&L-properties of the natural numbers and their collectivities:

A. Covariant (contravariant) "succession" and "precede" operations and addition operation PA's:

\((P_0(x) \in \text{Even}(\text{Pre}_0(x)) \to (P_0(x+1) \in \text{Odd}(\text{Pre}_0(x))) \land (S_0(x) \in \text{Even}(\text{Suc}_0(x)) \to (S_0(x+1) \in \text{Odd}(\text{Suc}_0(x))))\)

B. On covariant (contravariant) "succession" and "precede" operations and multiplication operation PA's:

i.- \((P_0(x)) · (S_0(y)) \in \text{Even}^a(Suc_{x+y}(0)) \land \text{Even}^a(Suc_x(y)) \land \text{Even}^a(Suc_y(x)) \land \text{Even}^b(Suc_{x+y}(0)) \land \text{Even}^b(Suc_x(y)) \land \text{Even}^b(Suc_y(x)) \land \text{Even}^a(\text{Pre}_0(x+y)) \land \text{Even}^b(\text{Pre}_0(x+y)) \to y = x + 1 \in \mathbb{N}_{0<\text{NT-PA}} \cap \mathbb{N}_{0<\text{NT-PA}}\)

ii.- \((P_0(x)) · (S_0(y)) = (P_0(x)) · (P_0(y)) \land x·y \to y = x+1 \in \mathbb{N}_{0<\text{NT-PA}} \cap \mathbb{N}_{0<\text{PA}}\)

iii.- \((P_0(x)) · (S_0(y)) = (P_0(x)) · (P_0(y)) \land x·x + x \to y = x+1 \in \mathbb{N}_{0<\text{NT-PA}} \cap \mathbb{N}_{0<\text{NT-PA}}\)

On the other hand and as direct consequence of both: first, "intended first logic des-embedding" of axiom 6 of PA \((\forall x \forall y(x \in \mathbb{N}_{0<\text{PA}})\{x·S(y) = x·y + x\}), \) and second, entailment from First Fundamental NT-FS&L Theorem; we are able to state:

\(\forall x \forall y \forall n \left( x, n \in \mathbb{N}_{0<\text{NT-PA}} \right), y \in \left\{ \text{Suc}_1(x) \right\} \land \left( \mathbb{N}_{0<\text{NT-PA}} \cap \mathbb{N}_{0<\text{NT-PA}} \right) \left( (x·y+x) \land x·(y+1) \to \right)\)
\[
x \cdot S_1(x) = 2 \cdot \Sigma [(x)] = 2 \cdot (0+1+2+3+4+5 \ldots + (n)) \land x \cdot (2 \cdot n+1) \rightarrow x = S_0 (2 \cdot n +0)
\]

\[
\Rightarrow
\]

\[
(x \cdot S_1(x) \in \text{Even-PN}_0(x \cdot x + x) \cap \text{Even-PN}_0(x \cdot (x+1))_{x \in \mathbb{N}})
\]

5. **NT-FS&L CONSIDERATIONS, DEFINITIONS AND THEOREMS ABOUT PRIME AND NATURAL NUMBERS, AND ON THEIR ADDITION AND MULTIPLICATION OPERATIONS (ELEMENTS and COLLECTIVITIES)**

The next set of well formed formula (also wff) and wfs-r statements, by the natural number N(0⊂NT-PA) set summarizes, “by comprehension and by extension; sense and intension”) some of the identities of the main collectivities of natural numbers found by “intentional de-embedding methodology” from NT-FOL-PA’s axiomatic (briefly referred as NT-PA):

\[
\forall x (x \in \mathbb{N}(0 \subset NT-PA))
\]

\[
\{ S_0(0), S_1(0), S_2(0), \ldots, S_x(0), \ldots \} \subseteq \mathbb{N}^\alpha (0 \subset NT-PA)
\]

\[
\{ S_0(0), S_0(1), S_0(2), \ldots, S_0(x), \ldots \} \subseteq \mathbb{N}^\alpha (0 \subset NT-PA)
\]

\[
\{ P_0(0), P_0(1), P_0(2), \ldots, P_0(x), \ldots \} \subseteq \mathbb{N}^\beta (0 \subset NT-PA)
\]

\[
\{ \ldots, P_0(x), \ldots, P_0(2), P_0(1), P_0(0) \} \subseteq \mathbb{N}^\beta (0 \subset NT-PA)
\]

&

\[
\{ S_0(0), \ldots, S_{x+1}(0), S_{x+2}(0), \ldots, S_{x+x}(0), \ldots \} \subseteq Suc^\alpha_x(0)
\]

\[
\{ S_0(x), \ldots, S_0(x+1), S_0(x+2), \ldots, S_0(x+x), \ldots \} \subseteq Suc^\beta_0(x)
\]

\[
\{ P_0(0), P_0(1), P_0(2), \ldots, P_0(x), \ldots P_0(x+x), \ldots \} \subseteq Pre^\alpha_0(x)
\]

Additionally, for example, when referred to zero by means of the customary decimal numerical system, we are able to state \(\forall x (x \in \mathbb{N}(0 \subset NT-PA))\):

\[
\{ 0, 1, 2, 3, \ldots, x, \ldots \} \subseteq \mathbb{N}^\alpha (0 \subset NT-PA)
\]

\[
\{ \ldots, x, \ldots, 3, 2, 1, 0 \} \subseteq \mathbb{N}^\beta (0 \subset NT-PA)
\]

\[
\{ 0, 1, 2, 3, \ldots, x \} \subseteq Suc^\alpha_x(0)
\]

\[
\{ \ldots, x, \ldots, 3, 2, 1, 0 \} \subseteq Suc^\beta_0(x)
\]

\[
\{ 0, 1, 2, 3, \ldots, x \} \subseteq Pre^\alpha_0(x)
\]

\[
\{ x, \ldots, 3, 2, 1, 0 \} \subseteq Pre^\beta_0(x)
\]

Therefore, in NT-FOL-PA, the following statements are wff and wfs-r statements:74
∀x(x ∈ N(0∈NT-PA))

\{ S_0(0), S_1(0), S_2(0), ..., S_x(0), ... \} ⊆ N^e(0∈NT-PA)
∧ ∨
\{ S_0(0), S_0(1), S_0(2), ..., S_0(x), ... \} ⊆ N^e(0∈NT-PA)
∧ ∨
\{ P_0(0), P_0(1), P_0(2), ..., P_0(x), ... \} ⊆ N^e(0∈NT-PA)
∧ ∨
\{ 0, 1, 2, 3, ..., x, ... \} ⊆ N^e(0∈NT-PA)

Hence, in NT-FOL-PA, the identity by comprehension (by means of intended de-embed logic operation and/or) of a natural number as “element” \((∀x)(x ∈ N(0∈NT-PA))\) can be well represented by:

\(0 = 0) ∧ ∨ (1 = 1)∧ ∨ (2 = 2)∧ ∨ (3 = 3) ∧ ∨ (4 = 4) ∧ ∨ ... ∧ ∨ (x = x) ...
∧ ∨
\((x+0 = 0+x) ∧ ∨ (x+1 = 1+x) ∧ ∨ (x+2 = x+2) ∧ ∨ ... ∧ ∨ (x+x = x+x) ...
∧ ∨
\((x+0 = 0+x) ∧ ∨ (x+1 = 1+x) ∧ ∨ (x+2 = x+2) ∧ ∨ ... ∧ ∨ (x+x = 1+x+1) ...
∧ ∨
\∧ ∨ (x = 1·x) ... ∧ ∨ (x+x = 2·x) ... ∧ ∨ (x+x +x = 3·x) ...

Hence;

\(\forall x \ (x ∈ N(0∈NT-PA))(N(0∈NT-PA) :=

\{ (0=0+0) ∧ (1=1+0) ∧ (2=2+0) ∧ (3=3+0) ∧ (x+0=1·x+0) ∧ (x+1=1·x+1) ∧
(x+2=1·x=x+1·x+2) ∧ ((x+x)=2·x+0) ∧ ((x+x)+1=2·x+1) ∧ ((x+x+x)+0=3·x+0);
((x+x+x)+1=3·x+1) ∧ ((x+x+x+x)+0=4·x+0) ∧ ((x+x+x+x+x)+0=5·x+0) ∧
((x+x+x+x+x+x)+0=6·x+0) ∧ ... ∧ ... \)

→

\(Suc^α_0(x) :=
\{ 0, 1, 2, 3, ..., x+0, x+1, x+2, ..., (x+x), (x+x)+1, ...(x+x+x),(x+x+x)+1, ... \}
∧ ∨
\{ 0, 1, 2, 3, ..., 1·x , 1·x+1, 1·x+2 ..., 2·x+0, 2·x+1, ..., 3·x+0, 3·x+1, ..., ... \})
∧

\(Suc^β_1(0) :=
\{...(x+x+x)+1, (x+x+x), ..., (x+x)+1, (x+x), ..., x+2, x+1, x+0, ..., 3, 2, 1, 0 \}
{ ..., 3·x+1, 3·x+0, ..., 2·x+1, 2·x+0, ..., 1·x+2,1·x+1, 1·x, ..., 3, 2, 1, 0 }\)
In view of all the above, the elements (also, called until now, as natural numbers or integers) that conform non-dense, parity complete ordered \( \text{Pre}^\alpha_0(x) \), \( \text{Suc}^\alpha_0(x) \) and \( \text{Suc}^\alpha_x(0) \) collectivities of natural numbers referred to natural number zero are the natural number concepts referred as the elements conforming the set of the natural numbers herein called \( \mathbb{N}(0 \subset \text{NT-PA}) \).

Said natural numbers and collectivities referred to natural number zero, results from an "intentional de-embedding" (sense and intension of NT-SF&L) operation by means of the next two operational processes of the inductive-deductive logical apparatus of NT-SF&L language:

i.- First order logic translation (briefly, FOL-T) of the set of the higher order logic axioms supporting \( \mathbb{N}(0 \subset \text{PA}) \), (hereinafter symbolized by \( \text{SPA}(0 \subset \text{PA}) \)).

ii.- First order logic inductive entailment (briefly, FOL-I) from the set supporting \( \mathbb{N}(\text{FOL-SPA}(0 \subset \text{PA})) \) axiomatic definition, as well as of their elements and operations, and subsets.

**HLPrime-1-NT Theorem:**

\[
\forall x \left( x \in \text{Pre}^\alpha_0(x) \land \text{Suc}^\alpha_0(x) \land \text{Suc}^\alpha_x(0) \land \mathbb{N}^\alpha(0 \subset \text{NT-PA}) \right)
\]

\[
\left( \left( 1. \left( \{ e-0, o-1, e-2, o-3, ..., e-(x), o-(x+1), e-(x+2), ..., e-(x+x), o-(x+x+1) \}, \right) \subseteq \mathbb{N}^\alpha(0 \subset \text{PA}) \right) \rightarrow \\
\left( \{ ..., o-(x+x+1), e-(x+x), ..., e-(x+2), o-(x+1), e-(x), ..., e-4, o-3, e-2, o-1, e-0 \} \subseteq \mathbb{N}^\beta(0 \subset \text{PA}) \right) \right)
\]
2. \( \{ e\cdot 0, o\cdot 1, e\cdot 2, o\cdot 3, e\cdot 4, \ldots, e\cdot (x+x), o\cdot (x+x+1), e\cdot (x+x+2), \) \\
\( o(x+x+3), e\cdot (x+x+4), \ldots \} \subseteq Suc^\alpha x(0) \rightarrow \)
\( \{ ..., e\cdot (x+x+4), o(x+x+3), e\cdot (x+x+2), o\cdot (x+x+1), e\cdot (x+x), \) \\
\( ..., e\cdot 4, o\cdot 3, e\cdot 2, o\cdot 1, e\cdot 0 \} \subseteq \{ Suc^\beta x(0) \} \)
\&
3. \( \{ e\cdot 0, o\cdot 1, e\cdot 2, o\cdot 3, e\cdot 4, \ldots, e\cdot (x+x), o\cdot (x+x+1), e\cdot (x+x+2), \) \\
\( o(x+x+3), e\cdot (x+x+4), \ldots \} \subseteq Pre^\alpha_0 (x) \rightarrow \)
\( (Pre^\beta_0 (x) \subseteq \{ ..., e\cdot (x+x+4), o(x+x+3), e\cdot (x+x+2), o\cdot (x+x+1), e\cdot (x+x), \) \\
\( ..., e\cdot 4, o\cdot 3, e\cdot 2, o\cdot 1, e\cdot 0 \} ) \rightarrow \)
\( \left( N(FOL-SP(0 \subset \mathcal{PA})) \subseteq N(SPA(0 \subset \mathcal{PA}) \rightarrow N(0 \subset NT-\mathcal{PA}) \subseteq N(FOL-SP(0 \subset \mathcal{PA}))) \right) \)

(N.T.-Pr-th. 1)\(^75\)

**FOURTH FOUNDAMENTAL NT-FS&L THEOREM**

\( \forall x \ (x \in N(0 \subset NT-\mathcal{PA})) \)
\( \left( (\sim^\alpha (0 \subset NT-\mathcal{PA})) = \{ e\cdot 0, o\cdot 1, e\cdot 2, o\cdot 3, \ldots, e\cdot (x+x), \ldots \} \right) \Rightarrow \)
\( Pre^\beta_0 (x) \land Suc^\alpha x(0) \subseteq \sim (0 \subset NT-\mathcal{PA}) \rightarrow \)
\( \left( (e\cdot P_0 (x+x) = e\cdot (x+x) \land e\cdot S_{x\times x}(0)) \land (o\cdot P_0 (x+x+1) = o\cdot (x+x+1) \land o\cdot S_{x\times x+1}(0)) \right) \Rightarrow \)
\( \left( (e\cdot P_0 (x) = e\cdot S_x(0) \land e\cdot (x) \land (o\cdot P_0 (x) = o\cdot S_x(0) \land o\cdot (x)) \right) \land \)
\( \left( \#\cdot Pre^\alpha_0 (x) = x+1 \Rightarrow x \in Even(Suc_0 (x)) \rightarrow (x+1) \in Odd(Suc_0 (x+1)) \right) \)
\( \lor \)
\( \#\cdot Pre^\alpha_0 (x) = x+1 \Rightarrow x \in Odd(Suc_0 (x)) \rightarrow (x+1) \in Even(Suc_0 (x+1)) \right) \)
\( \land \)
\( \left( (x+1) \in Even(Suc_0 (x+1)) \rightarrow (x) \in Odd(Suc_0 (x)) \right) \)
\( \land \)
(x) ∈ Even(Suc_0(x)) → (x+1) ∈ Odd(Suc_0(x+1)))

(NT-FS&L F.Th. 4)

Please, take in account that:

1. (∀x ∈ N (0 ⊂ NT-PA))

0 ∈ Suc_0(x) ∩ N (0 ⊂ NT-PA) & 0 ∉ N (0 ⊂ NT-PA) ∧ Suc_1(x) ∧ Suc_1(1)

∧

Suc_0(1) ∧ Suc_1(x) ⊆ N (0 ⊂ NT-PA) ∧ N (0 ⊂ NT-PA)

∧

Suc_x(x) ∧ Suc_0(x+x) ∧ Suc_x+x(0) ⊆ N (0 ⊂ NT-PA) ∧ N (0 ⊂ NT-PA)

2. (∀x ∈ N (0 ⊂ NT-PA))

0 ∈ Even(Suc_0(x)) ∧ 1 ∈ Odd(Suc_0(x)) ⇒

((0=e_0) ∧ 1=o_1) & (0=e_0) ∧ 1=o_1)

→

(x+x+0) = (x+x) + 0 ∈ Even(Suc_0(x)) ∧ (x+x+1) = (x+x)+1 ∈ Odd(Suc_0(x))

&

{0, 1, ..., (x+0), (x+1), ..., ..., (x+x), (x+x+1), ..., ...}

3. (∀x ∈ N (0 ⊂ NT-PA))

{1. ((S_0(x) ∧ S_0(x)) ∈ Even(Suc_0(x)) V S_0(x) ∧ S_0(x) ∈ Odd(Suc_0(x)))

∧

(S_0(x)+1) ∈ Even(Suc_0(x)) V (S_0(x)+1) ∈ Odd(Suc_0(x))

&

2. ((S_0(x+x) ∧ S_0(x) ∧ S_{x+x}(0)) ∈ Even(Suc_0(x))

∧

(S_0(x+x+1) ∧ S_{x+1}(x) ∧ S_{x+1}(x) ∧ S_{x+x+1}(0) ∧ S_{x+x+1}(1) ∧ S_{x+1}(x+1) ∧ S_1(x+x) ∈ Odd(Suc_0(x))

∧

((S_0(x+x)+1 ∧ S_0(x)+1 ∧ S_{x+x}(0)+1 ∈ Odd(Suc_0(x)))) 76

HLPrime-2-NT Theorem:

∀x (x ∈ N (0 ⊂ NT-PA))
A. Succession and covariance referred to the collectivity:

\[
[Suc^\alpha_x(0) := \{ S^\alpha_0(0), S^\alpha_0(1), S^\alpha_0(2), S^\alpha_0(3), \ldots, S^\alpha_0(x) \} \]
\land
\]

\[
Suc^\alpha_x(0) := \{ S^\alpha_0(0), S^\alpha_1(0), S^\alpha_2(0), S^\alpha_3(0), \ldots, S^\alpha_x(0) \} \]
\rightarrow

\[
Suc^\alpha_x(0) := \{ e-0, o-1, e-2, o-3, \ldots, x \} \Rightarrow Pre^\alpha_0(x) := \{ e-0, o-1, e-2, o-3, \ldots, x \}
\rightarrow

\[
Pre^\alpha_0(x) := \{ e-0, o-1, e-2, o-3, \ldots, x \} = \{ P^\alpha_0(0), P^\alpha_0(1), P^\alpha_0(2), P^\alpha_0(3), \ldots, P^\alpha_0(x) \}
\text{(Nomenclature set referred to natural number 0 as index-enumerator)}
\land

\[
\{ p^\alpha_x(x), \ldots, p^\alpha_3(x), p^\alpha_2(x), p^\alpha_1(x), p^\alpha_0(x) \}
\text{(Nomenclature set referred to natural number (x) as enumerator, which in turn is referred to natural number zero)}
\]

B. Succession and contravariance referred to the collectivity:

\[
[Suc^\beta_x(0) := \{ S^\beta_0(0), \ldots, S^\beta_3(0), S^\beta_2(0), S^\beta_1(0), S^\beta_0(0) \} \]
\land
\]

\[
Suc^\beta_x(0) := \{ s^\alpha_0(x), \ldots, s^\alpha_3(0), s^\alpha_0(2), s^\alpha_0(1), s^\alpha_0(0) \} \]
\rightarrow

\[
Suc^\beta_x(0) := \{ x, \ldots, o-3, e-2, o-1, e-0 \} \Rightarrow Pre^\beta_0(x) := \{ x, \ldots, o-3, e-2, o-1, e-0 \}
\rightarrow

\[
Pre^\beta_0(x) := \{ x, \ldots, o-3, e-2, o-1, e-0 \} = \{ p^\beta_0(x), \ldots, p^\beta_0(3), p^\beta_0(2), p^\beta_0(1), p^\beta_0(0) \}
\text{(Nomenclature set referred to natural number 0 as index-enumerator)}
\land

\[
\{ p^\beta_0(x), p^\beta_1(x), p^\beta_2(x), p^\beta_3(x), \ldots, p^\beta_x(x) \}
\text{(Nomenclature set referred to natural number (x) as enumerator, which in turn is referred to natural number zero)}
\]

Hence, \( \forall x, x \in Suc^\alpha_x(0) \land Suc^\beta_x(0) \land Pre^\alpha_0(x) \land Pre^\beta_0(x) \land \mathcal{N}(0\text{CNT-PA}) \)

\[
\left( (S^\alpha_x(0) + S^\alpha_x(0) = 0 \land P^\alpha_0(x) + P^\beta_0(x) \rightarrow e-0 = e-S^\alpha_0(x) \land e-P^\alpha_0(x) \land e-P^\beta_0(x) \right) \rightarrow
\]

\[
(x=0 \land 0+0=0) \Rightarrow 0 \in Even(x) \}
\]
\[ [S_x(0) = S^\alpha_x(0) \land S^\alpha_0(x) \land S^\beta_0(x) \land S^\beta_x(0) \rightarrow P_0(x) = P^\beta_0(x) \land P^\alpha_0(x)] \]

\[ \rightarrow \]

1. \( S^\alpha_0(x) + S^\beta_x(0) \land S^\beta_0(x) + S^\alpha_x(0) = x \)

\&

\[ \forall x \forall y (y \in P^\alpha_x(0))(x = P^\alpha_0(y) + P^\beta_0(y) = P^\alpha_0(x) \land P^\alpha_y(y+y) \rightarrow x=y) \]

2. \( S^\alpha_0(x) + S^\alpha_x(0) \land S^\beta_0(x) + S^\beta_x(x) = x + x \)

\&

\[ \forall x \forall y (y \in P^\alpha_x(0))(x = P^\alpha_0(y) + P^\alpha_0(y) = P^\beta_0(y) \land P^\beta_y(y+y) \rightarrow x=y+y) \]

\[ S^\alpha_0(x) + S^\alpha_x(0) \land S^\beta_0(x) + S^\beta_x(0) \in \text{Even}(x), \forall x (x \in N(0 \subset PA)) \]

[In multiplicative notation, only alphabetic meaning of symbol “\( \cdot \)“:
\[ x + x = 2 \cdot x = 2 \cdot S^\alpha_x(0) \land 2 \cdot S^\beta_x(0) \land 2 \cdot P^\beta_0(x) \land P^\alpha_0(x) + P^\beta_0(x) \land 2 \cdot P^\beta_x(2 \cdot x) \]

Hence, \( (\forall x \in N(0 \subset NT-PA)) \)

\[ Suc^\alpha_x(0) + Suc^\beta_x(0) := \{x \mid x \in PN_0(x), \forall x \in Even(x) \lor Odd(x) \}
\]

\&

\[ Suc^\beta_x(0) + Suc^\alpha_x(0) := \{x \mid x \in PN_0(x), \forall x \in Even(x) \lor Odd(x) \}
\]

\[ \rightarrow \]

\[ Suc^\alpha_x(0) + Suc^\beta_x(0) = Suc^\beta_x(0) + Suc^\alpha_x(0) \]

\[ \]

\[ Suc^\alpha_x(0) + Suc^\beta_x(0) := \{x+x \in Even(x) \mid x \in PN_0(x), \forall x \in Even(x) \lor Odd(x) \}
\]

\&

\[ Suc^\beta_x(0) + Suc^\beta_x(0) := \{x+x \in Even(x) \mid x \in PN_0(x), \forall x \in Even(x) \lor Odd(x) \}
\]

\[ \rightarrow \]

\[ Suc^\alpha_x(0) + Suc^\alpha_x(0) = Suc^\beta_x(0) + Suc^\beta_x(0) \subset Even(x) \]

\[ (NT-Primes-th. 2) \]

Additionally, we can define a new operation, which will be herein after named as

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“the scalar multiplication of a natural number times a set of successors” of natural numbers referred to natural number \( 0 \) by the next statement:

\[
2 \otimes \{ S^\alpha_x(0) \} := \text{Suc}^\alpha_x(0) \oplus \text{Suc}^\alpha_x(0) \\
&
2 \otimes \{ S^\beta_x(0) \} := \text{Suc}^\beta_x(0) \oplus \text{Suc}^\beta_x(0)
\]

\[
\text{Suc}^\alpha_x(0) \oplus \text{Suc}^\alpha_x(0) = 2 \otimes \{ S^\alpha_x(0) \} = \{ 2 \cdot S^\alpha_x(0) \} \\
\wedge
\text{Suc}^\beta_x(0) \oplus \text{Suc}^\beta_x(0) = 2 \otimes \{ S^\beta_x(0) \} = \{ 2 \cdot S^\beta_x(0) \}
\]

\[
\forall x \forall y \left( n, x \in N(0 \subset PA) \right) \left( n \otimes \{ S_x(0) \} = \{ n \cdot S^\alpha_x(0) \} \wedge \{ n \cdot P^\alpha_x(0) \} \right)
\wedge
\forall x \forall y \left( n, x \in N(0 \subset NT-PA) \right) \left( n \otimes \{ S_x(0) \} = \{ (S^\alpha_x(0)) \cdot n \} \wedge \{ (P^\alpha_x(0)) \cdot n \} \right)
\]

First Highlighted NT-FS&L Considerations and Observations on first order logic (FOL) and Peano Arithmetic axiomatic of the collectivity named set of the natural numbers and their Arithmetic.

The set of “statements” (statements) conforming the Peano’s axiomatic of natural numbers and of their Arithmetic, when expressed in first order Logic, herein symbolized by \( \text{FOL-SPA}(0 \subset PA) \), refers to:

i.- The two allowed identities, as elements and as successors of elements, of the natural numbers that conforms the collectivity defined by comprehension as “set of the natural numbers, natural number zero included as element” (briefly, symbolized by \( \mathcal{N}(\text{FOL-SPA} \subset PA) \)). (Hereinafter, the subset conformed by Axioms 1 and 2).

ii.- The “set of allowed exchanges of above identities” induced by two arithmetic operations of the natural numbers, the addition and the multiplication, respectively. (Hereinafter; by addition the subset conformed by Axioms 3 and 4, and by multiplication, the subset conformed by Axioms 5 and 6).

iii.- An induction logical & mathematical schema, which is engrained in the FOL logical substitution operation, and allows and ensures logic entailment processes from the elements of \( \mathcal{N}(\text{FOL-SPA} \subset PA) \) and \( \text{FOL-SPA}(0 \subset PA) \), the natural numbers and the statements that hold “truth”, respectively. (Hereinafter, the subset conformed by Axiom 7). \(^{78}\) “intended first logic de-embedding”.

\( \text{FOL-SPA}(0 \subset PA) : \forall x \forall y \left( x, y \in \mathcal{N}(\text{FOL-SPA}(0 \subset PA)) \right) \)

1. On the two identities of the natural numbers; elements and successors
1. On the two identities of the natural numbers; elements as successors of a natural number:

1.1. ∀x (S(x)=x)
1.2. ∀x∀y (S(x) = S(y) → x=y)

2. On the addition operation and the exchange of identities of natural numbers:

2.1. ∀x (x + 0 = x)
2.2. ∀x∀y (x + S(y) = S(x+y))

3. On the multiplication operation (between a successor and a natural number, not between two natural numbers) and the exchange of identities of natural numbers:

3.1. ∀x (x·0=0)
3.2. ∀x∀y (x·S(y) = x·y + x)

4. On induction schema by substitution operation:

4.1. ∀y1 ... ∀yn ((φ(x|0) & ∀x(φ→φ(x|S(x)))) → ∀x φ), whenever φ is a formula whose free variables (also named occurrences) are among x0, y1, ..., yn.

\( NT\text{-FOL-SPA}(0 \subset \text{PA}); \forall x \forall y \ (x, y \in \mathbb{N} (0 \subset \text{NT-PA})) \)

1. On the two identities of the natural numbers; elements as successors of a natural number:

1.1. ∀x (S0(x) = x)
1.2. ∀x∀y (S0(x) = S0(y) → x=y)

2. On the addition operation and the exchange of identities of natural numbers:

2.1. ∀x (x + 0 = x)
2.2. ∀x∀y (x + S0(y) = S0(x+y))

3. On the multiplication operation (between a successor and a natural number, not between two natural numbers) and the exchange of identities of natural numbers:

3.1. ∀x (x·0=0)
3.2. ∀x∀y (x·S0(y) = x·y + x)

4. On induction schema by substitution operation:

4.1. ∀y1 ... ∀yn ((φ(x|0) & ∀x(φ→φ(x|S0(x)))) → ∀x φ), whenever φ is a formula whose free variables (also named occurrences) are among x0, y1, ..., yn.
5. **On the two identities of the natural numbers; elements as predecessors of natural numbers:**

1.1. \( \forall x (P_0(x) = x) \rightarrow x \neq 0 \)
1.2. \( \forall x \forall y (P_0(x) = P_0(y) \rightarrow x = y) \)

6. **On the addition operation and the exchange of identities of natural numbers:**

2.1. \( \forall x (x + 0 = x) \)
2.2. \( \forall x \forall y (x + P_0(y) = P_0(x+y)) \)

7. **On the multiplication operation (between a successor and a natural number, not between two natural numbers!) and the exchange of identities of natural numbers:**

3.1. \( \forall x (x \cdot 0 = 0) \)
3.2. \( \forall x \forall y (x \cdot P_0(y) = x \cdot y + x) \)

8. **On induction schema by substitution operation:**

4.1. \( \forall y_1 \ldots \forall y_n ((\varphi(x|0) \& \forall x(\varphi \rightarrow \varphi(x|P_0(x)))) \rightarrow \forall x \varphi) \), whenever \( \varphi \) is a formula whose free variables (also named occurrences) are among \( x_0, y_1, \ldots, y_n \).

Thus, we are able to represent ("substitution operation" is allowed as "logic operation" by induction statement 7 of initial PA’s, which represent a successor statement of zero statement) in first order language the next definition about the collectivity named (called) "set of axioms of PA’s" (abbreviated as SPA(0⊂PA) definition by extension and comprehension) in which the elements are, in turn, "wfs" and "wfs-r statements: \( \forall x (x \in \mathcal{N}(0\subset PA) \cap \mathcal{N}(0\subset NT-PA)) \),

1. **SPA(0⊂PA) := \{ S(0), S(1), S(2), S(3), S(4), S(5), S(6), S(7) \} \)
   \&
   \{ \{ 1, 2, 3, 4, 5, 6, 7 \} \}
   \&
   \{ \#-SPA’s_{(0}\subset PA) = 7, (7 \in \mathcal{N}(0\subset PA)) \} \)

2. **SPA^α (0⊂FOL-PA) := \{ S(11), S(12), S(21), S(22), S(31), S(32), S(41) \} \)
   \&
\[
\left\{ \{ P_0(11), P_0(12) \}, P_0(21), P_0(22), P_0(31), P_0(32), P_0(41) \} \right\}
\]

\&
\[
\left\{ \{11, 12, 21, 22, 31, 32, 41\} \right\}
\]

\&
\[
\left\{ \#: \text{SPA}(0 \subset \text{NT}-\text{PA}) = 7, \ (7 \in \mathbb{N}(0 \subset \text{PA})) \right\}
\]

Please, allow us to state both:

\textit{i.-} The next statement is “true” when referred to “statements” as statements referred to the elements of SPA\(^\alpha\) (0 \subset \text{FOL-PA}) and/or SPA\(^\alpha\) (0 \subset \text{PA}) by means of SPA\(^\alpha\) (0 \subset \text{NT-PA})

\[
\left\{ \{ S(0), S(1), S(2), S(3), S(4), S(5), S(6), S(7) \} \right\}
\]

\[
= \left\{ \{ P_0(11), P_0(12), P_0(21), P_0(22), P_0(31), P_0(32), P_0(41) \} \right\}
\]

\[
\rightarrow \quad (S(0) = P_\alpha^\alpha(11)) \land (S(1) = P_\alpha^\alpha(12)) \land (S(2) = P_\alpha^\alpha(21)) \land (S^\alpha(3)) = (P_\alpha^\alpha(22)) \land \\
S(4) = P_\alpha^\alpha(22) \land S(5) = P_\alpha^\alpha(31) \land S(6) = P_\alpha^\alpha(32) \land S(7) = P_\alpha^\alpha(41)
\]

\textit{ii.-} The next statement is “false” when referred to “natural numbers” as elements of \(\mathbb{N}^\alpha_{0 \subset \text{PA}}\) and/or \(\mathbb{N}^{\alpha}(0 \subset \text{FOL-PA})\) and/or SPA\(^\alpha\) (0 \subset \text{NT-PA}) and/or SPA(0 \subset \text{PA})

\[
\left\{ \{ S(0), S(1), S(2), S(3), S(4), S(5), S(6), S(7) \} \right\}
\]

\[
= \left\{ \{ P_0(11), P_0(12), P_0(21), P_0(22), P_0(31), P_0(32), P_0(41) \} \right\}
\]

\[
\rightarrow \quad (\{(0) = (11)\} \land (\{(1) = (12)\} \land (\{(2) = (21)\} \land (\{(3) = (22)\} \land \\
(4) = (22) \land (\{(5) = (31)\} \land (\{(6) = (32)\} \land (\{(7) = (41)\})
\]

As a direct consequence, the statement referred to the collectivity SPA\(^\alpha\)(0 \subset \text{NT-PA}) set
∀x(x ∈ ℳα(0⊂PA)) and/or ℳα(0⊂FOL-PA) and/or SPAα(0⊂NT-PA))

(∀x (S(x)≠0) → S(x) = 0),
is “false” in NT-FS&L. Please, take in account that for the last statement to “be true”, for example, requires that “the conservation of parity” has no entailment.

((o-0) = (11)) → ((o-1)=(e-12)) & ((o-5)=(o-31))

Second Highlighted NT-Considerations on Succession, Precede, Covariance, Contravariance, Partition, Addition, Multiplication of even and odd natural numbers

In NT-PA’s (NT-FS&L), which incorporate the first order logic framework, the former statements of PA’s axiomatic (1-7), when we are using notation and formulation referred to natural number zero and by using “precede” operation as intended" (sense and intension) operation, should be translated as:"81

1. On the element zero as natural number concept and “gauge” reference). 

1.1. ∀x [(0 = S0(0) ∧ P0(0) ∧ Px(x)) ∨ ((S1(x) ≠ 0 → P1(x) ≠ 0) ∧ (P1(x) ≠ 0 → S1(x) ≠ 0))].

1.2. ∀x [ [(0 = S0(x) ∧ P0(x) → x = 0] ∧ [(S1(x) ≠ 0 → P1(x) ≠ 0) ∧ (P1(x) ≠ 0 → S1(x) ≠ 0)]].

1.1. ∀x∀y [S0(x) = S0(y) → x = y ]

1.2. ∀x∀y [(S0(x) = S0(y) ∧ (P0(x) = P0(y)) → x = y ]

2. On addition operation:

2.1. ∀x [(x + 0 = x) ∧ (0 + x = x)]

2.2. ∀x∀y [(x + Sy(0) = Sx+y(0) ∧ S0(x+y) ∧ P0(x+y)) → (x + P0(y) = S0(x+y) ∧ Sx+y(0))]]

3. On multiplication operation:

3.1. ∀x [(x·0 = 0) ∧ (0·x = 0)]

3.2. ∀x∀y [((x·S0) = (x·S0) + x) ∧ (x·S0(y)) ∧ (x·P0(y))]

4. On induction schema by substitution operation:

4.1. ∀y₁ ... ∀yn ((φ(x|0) & ∀x(φ→φ(x|S(x)))) → ∀x φ), whenever φ is a formula whose free variables (also named occurrences) are among x₀, y₁, ..., yₙ.
Therefore, only in terms of the precede operation referred to zero (axioms 3-1 and 3-2 NT-PA) can be "expanded" (also, translated) in NTS&L:

\forall x \forall y \ (x, y \in \mathbb{N}(\emptyset \cup \text{NT-PA}))

3.1 \forall x ((x \otimes \text{Suc}_a^\alpha(0) \subseteq (\text{Suc}_a^\alpha(0) \otimes x) \land 1 \otimes \text{Suc}_a^\alpha(0) \land x \otimes \text{Pre}_a^\alpha(0)) \rightarrow x=1)\\
3.2 \forall x ((1 \otimes \text{Suc}_a^\alpha(0) \subseteq (\text{Suc}_a^\alpha(0) \otimes 1) \land \text{Suc}_a^\alpha(1 \cdot x) \land \text{Pre}_a^\alpha(1 \cdot x))\\
3.3 \forall x ((1 \otimes \text{Suc}_a^\alpha(0) \subseteq (\text{Suc}_a^\alpha(0) \otimes 2) \land \text{Suc}_a^\alpha(2 \cdot x) \land \text{Pre}_a^\alpha(2 \cdot x))\\
3.4 \forall x (x \cdot 0 = 0 \land 0 \cdot x \land P_x(0) \cdot P_0(0) \land P_0 \rangle_0 \land P_{x \cdot x}(x \cdot x) \land P_{x \cdot x \cdot x}(x \cdot x \cdot x))\\
3.5 \forall x ((x \cdot S_0(x) = x \cdot P_0(x) \land x \cdot P_a(x \cdot x) \land x \cdot (P_a(x) + x \cdot x)) \land (x \cdot P_a(x) + x \cdot x) \land x \cdot 0 + x \cdot x) \land (x \cdot S_0(x) = P_0(x) \cdot P_0(x) \land x \cdot x)\\
3.6 \forall x \forall y ((x \cdot S_0(y) = (x \cdot y) \cdot x \cdot P_0(y) \land x \cdot P_a(y \cdot y) \land x \cdot P_a(y \cdot y) \rightarrow (x \cdot S_0(y) = x \cdot 0 + x \cdot y) \land x \cdot y) \land (x \cdot S_0(y) = P_0(x) \cdot P_0(y) \land P_0(y) \cdot P_0(x) \land x \cdot y)\\
3.7 \forall x \forall y ((x \cdot S_0(x) = x \cdot S_0(y) \rightarrow P_0(x) \cdot P_0(x) = P_0(x) \cdot P_0(y) \land (x \cdot P_0(x) = x \cdot P_0(y) \land (P_0(x) \cdot P_0(x) = x \cdot x)) \land (P_0(x) = P_0(y) \land (x = y)\\
3.8 \forall x ((x \cdot S_1(x) = x \cdot x \cdot x \land x \cdot (x \cdot 1) \land (P_0(x) \cdot P_0(x) + P_0(x)) = (P_0(x) \cdot P_0(x) + 1) = (P_0(x) \cdot P_0(x) \cdot P_0(x) + P_0(x) \cdot P_0(x) + 1 \cdot P_0(x)) \land x \cdot S_1(x) = P_0(x) \cdot P_1(x \cdot 1) \land (P_0(x) \cdot P_0(x) + 1 \cdot P_0(x) \rightarrow x \cdot S_1(x) = (1 \cdot x + x \cdot x) \land (x \cdot x + 1 \cdot x)\\
3.9 \forall x \forall y ((x \cdot S_1(y) = x \cdot (y + 1) \land (P_0(x) \cdot P_0(x + 1)) = (P_0(x) \cdot P_0(x + 1)) \land (P_0(x) \cdot P_0(y) + 1 \cdot P_0(x) \rightarrow x \cdot S_1(y) = (P_0(x) \cdot P_0(x + 1)) \land P_0(x) \cdot P_0(y) + 1 \cdot P_0(x) \land x \cdot S_1(y) = (1 \cdot x + x \cdot y) \land (x \cdot y + 1 \cdot x)\\
3.10 \forall x \forall y ((x \cdot S_1(x) = x \cdot S_1(y) \rightarrow x \cdot S_1(x) = ((P_0(x) \cdot P_1(x + 1)) \land (P_0(x) \cdot P_0(x) + 1 \cdot P_0(x)) \land x \cdot S_1(y) = ((P_0(x) \cdot P_0(y + 1)) \land P_0(x) \cdot P_0(y) + 1 \cdot P_0(x)) \land
Please, before defining prime natural numbers, let us point out the next NT-PA considerations and observations about “parity and imparity, covariance and contravariance, in relation with the operations addition, multiplication of both; natural numbers and their “successor and predecessor” identities, and with theirs collectivities.

1. \( \forall x \in \mathbb{N}(0\subset NT-PA) \)
   \[
   \text{Even-Subset}\mathbb{N}^\alpha(0\subset NT-PA)(x+x) \subseteq \text{Even-Subset}\mathbb{N}^\alpha(0\subset NT-PA)(x+x+1)
   \]
   \[
   \text{Even-Subset}\Pre^\alpha_0(x+x) \subseteq \text{Even-Subset}\Pre^\alpha_0(x+x+1)
   \]
   \[
   \text{Even-Subset}\Pre^\alpha_0(x+x+1) = \#\text{-Even-Subset}\Pre^\alpha_0(x+x)
   \]

2. \( \forall x \in \mathbb{N}(0\subset NT-PA) \)
   \[
   \text{Even-Subset}\Pre^\alpha_0(x+x) \subseteq \{ e\cdot(0+0), e\cdot(1+1), e\cdot(2+2), ..., e\cdot(x+x) \}
   \]
   \[
   \{ 0, 2, 4, 6, 8, 10..., e\cdot(x+x) \} \subseteq \{ 0, 2, 4, 6, 8, 10..., e\cdot(x+x) \}
   \]
   \[
   \text{Even-Subset}\Pre^\alpha_0(x+x) \subseteq \Pre^\alpha_0(2\cdot x) \wedge 2 \otimes \Pre^\alpha_0(x)
   \]
   \[
   \#\text{-Even-Subset}\Pre^\alpha_0(x+x) = \#\text{-Pre}^\alpha_0(2\cdot x) \wedge \#2 \otimes \Pre^\alpha_0(x)
   \]

3. \( \forall x \in \mathbb{N}(0\subset NT-PA) \)
   \[
   \text{Even-Subset}\Pre^\alpha_0(x+x+1) \subseteq \Pre^\alpha_0(2\cdot x) \wedge 2 \otimes \Pre^\alpha_0(x)
   \]
   \[
   \#\text{-Even-Subset}\Pre^\alpha_0(x+x+1) = \#\text{-Pre}^\alpha_0(2\cdot x) \wedge \#2 \otimes \Pre^\alpha_0(x)
   \]
   \[
   \#\text{-\text{Subset}\Pre^\alpha_0(x+x)+1} = \text{\text{Subset}\Pre^\alpha_0(x+x+1)}
   \]
Odd-Subset- Pre\(^{α_0}\) (x+x) \(\subseteq\) \{ e-0+0+1, e-(1+1)+1, e-(2+2)+1, ..., e-(x+x)+1 \}
\&
\{ 1, 3, 5, 7, 9, 11..., (e-(x+x))+1 \} \subseteq Odd-Subset- Pre\(^{α_0}\) ((x+x)+1)
\&
\{ 0+1, 2+1, 4+1, 6+1, 8+1, 10+1..., e-(x+x)+1 \}
\&
\{ 1, 3, 5, 7, 9, 11..., e-(x+x)+1 \}
\[\begin{align*}
P_x(x) \land P_{x+1}(x+1) \land P_{x+x}(x+x) \land P_{x+x+1}(x+x+1) \land \not\in Odd-Pre^{α_0}((x+1)) \\
\end{align*}\]
\[\begin{align*}
Even-Subset- Pre^{α_0}(x+x) \land Even-Subset- Pre^{α_0}(x+x+1) \subseteq Pre^{α_0}(x+x) \\
Even-Subset- Pre^{α_0}(x+x) \cup Odd-Subset- Pre^{α_0}(x+x+1) \subseteq Pre^{α_0}(x+x+1) \\
Pre^{α_0}(x+x) \subseteq Pre^{α_0}(2\cdot x) \land 2 \otimes Pre^{α_0}(x) \\
Pre^{α_0}(x+x) \subseteq 1 \otimes Pre^{α_0}(x+x) \land 2 \otimes Pre^{α_0}(x) \\
1 \otimes Pre^{α_0}(x+x) \subseteq 1 \otimes Pre^{α_0}(x+x+1) \\
\end{align*}\]
\[\begin{align*}
P_0((x+x+1) \not\in 1 \otimes Pre^{α_0}(x+x) \land 2 \otimes Pre^{α_0}(x) \\
P_0((x+x+1) \not\in 1 \otimes Pre^{α_0}(x+x) \land y \otimes Pre^{α_0}(x) (\forall y \neq 1, y \in N(0\subset NT-PA)) \\
P_0((x+x+1) \subseteq 1 \otimes (1 \otimes 2 \otimes Pre^{α_0}(x))) \\
\end{align*}\]

Hence: \(^{82}\)

- Every natural even number \(x\) represented by its identity of predecessor and/or successor of a natural number referred to the even-natural number \(x+x\) (\(P_0(x+x)\)), which is referred in turn to natural number 0, is an element of the collectivity \(1 \otimes Pre^{α_0}(x+x) \subseteq 1 \otimes Pre^{α_0}(x+x) \land 2 \otimes Pre^{α_0}(1\cdot x)\)

- Every natural odd number \(x+1\) represented by its identity of predecessor and/or successor of a natural number referred to the even-natural number \(x+x\) (\(P_1(x+x)\)), which is referred in turn to natural number 0, is an element of the collectivity \(1 \otimes Pre^{α_0}(x+x) \subseteq (1 \otimes (1 \otimes 2 \otimes Pre^{α_0}(x) \land 2 \otimes Pre^{α_0}(x))\)

- Every natural odd number \(x+1\) represented by its identity of predecessor and/or successor of a natural number referred to the odd-natural number \(x+x+1\) (\((P_0(x+x+1) \land P_1(x+1+x+1))\)), which is referred in turn to natural number 0, is not an element of either a collectivity \(1 \otimes Pre^{α_0}(x+x) \land y \otimes Pre^{α_0}(1\cdot x) (\forall y \neq 1, y \in N(0\subset NT-PA))\) or, a collectivity \(1 \otimes Pre^{α_0}(x+x) \land 1 \otimes Pre^{α_0}(x\cdot y) (\forall y \neq 1, y \in N(0\subset NT-PA))\)
4. Please, allow us to remind the reader that NT-FS&L F.Th 2, is a true statement,⁸⁳ if and only if,

\[ \forall x \forall y \left( x \in \text{Pre}^{\alpha}_0 (x+1) \land \mathcal{N}^{\alpha} (0 \subset \text{NT-PA}) \land y \in \mathcal{N}(0 \subset \text{NT-PA}) \land \text{Pre}^{\alpha}_0 (x) \land \text{Pre}^{\alpha}_0 (x+x+1) \right) \land \mathcal{P}(\mathcal{N}(\text{Pre}^{\alpha}_0 (y))) \]

\[ x = P_0(x) \]

\[ x \cdot \left( 1 \otimes P_{x+x}(x+x+1) = \left( 2 \otimes \sum [(x)] \right) \right) \rightarrow y = P_0(x) + 1 \land P_{x+x}(x+x+1), \]

\[ \forall y \in \text{Sucd}(P_0(x)) \land \left( \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \subset \text{NT-PA}) \right) \]

\[ x = P_0(x) \]

\[ P_0(x) \cdot \left( 1 \otimes P_{x+x}(2 \cdot x+1) = \left( 2 \otimes \sum [(x)] \right) \right) \rightarrow y = P_0(x) + 1 \land P_{x+x}(x+x+1), \]

\[ \forall y \in \left( \text{Pre}^{\alpha}_0 (x) \cap \text{Pre}^{\alpha}_0 (P_0(x+1)) \land \text{Pre}^{\alpha}_0 (P_1(x+2)) \land \text{Pre}^{\alpha}_0 (P_{x+x}(x+x+1)) \right) \land \]

\[ \forall x \in \text{Pre}^{\alpha}_0 (P_0(x)) \land \text{Pre}^{\alpha}_0 (P_1(x+1)) \land \text{Pre}^{\alpha}_0 (P_{x+x}(x+x+1)) \land \]

\[ x, y \left( \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \subset \text{NT-PA}) \right) \]

Additionally, we have proof that in NT-FS&L, is in turn true that:

\[ \forall x \in \mathcal{N}^{\alpha} (0 \subset \text{NT-PA}) \left( e^{-}(x \in \text{Even}(P_0(x))) \rightarrow \right) \]

1. \( o \cdot y \in \text{Even} \left( \left( (P_0(x+1) \cap \text{Even}(P_0(2 \cdot x)) \cap \text{Even}(P_0(2 \cdot x+1)) \right) \right) \land \]

2. \( o \cdot y = P_0(x+1) \in \text{Odd}(P_1(x+2)) \land o \cdot y \notin \text{Pre}^{\alpha}_1 (x+1) \land \text{Pre}^{\alpha}_1 (x+x+1) \land \text{Odd}(P_0(2 \cdot x)) \land \text{Odd}(P_0(2 \cdot x+1)) \)

**HLPrime-1-NT Definition:**⁸⁴ A natural number \( y \) (\( y \in (\text{Pre}^{\alpha}_0 (x) \cap \text{Pre}^{\alpha}_0 (P_0(x+1)) \land \text{Pre}^{\alpha}_0 (P_1(x+2)) \land \text{Pre}^{\alpha}_0 (P_{x+x}(x+x+1)) \land \text{Sucd} (P_0(x)) \land \mathcal{P}(\mathcal{N}(\text{Pre}^{\alpha}_0 (y))) \land \left( \mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \subset \text{NT-PA}) \right) \)) is named (also, called renamed, nicknamed and defined) both: “prime predecessor of even natural number \( x \)” and will be represented as \( p \cdot \text{Pre}^{\alpha}_0 (x) \) (\( x \in \mathcal{N}(0 \subset \text{PA}) \land \mathcal{P}(\mathcal{N}^{\alpha}_0 (x) \land x = p \cdot \text{Pre}^{\alpha}_0 (x) \)) or “prime natural number \( y \), and then represented by “\( p \)” and formulated as:

\[ y = p \land \left( 1 \otimes P_{x+x}(x+x+1) \right) \]

if and only if, then:
\[
x \cdot y = x \cdot (1 \otimes P_{x \times x}(x+x+1))
\]

\[
(y = (1 \otimes P_{x \times x}(x+x+1)) \land p \cdot P^0 \alpha_0(p) \land p \cdot P^0 \alpha_0(y) \land p \cdot P^0 \alpha_0(y)) \rightarrow x = 1
\]
\[
\land (y = 1 \land p \land (1 \otimes P_{x \times x}(x+x+1)) \rightarrow x = 0, x \notin (\mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \subset \text{NT-PA})))
\]

When NT-FS\&L F.Th 2 is a true statement (wfs and wfs-\text{rs}) and every natural number is referred to its predecessor (and/or successor) identity is, in turn, referred to the natural number zero concept (0) identity: \text{85}

**HLPr-2-NT Definition:** The set of natural numbers \( p \ (p \in \text{Pred}^\alpha_0(x) \land p \in \mathcal{N}(0 \subset \text{NT-PA})) \), which is comprised by “prime predecessors of every natural number \( x \)” \( (x \in \mathcal{N}(0 \subset \text{NT-PA}) \land \mathcal{P} \mathcal{N}^0(x), x = p \cdot P^\alpha_0(x) \land p \cdot S^\alpha_0(0) \land p \cdot S^\alpha_0(x)) \) is named (also, called renamed, nicknamed and defined) the “set of primes of natural number \( x \)” if an only if the next statement is a wfs and wfs-r statement and every natural number as their predecessor (and/or successor) identity is, in turn, referred to natural the number zero concept (0) identity:

\[
\text{Primes}(p \cdot P^\alpha_0(x)) := \{ x = p \land (1 \otimes P_{x \times x}(x+x+1)) \in (\mathcal{N}(0 \subset \text{PA}) \cap \mathcal{N}(0 \subset \text{NT-PA})) \mid p = x \land p \cdot P^\alpha_0(x) \land (\text{Pred}^\alpha_0(x) \cap \text{Pred}^\alpha_0(P_0(x+1)) \land \text{Pred}^\alpha_0(P_1(x+2)) \land \text{Pred}^\alpha_0(P_{x \times x}(x+x+1)) \land \text{Sucd}(P_0(x)) \land \mathcal{P} \mathcal{N}^0((P_0(y)) \), \forall x \in \mathcal{N}(0 \subset \text{NT-PA})
\]

(def-Pr. 2)

**NT-FS\&L re-PRESENTATION (Decimal Number System) OF SET OF THE PRIMES NATURAL NUMBER SET in the Decimal System:**

\[
\text{Primes}(p \cdot P^\alpha_0(x)) := \{ 1, 2, 3, 5, 7, 11, 13, 17, \ldots \}
\]

On the other hand, in NTFS\&L is able to establish a complete (two equivalence classes or quotient sets; also, referred as “ a two color partition”, two collectivities whose union is an “all” and the intersection is the non-numerable and non-countable previously referred and defined set \( \Theta := \{ \emptyset | \emptyset \notin \mathcal{N}(0 \subset \text{PA}) \land \mathcal{N}(0 \subset \text{PA}) \} \) partition of the natural number \( \mathcal{N}(0 \subset \text{NT-PA}) \) set collectivity.

Hence,

\[
\text{Class I} := \{ x \in \mathcal{N}(0 \subset \text{NT-PA}) \mid x \in \text{Primes}(p \cdot P^\alpha_0(x)) \land P(0 \subset \text{NT-PA}) \} \land \text{P}(0 \subset \text{NT-PA}) := \text{Class I}
\]

\[
\text{Class II} := \{ x \in \mathcal{N}(0 \subset \text{NT-PA}) \mid x \notin \text{Primes}(p \cdot P^\alpha_0(x)) \land P(0 \subset \text{NT-PA}) \}
\]

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\[ \mathcal{NP}(0\cap\text{NT-PA}) := \text{Class II} \]
(def-Pr. 2 bis)

Hence, in NT-FS&L the new identity of \(\mathcal{N}(0\cap\text{NT-PA})\) is well symbolically represented and referred by the following statement:

\[ \mathcal{N}(0\cap\text{NT-PA}) := \{ x \in \mathcal{N}(0\cap\text{NT-PA}) \mid x \in \text{Class I} \cup \text{Class II} \} , \forall x \in \mathcal{N}(0\cap\text{NT-PA}) \]

Finally, as a direct consequence, we are able to state:

1. \( \forall x \forall p \ (x \in \mathcal{N}(0\cap\text{NT-PA}), p \in \mathcal{P}(0\cap\text{NT-PA})) \) \( (\mathcal{PN}_0(x) = \mathcal{PN}_0(p) \rightarrow x = p) \)
2. \( \forall p \ (p \in \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{P}(0\cap\text{NT-PA})) \) \( (\mathcal{PN}_0(x) = \mathcal{PN}_0(p) \rightarrow x = p) \)
3. \( \bigcup \mathcal{PN}_0(p) \}_{p \in \mathcal{P}(0\cap\text{NT-PA})} \subseteq \mathcal{P}(0\cap\text{NT-PA}) \rightarrow \)
   \( ((\mathcal{P}(0\cap\text{NT-PA}) \subset \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{N}(0\cap\text{NT-PA})) \rightarrow \)
   \( \mathcal{N}(0\cap\text{NT-PA}) \subseteq \mathcal{N}(0\cap\text{NT-PA}) \cup \mathcal{PN}_0(0) \) \)
   NT-th. 6 bis)
4. \( (\mathcal{PN}_0(0) \not\subseteq \mathcal{P}(0\cap\text{NT-PA}) \rightarrow \mathcal{P}(0\cap\text{NT-PA}) \cap \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{N}(0\cap\text{NT-PA}) \)
5. \( \forall p \ (p \in \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{P}(0\cap\text{NT-PA})) \) \( (p \in \text{Even } \mathcal{P}(0\cap\text{NT-PA}) \rightarrow p = 2) \)
   \( \land \)
   \( \forall p \ (p \in \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{P}(0\cap\text{NT-PA})) \) \( (p \in \text{Odd } \mathcal{P}(0\cap\text{NT-PA}) \rightarrow p \neq 2) \)
6. \( \forall p \ (p \in \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{P}(0\cap\text{NT-PA})) \) \( (#\text{- Even } \mathcal{P}(0\cap\text{NT-PA}) = 1 \rightarrow \)
   \( (#\text{- } \mathcal{P}(0\cap\text{PA}) = #\text{- } (\text{Odd } \mathcal{P}(0\cap\text{NT-PA}) + 1) \) \)
   \( \land \)
   \( (#\text{- } \mathcal{P}(0\cap\text{PA}) (x) = \mathcal{S}^\alpha_1 (#\text{- Odd } \mathcal{Primes}(p\cdot p^o(x))) \) \)
   \( \land \)
   \( (#\text{- } \mathcal{P}(0\cap\text{PA}) (x) = \mathcal{P}^\alpha_1 (#\text{- Odd } \mathcal{Primes}(p\cdot p^o(x))) + 2 \) \)
7. \( \forall p \ (p \in \mathcal{N}(0\cap\text{NT-PA}) \cap \mathcal{P}(0\cap\text{PA})) \)
   \( (\text{Even } \mathcal{P}(0\cap\text{PA}) \cap \text{Odd } \mathcal{P}(0\cap\text{PA}) = e\cdot \mathcal{PN}_0(2) \subseteq \mathcal{N}(0\cap\text{NT-PA}) \rightarrow \)
   \( e\cdot \mathcal{PN}_0(2) \cup \text{Odd } \mathcal{P}(0\cap\text{PA}) \subseteq \mathcal{P}(0\cap\text{NT-PA}) \)
   \( \& \)
   \( (e\cdot \mathcal{PN}_0(0) \cup e\cdot \mathcal{PN}_0(2)) \cup \text{Odd } \mathcal{P}(0\cap\text{PA}) \subseteq \mathcal{N}(0\cap\text{NT-PA}) \)

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**HLPr-3-NT Definition:** The natural number \( 2 \in \mathbb{N}(0\text{CNT-PA}) \land \text{Even}(x) \), which is in NT-FS&L the unique element of the “primordial of natural number two” \( e-PN_{0}^{a}(2) \) (briefly: \( PN(2) \)) is named (also, called renamed, nicknamed and defined) “the even-prime number” and hence, if an only if, the next statement is a wfs and wfs-r statement: \((\forall x \in \mathbb{N}(0\text{CNT-PA}))\)

\[
e-MN(2) \subseteq ((\mathbb{N}(0\text{CNT-PA}) \cap \mathbb{N}(0\text{CNT-PA})) \cap P(0\text{CNT-PA})) \rightarrow \\
(2 = p-P_{0}(2) \land p-S_{0}(1 +1) \land p-S_{1+0}(1) \land p-S_{1+1}(0) \land p-S_{2}(0) \land p-S_{1}(1)) \in \\
\text{Even-}(\mathbb{N}(0\text{CNT-PA}) \cap \mathbb{N}(0\text{CNT-PA})) \cap P(0\text{CNT-PA}) \land e-MN_{0}^{a}(2) \\
\land \\
((2 \cdot x = (1 +1) \cdot x) \rightarrow (1 \cdot x = (0\cdot 1 +1) \cdot x) \land (x \cdot 1 = x \cdot (1\cdot 0 +1)) \land (x \cdot 1 = x \cdot 1\cdot (0 +1)) \\
\land \\
(1 + 1 = 2 \rightarrow (2 + 2 = (2\cdot 2 + 0\cdot 2)) \land 2\cdot (2+0) \land ((2\cdot 1+1) +1 \land (3+1) \land (4)) ^{87} \\
\text{(def-Pr. 3)}
\]

\[
e-{PN}(2) \subseteq \{ \text{Even-}(\mathbb{N}(0\text{CNT-PA}) \cap \mathbb{N}(0\text{CNT-PA})) \cap P(0\text{CNT-PA})) \}
\text{(def-Pr. 3 bis)}
\]

\[
e-{PN}(2) := \{ P(0\text{CNT-PA}) \cap \text{Odd-Primes}(x) \}
\text{(def-Pr. 3 tris)}
\]

The natural number \( p \in \mathbb{N}(0\text{CNT-PA}) \cap P(0\text{CNT-PA}) \cap \text{Odd}(x) \), is named (also, called renamed, nicknamed and defined) “odd-prime number” and hence, element of the “set of odd -prime natural numbers” \( Odd-Primes(x) \) if an only if, the next statement is a wfs and wfs-r statement:

\[
\forall p(p \in P(0\text{CNT-PA}))(p \in Odd(x)) \rightarrow p \neq 2
\]

Hence, we can adopt the following definition for the prime natural prime set collectivity:

\[
P(0\text{CNT-PA}) := \{ (even-PN(2)) \cup (Odd-Primes(x) \subset \mathbb{N}(0\text{CNT-PA})) \}
\text{(def-Pr. 3 tris)}
\]

**HLPr-4-NT Definition:** The natural number \( x \in \mathbb{N}(0\text{CPA}) \land Odd(x) \), is named (also, called renamed, nicknamed and defined) “odd non-prime number” and hence, element of the “set of odd non-prime natural numbers” \( Odd-NP(x) \)
(briefly: o-\(\mathcal{NP}(x)\)), if an only if, the next statement is a wfs and wfs-r statement:

\[
\text{Odd-}\mathcal{NP}(x) := \{ x \in \text{Odd}(x) \mid x \not\in \mathcal{P}(0 \subset \mathcal{NP}) \land o-\text{Primes}(p-p^{\alpha}_0(x)) \},
\]

\[
\forall x \in \mathcal{N}(0 \subset \mathcal{NP}) \land \text{Odd}(x)
\]

\[
\forall x \forall p \ (x \in \mathcal{N}(0 \subset \mathcal{NP}), p \in \mathcal{P}(0 \subset \mathcal{NP}) \land \text{Suc}^{\alpha}_0(x) \land \text{Pre}^{\alpha}_0(x)
\]

(def-Pr. 4)

**HLPr-5-NT Definition:** The natural number \(x (x \in \mathcal{N}(0 \subset \mathcal{NP}) \land \text{Odd}(x))\), is named (also, called renamed, nicknamed and defined) “even non-prime number” and hence, element of the “set of even non-prime natural numbers” \(\text{Even-}\mathcal{NP}(x)\) (briefly: \(\mathcal{E-}\mathcal{NP}(x)\)) if an only if, the next statement is a wfs and wfs-r statement:

\[
\text{Even-}\mathcal{NP}(x) := \{ x \in \text{Even}(x) \mid x \not\in \mathcal{P}(0 \subset \mathcal{NP}) \land \text{Primes}(p-p^{\alpha}_0(x)) \},
\]

\[
\forall x \in \mathcal{N}(0 \subset \mathcal{NP}) \land \text{Even}(x)
\]

\[
\forall x \forall p \ (x \in \mathcal{N}(0 \subset \mathcal{NP}), p \in \mathcal{P}(0 \subset \mathcal{NP}) \land \text{Suc}^{\alpha}_0(x) \land \text{Pre}^{\alpha}_0(x)
\]

(def-Pr. 5)

**HLPrime-3-NT Theorem:** \((\forall x \in \mathcal{N}(0 \subset \mathcal{NP}))^{88}\)

The natural number \(2 (2 \in \mathcal{N}^{\alpha}(0 \subset \mathcal{NP}))\) is the unique natural number, which in turn is element of the set \(\mathcal{P}(0 \subset \mathcal{NP})\) and e-\(\mathcal{PN}_0(x)\).

\[
\forall x\ (x \in \mathcal{N}(0 \subset \mathcal{NP})) (x \in \text{e-}\mathcal{PN}_0(x) \cap \mathcal{P}(0 \subset \mathcal{NP}) \rightarrow
\]

\[
(x = 2 \Rightarrow (0+1+1) \land (1+1) \land 1 \cdot (0+1) \land (1 \cdot (1+1)) \land ((1+0) \cdot 1)) \\
\rightarrow \]

\[
x = S_2(0) \land S_1(1) \land S_0(2) \land S_0(1+1) \land S_1(0+1) \land S_1(1+0) \\
\]

(NT-Pr-th. 3)

Hence, \(\forall x, (x \in \mathcal{N}(0 \subset \mathcal{NP}))\),

The “one-time addition of natural number one (1)” is natural number two (2), which is the “one-successor of the number one (1)”, which is both: the “one-successor of natural number zero (0)” and the “one predecessor of number two (2)”.

**Please allow us to point the following statement:** \((\forall x \in \mathcal{N}(0 \subset \mathcal{NP}))^{89}\)

\[
(\text{Even}^{\alpha}_0 (x + x + 2) \subseteq \text{Even}^{\alpha}_0((x+1) + (x+1)) \rightarrow
\]

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\[ (\text{Even}^\alpha_0 (x + x) \subseteq \text{Even}^\alpha (x + x + 2) \rightarrow x = 1) \]

\&

\[ ((x+1) \in \text{Even}^\alpha_0 (x + 1) \lor \text{Odd}^\alpha_0 (x + 1)) \land ((x+x) \in \text{Even}^\alpha (S_x(0))) \]

(NT-Pr-th. 3 bis)

**HLPrime-4-NT Theorem:** \((\forall x \in N (0 \subset NT-P\alpha))\)

**Lemma I**

\[ o \cdot \mathcal{P}N_0 (0+(1)) = 1 \subseteq \text{Odd}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \]

\[ o \cdot \mathcal{P}N_0 (1+(0)) = 1 \subseteq \text{Odd}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \]

\[ e \cdot \mathcal{P}N_0 (0) \cup o \cdot \mathcal{P}N_0 (x) \subseteq \text{Odd}^\alpha_0 (x+1) \]

\[ \mathcal{P}N_0 (1) \subseteq \text{Odd}^\alpha_0 (x) \cap o \cdot \mathcal{N} \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \rightarrow \]

\[ \mathcal{P}N_0 (1) \subseteq \text{Odd}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \]

**Lema II**

\[ e \cdot \mathcal{P}N_0 (0+1+1 = 2) \subseteq \text{Even}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \land \]

\[ e \cdot \mathcal{P}N_0 (1+(1+0) = 2) \subseteq \text{Even}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset NT-P\alpha)(x) \land \]

\[ e \cdot \mathcal{P}N_0 (0) \cup e \cdot \mathcal{P}N_0 (2) \subseteq \text{Even}^\alpha_0 (x+2) \land \]

\[ e \cdot \mathcal{P}N_0 (2) \subseteq \text{Even}^\alpha_0 (x) \cap \mathcal{P}^\alpha_0 (0 \subset PA)(x) \rightarrow \]

\[ e \cdot \mathcal{P}N_0 (2) \subseteq \text{Even}^\alpha_0 (x) \cap e \cdot \mathcal{N} \mathcal{P}^\alpha_0 (0 \subset PA)(x) \land \]

**Lemma III**

\[ \text{Even}^\alpha (x + x + 2) \subseteq \text{Even}^\alpha_0 ((x+1) + (x+1)) \rightarrow \]

\[ (\text{Even}^\alpha_0 ((x + x) + 2) \subseteq \text{Even}^\alpha_0 (y + y) \rightarrow y = x + 1 \land S_1(x) \land \]

\[ S_0(x+1)) \]

(NT-Pr-th. 4)

**HLPr-6-NT Set of Definitions about NT-tensorial natural number nomenclature, formulation and notation:**

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Hereinafter:

\[ z[+x]t \ (\text{also} \ [x+]t \ z), \ [+x]t \ (\text{also} \ [x+]t \ ) \ \text{and} \ [+x] \ (\text{also} \ [x+]t \ ) \ \in \ \{ \text{Symbols and formulas of the alphabet, lexical, syntax, grammar and semantic of NT-FS&L, which are referring to the addition, succession and precede identities operations in turn to conserving and breaking parity, imparity, covariance, contravariance and primality properties of their symbolic locus-based representation identities} \} \]

\&

\[ z[-x]t \ (\text{also} \ [-x]t \ z), \ [-x]t \ (\text{also} \ [-x]t \ ) \ \text{and} \ [-x] \ (\text{also} \ [-x]t \ ) \ \in \ \{ \text{Symbols and formulas of the alphabet, lexical, syntax, grammar and semantic of NT-FS&L, which are referring to the multiplication, succession and precede, identities-operations in turn to conserving and breaking parity, imparity, covariance, contravariance and primality properties of their symbolic locus-based representation identities} \} \]

Then:

\[ \forall x \left( x \in \mathcal{N}^\alpha_0(1\oplus, 0\oplus, 0\in\text{NT-PA}) \right) \]

\[ \left( \{ \mathcal{N}(0\in\text{CPA}) \cap \mathcal{N}(0\notin\text{CPA}) \} \forall x \in (\mathcal{N}(0\in\text{CPA}) \cap \mathcal{N}(0\notin\text{CPA})) \subseteq \mathcal{P}\mathcal{N}^\alpha_0(1\oplus, 0\oplus, 0\in\text{NT-FOL-PA})(0) \right) \]

\[ \rightarrow \]

\[ \bigg( x \in \{ \mathcal{N}(0\in\text{CPA}) \ \mathcal{N}(1\oplus, 0\oplus, 0\in\text{NT-PA}) \bigg) \rightarrow 0 \in \{ \mathcal{P}\mathcal{N}^\alpha_0(1\oplus, 0\oplus, 0\in\text{NT-FOL-PA})(0) \bigg) \bigg) \]

Def-NT-6.1. \[ \forall x \forall y \forall z \forall t \left( x, y, z, t \in \mathcal{N}(1\oplus, 0\oplus, 0\in\text{NT-FOL-PA}) \right) ^{92} \]

Every natural number \( x \left( x \in \mathcal{N}(0\in\text{NT-PA}) \right) \) resulting from (also, obtained by) the addition of natural number \( z \left( z \in \mathcal{N}(0\in\text{NT-PA}) \right) \) and the natural number resulting from adding \( t \)-times \( \left( t \in \mathcal{N}(0\in\text{NT-PA}) \right) \) the natural number \( y \left( y \in \mathcal{N}(0\in\text{NT-PA}) \right) \), if an only if, the next statement is a wfs and wfs-r statement:

\[ \left( \left( x := z[+x]t \ \& \ y := 0[+x]t \right) \right) \Rightarrow \]

\[ \left( x = x[+1]_0 \ \& \ x = 0[+x]_1 \ \& \ x = x[+0]_t \ \& \ x = S_x(0) \right) \ & \]

\[ \left( y = y[+1]_0 \ \& \ y = 0[+x]_1 \ \& \ y = y[+0]_t \ \& \ y = S_y(0) \right) \ & \]

\[ \left( z = z[+z]_0 \ \& \ z = 0[+z]_1 \ \& \ z = z[+0]_t \ \& \ z = S_z(0) \right) \ & \]

\[ \left( t = t[+z]_0 \ \& \ t = 0[+z]_1 \ \& \ t = x[+0]_t \ \& \ t = S_t(0) \right) \]

{def-Pr. 6.1}
Def. NT-6.2. \(\forall x \forall y \forall z \forall t \ (x, y, z, t \in \mathcal{N}(1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}PA)) \ \{z[x]t, [x]t, [x] \in \}
\{Tensorial Symbols of the NTS&L Alphabet for natural number concept-identity\}

Hereinafter, every natural number \(x \ (x \in \mathcal{N}(0\mathbb{C}NT\text{-}PA))\) resulting from (obtained by) the multiplication of natural number \(z \ (z \in \mathcal{N}(0\mathbb{C}NT\text{-}PA))\) and the natural number resulting from the multiplication \(t\)-times \((t \in \mathcal{N}(0\mathbb{C}NT\text{-}PA))\) the natural number \(y \ (y \in \mathcal{N}(0\mathbb{C}NT\text{-}PA))\), if an only if, the next statement is a wfs and wfs-r statement:

\[
\left( (x := z[x]t \ \& \ y := 1[x]t) \right) \implies \\
(x = x[1]_1 \ \land \ x = 1[x]_1 \ \land \ x = x[1]_t \ \land \ x = S_x(0)) \ \& \\
(y = y[1]_1 \ \land \ y = 1[y]_1 \ \land \ y = y[1]_t \ \land \ y = S_y(0)) \ \& \\
(z = z[1]_1 \ \land \ z = 1[z]_1 \ \land \ z = z[1]_t \ \land \ z = S_z(0)) \ \& \\
(t = t[+1]_1 \ \land \ t = 1[1]x_1 \ \land \ t = x[1]_t \ \land \ t = S_t(0))
\]

(def-Pr. 6.2)

Hence, hereinafter the notation, nomenclature and formulation in the first order language NFS&L of every natural number and every addition symbolic-representation by the next wfs and wfs-r statement about of symbols and/or formulas in both \(\mathcal{N}(0\mathbb{C}NT\text{-}FOL\text{-}PA)\) and \(\mathcal{N}(1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}FOL\text{-}PA)\):

\[
\forall x \forall y \forall z \forall t \ (x, y, z, t \in \mathcal{N}(1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}PA) \ \& \ \mathcal{N}(0\mathbb{C}NT\text{-}FOL\text{-}PA))
\]

\[
(((z[+x]y = z[x+]y) \rightarrow ([+x]y = [x+]y)) \\
\mathcal{N}(z[x]t, z[+x]t, 1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}PA) \subseteq \mathcal{N}(0\mathbb{C}NT\text{-}FOL\text{-}PA))
\]

HLPrime-7-NT Theorem: \(\forall x \in \mathcal{N}(0\mathbb{C}NT\text{-}FOL\text{-}PA)\)

7.1.

\[
\mathcal{NP}^\alpha_0 \left( y[x]t ; z[+x]t , 1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}PA \right) \subseteq \\
\left\{ x \in \mathcal{N}(0\mathbb{C}NT\text{-}PA) \mid x = S_x(0) \ \& \ y[x]t \ \& \ x \notin \text{Primes}(p\text{-}P^\alpha_0(x)) \right\},
\]

\[
\forall x \forall y \forall t \forall p \ (x, y, t \in \mathcal{N}(0\mathbb{C}NT\text{-}PA), p \in P(0\mathbb{C}NT\text{-}PA) \ \cap \ S_{\text{Suc}}^\alpha_0(x))
\]

(NT-Pr-th. 7.1)

7.2.

\[
\mathcal{NP}^\alpha_0 \left( y[x]t ; z[+x]t ; 1\mathbb{0}, 0\mathbb{0}, 0\mathbb{C}NT\text{-}PA \right) \subseteq
\]
\{ x \in \mathcal{N}(0\cap\text{PA}) \mid x = S_x(0) \land y[x]t \land x \notin \text{Primes}(p \cdot P^\alpha_0(x)) \},
\forall x \forall y \forall t \forall p (x, y, t \in \mathcal{N}(0\cap\text{PA}), \ p \in \mathcal{P}(0\cap\text{PA}) \cap Suc^\alpha_0(x))
\}

(NT-Pr-th. 7.2)^{93}

7.3.

\mathcal{P}^\alpha(y[x]t; 1\oplus, 0\oplus, 0 \in \text{PA}) =
\{ x \in \mathcal{N}(0\cap\text{PA}) \mid x = S_x(0) \land y[x]t \land x \in \text{Class I} \},
\forall x \forall y \forall t (x, y, t \in \mathcal{N}(0\cap\text{PA}), x \in \mathcal{P}(0\cap\text{PA}) \cap Suc^\alpha_0(x))

\text{FIFTH FUNDAMENTAL NT-\textsc{FS}&\textsc{L} THEOREM}

\forall x \forall y (x \in \mathcal{N}(0\cap\text{PA}) \land y \in \mathcal{P}(0\cap\text{PA}))

\text{Part 1}
\mathcal{E}ven^\alpha_0(y+0) \subseteq \mathcal{E}ven^\alpha_0(x+x+0) \subseteq \mathcal{E}ven^\alpha_0((x+0)+(0+x)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0(y+0) \subseteq \mathcal{O}dd^\alpha_0(x+x+0) \subseteq \mathcal{O}dd^\alpha_0((x+0)+(0+x)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0(y+1) \subseteq \mathcal{E}ven^\alpha_0(x+x+1) \subseteq \mathcal{E}ven^\alpha_0((x+0)+(x+1)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0(y+1) \subseteq \mathcal{E}ven^\alpha_0(x+x+1) \subseteq \mathcal{E}ven^\alpha_0((0+x)+(x+1)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0((x+x)+1) \subseteq \mathcal{E}ven^\alpha_0(1+(x+x)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0(y+2) \subseteq \mathcal{E}ven^\alpha_0(x+x+2) \subseteq \mathcal{E}ven^\alpha_0((x+1)+(x+1)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0(y+2) \subseteq \mathcal{E}ven^\alpha_0(x+x+2) \subseteq \mathcal{E}ven^\alpha_0((0+x)+(x+2)) \rightarrow y = x+x
\mathcal{E}ven^\alpha_0((x+x)+2) \subseteq \mathcal{E}ven^\alpha_0(2+(x+x)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0(y+1) \subseteq \mathcal{O}dd^\alpha_0(x+x+1) \subseteq \mathcal{O}dd^\alpha_0((x+0)+(x+1)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0(y+1) \subseteq \mathcal{O}dd^\alpha_0(x+x+1) \subseteq \mathcal{O}dd^\alpha_0((0+x)+(x+1)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0((x+x)+1) \subseteq \mathcal{O}dd^\alpha_0(1+(x+x)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0(y+2) \subseteq \mathcal{O}dd^\alpha_0(x+x+2) \subseteq \mathcal{O}dd^\alpha_0((x+1)+(x+1)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0(y+2) \subseteq \mathcal{O}dd^\alpha_0(x+x+2) \subseteq \mathcal{O}dd^\alpha_0((0+x)+(x+2)) \rightarrow y = x+x
\mathcal{O}dd^\alpha_0((x+x)+2) \subseteq \mathcal{O}dd^\alpha_0(2+(x+x)) \rightarrow y = x+x
\&
\[ \text{Odd}^\alpha_0 (x) \subseteq \left[ \mathcal{PN}^\alpha_0 (p) \right]_{\forall p \in \mathcal{P}(0 \subset NT\text{-PA})} \cup o-\mathcal{NP}^\alpha_0 (x) \left[ \forall p \in \mathcal{P}(0 \subset NT\text{-PA}) \right] \]

\[ \mathcal{N}^\alpha_0 (y[x]t ; z[x]t , 1\Theta , 0\Theta , 0 \subset NT\text{-PA}) := \left[ \mathcal{PN}_0 (p) \right]_{\forall p \in \mathcal{P}(0 \subset NT\text{-PA})} \cup o-\mathcal{NP} (x) \cup \left[ \text{Even} (x) \right] \left[ \forall p \in \mathcal{P}(0 \subset NT\text{-PA}) \right] \]

**Part 2**

\[ \forall x \forall p \left( x \in \mathcal{N}(0 \subset NT\text{-PA}) \land p \in \mathcal{N}(0 \subset NT\text{-PA}) \cap \mathcal{P}(0 \subset NT\text{-PA}) \right) \]

\[ \mathcal{P}(0 \subset NT\text{-PA}) := \left\{ p \in \mathcal{N}(0 \subset NT\text{-PA}) \mid p \in \text{Primes}(p-p^\alpha_0(x)) \right\} \forall x \left( x \in \mathcal{N}(0 \subset NT\text{-PA}) \right) \]

\[ x = p+1 \rightarrow (p-p^\alpha_0(x)) = (p-p^\alpha_0(x+1)) \rightarrow S^\alpha_0(p+1) = S^\alpha_1(p) \land \]

\[ S^\alpha_0(p+1) = P^\alpha_0(x) \]

\[ \left\{ \mathcal{NP}^\alpha_0 \left\{ \left\{ x+x \right\} \right\} \right\}_{\forall (x+x) = p \left( x , p \in \mathcal{N}(0 \subset NT\text{-PA}) \right)} \cap \mathcal{NP}^\alpha_0 \left\{ \left\{ p^\alpha \right\} \right\}_{\forall (x+x) = p \left( x , p \in \mathcal{N}(0 \subset NT\text{-PA}) \right)} := \]

\[ \left\{ 1 , p \right\}_{\forall (x+x) = p \left( p \in \mathcal{N}(0 \subset NT\text{-PA}) \cap \mathcal{N}(0 \subset NT\text{-PA}) \right)} \]

\[ \mathcal{P}^\alpha_0(0 \subset NT\text{-PA}) \subseteq \left( \text{Even}^\alpha_0(p+p) \right) \forall p \in \mathcal{P}(0 \subset NT\text{-PA}) \subseteq \text{Even}^\alpha_0 (x) \forall x = p+1 \ p \in \mathcal{P}(0 \subset NT\text{-PA}) \]

\[ (\text{Even}^\alpha_0 (x)) \forall x = p+1 \ p \in \mathcal{P}(0 \subset NT\text{-PA}) \subseteq \mathcal{NP}^\alpha_0 \left\{ \left\{ x^N \right\} \right\}_{\forall (x+1) = p \left( x \in \mathcal{N}(0 \subset NT\text{-PA}) \right)} \]

\[ \rightarrow \]

\[ ( Even^\alpha_0 (x+x) ) \forall x = p+1 \left( x \in \mathcal{N}(0 \subset NT\text{-PA}) \right) \subseteq \mathcal{NP}^\alpha_0 \left\{ \left\{ x^N \right\} \right\}_{\forall (x+x) = 2x \left( x \in \mathcal{N}(0 \subset NT\text{-PA}) \right)} \]

\[ (NT\text{-FS&L F.Th. 5}) \]

**SIXTH FUNDAMENTAL NT-FS&L THEOREM**

\[ \forall x \forall y \forall z \forall t \forall n \forall p_1 \forall p_2 \left( x , t \in \mathcal{P}^\alpha_0 \left\{ y[x]t ; z[x]t ; 1\Theta , 0\Theta , 0 \subset NT\text{-PA} \right\} \cap \mathcal{N}^\alpha_0 \left\{ y[x]t ; z[x]t ; 1\Theta , 0\Theta , 0 \subset NT\text{-PA} \right\} \right) \]

\[ p_n \in \mathcal{P}_x^\alpha (1\Theta , 0\Theta , 0 \subset NT\text{-PA}) , n \in \mathcal{N}_x^\alpha \left\{ y[x]t ; z[x]t ; 1\Theta , 0\Theta , 0 \subset NT\text{-PA} \right\} \]

1. \[ \forall x \forall t \left( x , t \in \text{Even-Suc}_{x+x}(1) \land (\text{Odd} \left( \text{Pre}_{x+x}(1) \right)) \subseteq \mathcal{P}^\alpha_0 \left\{ y[x]t ; z[x]t ; 1\Theta , 0\Theta , 0 \subset NT\text{-PA} \right\} \right) \]

\[ \left( x + t = S_0(x+t) \land (o-S_0(x)) + S_0(t) \land (o-S_0(x)) + (o-S_0(x)) \land (e-2 \cdot S_0(x)) \right) \]
1.1. \( t = x \rightarrow (o \cdot S_0(x) = o \cdot S_0(t)) \land (o \cdot S_t(0) = o \cdot S_t(0)) \land o \cdot P_t(t+t) \land o \cdot P_x(x+x) \)
\[
\Rightarrow ((x = p_1) \rightarrow (y = p_1 \land p \cdot P_0(x) \land p \cdot P_x(x+x)))
\]
\[
x + t \in \text{Even}^\alpha (x+x) \land x \in \text{Odd-Pre}^\alpha_1(x+1)
\]
\[
x + t = 2 \cdot p_1 \land (x + x)
\]

1.2. \( t \neq x \rightarrow (o \cdot S_0(x) \neq o \cdot S_0(t)) \land (o \cdot S_t(0) \neq o \cdot S_t(0)) \land o \cdot P_t(t+t) \land o \cdot P_x(x+x) \)
\[
\Rightarrow ((x = p_1) \land (y = p_2) \rightarrow ((x = p \cdot P_0(x)) \land (t = p \cdot P_t(x) \land p \cdot P_t(x+x)))
\]
\[
x + t \in \text{Even}^\alpha_0(x+x) \land x \in \text{Odd-Pre}^\alpha_1(x+1))
\]
\[
(x + t) = (p_1 + p_2)
\]

2. \( \forall x \forall t (x, t \in \text{Even}^\alpha_0 \cdot \text{Suc}^\alpha_{(x+x+t)}(1) \land (\text{Odd}^\alpha_0 \cdot (\text{Pre}^\alpha_{x+x+t}(1)) \subseteq \mathcal{P}^\alpha_0 (y \cdot [x]_t ; z[+x]_t ; 1 \in \mathbb{N}) \cap \mathcal{N}^\alpha(y \cdot [x]_t ; z[+x]_t ; 1 \in \mathbb{N} \cap \mathbb{N}^\alpha)\)
\]

2.0. \( (\text{Odd}^\alpha_0 \cdot (\text{Pre}^\alpha_{x+x+t}(1)) \subseteq \text{Even}^\alpha_0 \cdot \text{Suc}^\alpha_{x+x+t}(1) \land (\text{Odd}^\alpha_0 \cdot \text{Pre}_{x+t}(1) \subseteq \text{Even}^\alpha_0 \cdot \text{Suc}^\alpha_{x+t}(1))\)
\[
(x + x + t) = 3 \cdot x \land (x + x + t) \land (x + x) + t \land (x) + (x + t) \land (t) + (x + x)
\]
\[
\Rightarrow
\]
\[
(t = x) \lor ((x+t) = e \cdot (x+x) \land e \cdot 2 \cdot x)
\]
\[
\Rightarrow
\]
\[
\exists (S_0(x+x+t) \land (S_0(x+x) + S_0(t)) \land (o \cdot S_0(x+x) + o \cdot S_0(x)) \land (o \cdot 3 \cdot S_0(x) \land (o \cdot (2+1) \cdot S_0(x)) \land (e \cdot (2) \cdot S_0(x) + o \cdot S_0(x)) \land (o \cdot (1+1+1) \cdot S_0(x)) \}
\]
\[
S_1((x + x + t) \land P_1((x + x + t)
\]

2.1. \( o \cdot (x + x + t) \)
2.1.1. \( t = x \rightarrow (o \cdot S_0(x) = o \cdot S_0(t)) \land (o \cdot S_t(0) = o \cdot S_t(0)) \land o \cdot P_t(t+t) \land o \cdot P_x(x+x) \)
\[ \Rightarrow ((x = p_1) \rightarrow (y = p_1 \land p \cdot P_0(x) \land p \cdot P_{x+x}(x) \land p \cdot P_x(x+x))) \]
\[ \land \]
\[ x + t \in Even_0^{\alpha} (x+x) \land x \in Odd_0^{\alpha} - Pre_1^{\alpha} (x+1) \]
\[ \land \]
\[ x + t = p_1 + p_1 \]
\[ \lor \]

2.1.2. \( t \neq x \rightarrow (o \cdot S_0(x) \neq o \cdot S_0(t)) \land (o \cdot S_t(0) \neq o \cdot S_t(0)) \land o \cdot P_t(t+t) \land o \cdot P_x(x+x) \)
\[ \land \]
\[ ((o \cdot S_0(x) + o \cdot S_t(x)) = (o \cdot S_0(t) + o \cdot S_t(t)) \rightarrow (o \cdot S_0(x) = P_t(x+x)) \land (o \cdot S_t(x) = o \cdot P_0(t)) \]
\[ \Rightarrow ((x = p_1) \land (y = p_2) \rightarrow ((x = p \cdot P_0(x)) \land (t = p \cdot P_t(x) \land p \cdot P_{x+x}(x)))) \]
\[ \land \]
\[ x + t \in Even_0^{\alpha} (x+x) \land x \in Odd_0^{\alpha} - Pre_1^{\alpha} (x+1) \]
\[ \land \]
\[ (x + t) = (p_1 + p_2) \]
\[ (NT-FS&L \ F. Th. 6) \]

**SEVENTH FUNDAMENTAL NT-FS&L THEOREM**

\[ \forall x_0 \forall x_1 \forall x_2 \ldots \forall x_n \ \forall p_0 \forall p_1 \forall p_2 \ldots \forall p_n \forall n(x_n \ p_n \ x, \ t \in \mathcal{F}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset NT-PA) \cap \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset NT-PA), \ p_n \in \mathcal{F}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset NT-PA), \ n \in \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset NT-PA) \cap \mathcal{N}_0^{\alpha} (0 \subset NT-PA) \]

1. 
[Subset] \[ \{ \mathcal{F}_0^{\alpha} [p_0, p_1, p_2, \ldots, p_n] \} \forall n, n \in \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset PA) \subseteq \]

[Subset] \[ \{ \mathcal{F}_0^{\alpha} [p_0, p_1, p_2, \ldots, p_n] \} \forall n, n \in \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset PA) \subseteq \]

[Subset] \[ \# - Subset \{ \mathcal{F}_0^{\alpha} [p_0, p_1, p_2, \ldots, p_n] \} \forall n, n \in \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset PA) = S_1(n) \rightarrow \]

[Subset] \[ \{ \mathcal{F}_0^{\alpha} [S_0(p_0), S_0(p_1), S_0(p_2), \ldots, S_0(p_n)] \} \forall n, n \in \mathcal{N}_0^{\alpha} (y[x]_t ; z[x]_t ; 1 \otimes, 0 \otimes, 0 \subset PA) \subseteq \]

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\[ \text{Subset}\{\mathcal{N}_0^\alpha[0, 1, 2, 3, \ldots, x_n]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\mathcal{N}_0^\alpha[0, 1, 2, 3, \ldots, P_1(x_n+1)]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} = S_1(x_n) \]

2.

\[ \text{Subset}\{\mathcal{P}^\alpha_0[p_0+p_0, p_1+p_1, p_2+p_2, \ldots, p_n+p_n]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\mathcal{P}^\alpha_0[S_0(p_0+p_0), S_0(p_1+p_1), \ldots, S_0(p_n+p_n)]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \]
\[ \text{Subset}\{\mathcal{P}^\alpha_0[S_0(p_0+p_0), \ldots, S_0(p_n+p_n)]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} = S_1(n) \]

\[ 2\emptyset \text{Subset}\{\text{Suc}^\alpha S_0(p_n)\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ 2\emptyset \text{Subset}\{\text{Suc}^\alpha P_1(x_n+1)\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \]
\[ \rightarrow \]
\[ \text{Subset}\{\text{Suc}^\alpha p_0, p_1, \ldots, p_n\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\text{Suc}^\alpha P_1(x_n+1)\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \]
\[ \land \]
\[ p_n = P_1(x_n+1)\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \]
\[ \rightarrow \]
\[ (p_n + p_n) \in \text{Even}^\alpha_0(x_n+x_n) \land (p \land x_n \in \text{Odd}^\alpha_0 - \text{Pre}^\alpha_1 (p - (x+1))) \]
\[ \land \]
\[ (S_0(p+p) = S_0(x+x) \land S_0(x) + S_0(p+x) \land S_0(x) + S_0(p + p))\forall x, p \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \land \mathcal{P}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \rightarrow x_n = p_n \]

3.

\[ \text{Subset}\{\mathcal{N}_0^\alpha[0.1, 2, 3, \ldots, x_n]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\mathcal{N}_0^\alpha[S_0(0), S_0(1), S_0(2), \ldots, S_0(x_n + x_n)]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\mathcal{N}_0^\alpha[P_1(0+1), P_1(1+1), \ldots, P_1((x_n+1) + (x_n+1))]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \subseteq \]
\[ \text{Subset}\{\mathcal{N}_0^\alpha[0.1, 2, 3, \ldots, p_n]\}\forall n \in \mathcal{N}^\alpha_0\{y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset PA\} \rightarrow \]
\[
\begin{align*}
\text{subset-}\{\text{Suc}_0^\alpha S_0(x_0)\}\forall n, n \in \mathcal{N}_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \subseteq \\
\text{subset-}\{\text{Suc}_0^\alpha P_1(x_n+1)\}\forall n, n \in \mathcal{N}_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \\
\text{subset-}\{P_0^\alpha [S_0(0), S_0(1), S_0(2), \ldots, S_0(p_3)]\}\forall n, n \in \mathcal{N}_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \\
\text{subset-}\{N_0^\alpha [P_1(1+1), P_1(2+1), \ldots, P_1(x_n+1)]\}\forall n, n \in \mathcal{N}_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \\
\end{align*}
\]

\[
\begin{align*}
&x_n = P_1(x_n+1)\forall n, n \in \mathcal{N}_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \\
&\quad \rightarrow \\
&\quad (x_n + x_n) \in Even_0^\alpha (x_n + x_n) \land (x_n \land p_n \in Odd_0^\alpha - Pre_1^\alpha (p-(x+1))) \\
&\quad \land \\
&\quad (S_0(p+p) = S_0(x+x) \land S_0(x) + S_0(x+x) \land S_0(x) + S_0(x) + S_0(x+p+p)) \\
&\forall x, \forall p (x, p \in P_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}] \land N_0^\alpha [y[x]; z[x]_t; z_0[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{PA}]) \quad x_n = p_n
\end{align*}
\]

\textit{Prime Numbers Fundamental NT-FS&L Inductive Theorem}

\[
\begin{align*}
&\forall x_0 \forall x_1 \forall x_2 \ldots \forall x_n, \forall p_0 \forall p_1 \forall p_2 \ldots \forall p_n (x_n p_n x, t \in P_0^\alpha [y[x]; z[x]_t; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}] \land N_0^\alpha [y[x]; z[x]_t; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}]) = p_n \in P_0^\alpha [y[x]; z[x]_t; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}, \forall n, n \in N_0^\alpha (0\subset\text{NT-PA})] \\
&1. \forall p_n (p_n \in P_0^\alpha [y[x]; z[x]_t; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}], \forall n, n \in N_0^\alpha (y[x]; z[x]_t; z[x]_t; 0\subset\text{NT-PA})) \\
&\quad p_n = p-S_0^\alpha (x_0) \in Even_0^\alpha (S_0(x_n + 1)) \land Even_0^\alpha (S_0(x_n + 1)) \rightarrow \\
&\text{subset-}\{P_0^\alpha [p-S_0^\alpha (p_0), p-S_0^\alpha (p_1), \ldots, p-S_0^\alpha (p_n)]\}\forall n, n \in N_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}] \\
&\subseteq \\
&\text{subset-}\{N_0^\alpha [p-S_0^\alpha (x_0), p-S_0^\alpha (x_1), \ldots, p-S_0^\alpha (x_n)]\}\forall n, n \in N_0^\alpha [y[x]; z[x]_t; 1\emptyset, 0\emptyset, 0\subset\text{NT-PA}] \\
&\subseteq
\end{align*}
\]

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\[\{N_{\alpha}^\alpha[p-P_{\alpha}^{\alpha}(x_{0}+1), p-P_{\alpha}^{\alpha}(x_{1}+1), \ldots, p-P_{\alpha}^{\alpha}(x_{n+1})]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\wedge\]

\[\{N_{\alpha}^\alpha[Suc_{\alpha}^{\alpha}(p-P_{\alpha}^{\alpha}(x_{n}+1))]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\subseteq\]

\[\{N_{\alpha}^\beta[Pre_{\beta}^{\beta}(p-P_{\alpha}^{\alpha}(x_{n}+1))]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\rightarrow\]

\[\{P_{\alpha}^{\alpha}[p_{0}, \ldots, p_{n}, p_{0}]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\forall p_{n} (p_{n} \in \mathcal{P}_{\alpha}^{\alpha} \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{PA}), \forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 0\subset\text{NT-PA}\]

\[p_{n} + p_{n} = S_{\alpha}^{\alpha}(p_{n} + p_{n}) \in Even_{\alpha}^{\alpha} S_{0}(2 \cdot p_{n} + 1) \rightarrow\]

\[\{P_{\alpha}^{\alpha}[p-S_{\alpha}^{\alpha}(p_{0}), p-S_{\alpha}^{\alpha}(p_{1}), \ldots, p-S_{\alpha}^{\alpha}(2 \cdot p_{n} + 1)]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\subseteq\]

\[\{N_{\alpha}^\alpha[p-S_{\alpha}^{\alpha}(x_{0}), \ldots, p-S_{\alpha}^{\alpha}(x_{2n}+1)]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\subseteq\]

\[\{N_{\alpha}^\alpha[p-P_{\alpha}^{\alpha}(x_{0}+1), \ldots, p-P_{\alpha}^{\alpha}(x_{2n}+1)]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\wedge\]

\[\{N_{\alpha}^\alpha[Suc_{\alpha}^{\alpha}(p-P_{\alpha}^{\alpha}(x_{2n}+1))]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\subseteq\]

\[\{N_{\alpha}^\beta[Pre_{\beta}^{\beta}(p-P_{\alpha}^{\alpha}(x_{2n}+1))]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\rightarrow\]

\[\{P_{\alpha}^{\alpha}[p_{0}, p_{1}, p_{2}, \ldots, p_{n}, p_{n+1}, \ldots, p_{2n}]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

\[\wedge\]

\[\{P_{\alpha}^{\beta}[p_{2n}, \ldots, p_{n+1}, p_{n}, \ldots, p_{2}, p_{1}, p_{0}]\}\forall n, n \in \mathcal{N}_{\alpha}^\alpha \mid y[x]t; z[+x]t; 1\otimes, 0\otimes, 0\subset\text{NT-PA}\]

In view of all the above, the next well symbolically represented and formulated statements hold true in the herein described NT-FS&L language: \(\forall x \forall y \forall t \forall p \text{ General}\)
nomenclature, formulation and notation (Alphabet valid for any numerical system referring the natural number concept)

\[ \forall x, \forall n (x = S_0(x) \land n = S_0(x+1) \forall n, n \in \mathbb{N}^\alpha [1, 0], \) \land \] \[ \forall y, \forall z, \forall t (x, x_n, y, z, t, n = S_0(x+1) \forall n, n \in \mathbb{N}^\alpha [1, 0], 0 \subseteq PA) \] \[ \subseteq \mathbb{N}^\alpha (x_n, z[x+1], z[x+1], z[x+1], z[x+1], z[x+1], z[x+1], z[x+1], z[x+1], t, (0 \subseteq NT-PA)) 95.96\]

1.1. \( \text{Suc}_0^\alpha (x) = \{ S_0(x), \ldots, (S_0(x)+1), (S_0(x)+2), \ldots, (S_0(x)+x), \ldots \} \) \&

1.2. \( \text{Suc}_0^\alpha (0, x) = \{ S_0(0), \ldots, S_{x+1}(0), S_{x+2}(0), \ldots, S_{x+x}(0), \ldots \} \) \&

1.3. \( \text{Pre}_0^\alpha (x) = \{ o-P_0(0), o-P_0(1), \ldots, P_0(x), \ldots \} \) \&

1.4. \( \mathbb{N}_0^\alpha (\text{NT-PACNT-FS&L}) = \{ e-P_0(x), o-P_0(1), \ldots, P_0(1), P_0(0) \} \)

1.5. \( \mathbb{N}_0^\alpha (\text{NT-PACNT-FS&L}) = \{ \ldots, P_0(x), \ldots, P_0(2), P_0(1), P_0(0) \} \)

1.6. \( \text{Suc}_0^\alpha (x) = \{ e-S_0(x), o-S_0(x+1), e-S_0(x+2), \ldots, e-S_0(x+x), \ldots \} \}

\[ \forall x_0, \forall x_n (x_n \in \{ \text{Subset -Even}_0^\alpha (x_n) \land \text{Subset -Even}_0^\alpha (x_n+1) \land \text{Subset -Even}_0^\alpha (x_n+1) \subseteq \mathbb{N}_0^\alpha (0 \subseteq PA) \subseteq \{ \text{NT- FS&L} \}) \}, \]

\[ \forall n (\{ \text{Subset -Odd}_0^\alpha (x_n+1) \subseteq \mathbb{N}_0^\alpha (0 \subseteq PA) \subseteq \{ \text{NT- FS&L} \}) \forall x, \]

\[ \forall x_n (\{ \text{Subset -Suc}_0^\alpha (x_n+1) \subseteq \mathbb{N}_0^\alpha (0 \subseteq PA) \subseteq \{ \text{NT- FS&L} \}) \forall x, \]

\[ o-p_n = p_n-P_1(x_n+1) \land x_n = e- S_0(x_n) \land e-S_0(x_n+0) \land e-S_0(x_n+x_n) \land e- S_{x_n}(x_n+n) \land e- (S_{x_n}(x_n)+S_0(n)) \land e-(S_{x_n}(x_n)+S_1(n+1)) \land e- (S_{x_n}(x_n))+P_1(n+1) \]

\[ \rightarrow \]

\[ o-p_n = p_n-S_1(x_n+1) = p_n-S_1(x_n+1) = p_n-(S_0(x_n+2)) = p_n-(S_0(x_n)+2) = p_n-(S_0(x_n)+2) = p_n-(S_0(x_n)+1) = p_n-(S_0(x_n)+2) \rightarrow o-p_n \subseteq \{ \text{Subset -Pre}_0^\alpha p_n-(P_0(x_n+2)) \} \]

\[ x_n \subseteq \{ \text{Subset -Pre}_0^\alpha p_n-(S_2(x_n)) \} \land \{ \text{Subset -Even}_0^\alpha (x_n+x_n) \subseteq \mathbb{N}_0^\alpha (0 \subseteq NT-PA) \subseteq \{ \text{NT- FS&L} \}, \]

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\[(o \cdot p_n + o \cdot p_n) = e \cdot (x_n) \land 2 \cdot (o \cdot p_n) \land e \cdot (x_n + x_n) = e \cdot (2 \cdot x_n) \implies (e \cdot x_n \neq o \cdot p_n) \lor (e \cdot (2 \cdot p_n)) = e \cdot (2 \cdot x_n) = e \cdot (x_n + x_n) \implies
\]

2.1.1. \((e \cdot x_n \neq o \cdot p_n) \implies ((o \cdot p_n = p_n \cdot P_1(x_n+1)) \land (e \cdot x_n = p_n \cdot S_0(x_n)) \implies e \cdot x_n = p_n \cdot S_0(x_n) \land p_n \cdot S_n(x_n + x_n)
\]
\[
\land p_n \cdot P_0(x_n) \land p_n \cdot P_n(x_n + x_n) \land (o \cdot p_n + o \cdot p_n) \land 2 \cdot (o \cdot p_n)
\]
\[
\implies e \cdot x_n = e \cdot (o \cdot p_n + o \cdot p_n) \land e \cdot (P_1(p_n+1) + P_1(p_n+1)) \land 2 \cdot S_0(x_n)
\]
\[
\land S_0(x_n) + S_0(x_n) \land e \cdot (S_1(p_n+1) + S_1(p_n+1)) \land e \cdot (S_2(p_n) + S_2(p_n))
\]

2.1.2. \(e \cdot (2 \cdot p_n)) = e \cdot (2 \cdot x_n) = e \cdot (x_n + x_n) \implies (o \cdot p_n + o \cdot p_n) = (p \cdot P_1(x_n+1) + p \cdot P_1(x_n+1)) \land e \cdot (p \cdot P_1(x_n+1)
\]
\[
+(x_n+1)) \land e \cdot (p \cdot P_1(x_n+2) \land e \cdot (p \cdot P_1(x_n+1)) = 2 \cdot (p \cdot P_1(x_n+1)) \land 2 \cdot S_0(x_n)
\]
\[
\land e \cdot (S_0(x_n) + S_0(x_n)) \land e \cdot (S_2(p_n) + S_2(p_n))
\]

3.1. \(\forall x_n. (x_n \in \{\text{Subset -Odd}^\alpha 0(x_n+1) \land \text{Subset -Odd}^\alpha 0(x_n + x_n + 1))
\]
\[
\subseteq N^\alpha 0(0 \subset PA) \subseteq \{\text{NT - FS\&L}\})
\]
\[
\forall p_n \in \{\text{Subset -Odd}^\alpha 0(x_n+1) \land \text{Subset -Odd}^\alpha 0(2 \cdot x_n + 1) \subseteq P^\alpha 0(0 \subset PA) \subseteq \{\text{NT - FS\&L}\})\forall x_n,
\]
\[
\forall n \in \{\text{Subset -Suc}^\alpha 0(x_n+1) \subseteq N^\alpha 0(0 \subset PA) \subseteq \{\text{NT - FS\&L}\})\forall x
\]
\[
o \cdot p_n = p_n \cdot P_1(x_n+1) \land x_n = e \cdot S_0(x_n) \land e \cdot S_0(x_n+0) \land e \cdot S_n(x_n + x_n)
\]
\[
\land e \cdot S_n(x_n + n) \land e \cdot (S_n(x_n) + S_0(n)) \land e \cdot (S_n(x_n) + S_1(n+1))
\]
\[
\land e \cdot (S_n(x_n)) + P_1(n+1)
\]
\[
\implies \text{Subset -Odd}^\alpha 0(x_n + x_n + 1) \subseteq \text{Subset -Odd}^\alpha 0((x_n + x_n) + 1) \subseteq
\]
\[
\text{Subset -Odd}^\alpha 0(x_n + x_n) \cup \text{Subset -Odd}^\alpha 0(1) \land
\]
\[
\text{Subset -Odd}^\alpha 0(x_n) \cup \text{Subset -Odd}^\alpha 0(x_n + 1) \rightarrow
\]
\[
x_n \in \{\text{Subset -Odd}^\alpha 0(x_n+1) \cup (\text{Subset -Odd}^\alpha 0(e \cdot (x_n+x_n)) \subseteq \text{Subset -Even}^\alpha 0(x_n + x_n))\}
\]
\[
\rightarrow
\]
\[
\exists ((o \cdot p_n + o \cdot p_n + o \cdot p_n) = 3 \cdot o \cdot p_n = o \cdot p_n + e \cdot (o \cdot p_n + o \cdot p_n) = (x_n + x_n) + o \cdot p_n
\]
\[
= (o \cdot p_n + x_n) + o \cdot p_n = e \cdot (o \cdot x_n + o \cdot x_n) + o \cdot p_n = (o \cdot x_n + o \cdot x_n) + o \cdot x_n
\]
\[
= o \cdot x_n + o \cdot x_n + o \cdot x_n = 2 \cdot (p_n \cdot x_n) + 1 \cdot p_n \cdot x_n
\]

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Further numerable and countable NT FS&L Theorems by means of “induction” (Entailment ensured from the above analyzed cases 2.1.1 & 2.1.2 from NT-PA.)

Further numerable and countable NT FS&L-Theorems by means of “induction” (Entailment ensured from the above analyzed cases from NT-FOL-PA).

6. NT-DEMONSTRATION (PROOF) OF THE FAMILY OF CONJECTURES KNOWN AS “GOLDBACH’S”

Please, for all of the above, let us state in NT-SF& that:

∀x,x_n (x = S_0(x) \land n = S_0(x+1) \forall n, n \in N, y[\cdot x]; z[+x]t; 1\otimes, 0\otimes, NT-PA \subseteq NT-FS&L),

∀y,y,z, \forall t

[x, x_n, y, z, t, n = S_0(x+1) \forall n, n \in N, \otimes_{+0} \{y[\cdot x]; z[+x]t; 1\otimes, 0\otimes, 0\subseteq PA\} \subseteq

\{\mathcal{N}_0(x_n, z[+x]t, z[+x]t, z[+x]t, z[+x]t, z[+x]t, z[+x]t, z[+x]t, z[+x]t \subseteq (0 \subseteq PA))\}

{\mathcal{N}_{\alpha}(x_n) \land \mathcal{N}_{\alpha}(x_n) \subseteq \{\mathcal{N}_{\alpha}(0 \subseteq PA) \subseteq \{NT-\ \FS&L\} \cap \{\mathcal{N}_{\alpha}(0 \subseteq PA) \subseteq \{NT-\ \FS&L\} \}

\Rightarrow

\mathcal{P}_{\alpha}(1) \land \mathcal{P}_{\alpha}(2) \land \mathcal{P}_{\alpha}(3) \land \mathcal{P}_{\alpha}(5) \land \mathcal{P}_{\alpha}(7)

\subseteq

\{\mathcal{N}_{\alpha}(0 \subseteq PA) \subseteq \{NT-\ \FS&L\} \cap \{\mathcal{P}_{0}(0 \subseteq PA) \subseteq \{NT-\ \FS&L\} \}

Quod erat demonstrandum

All of the above, we are able to offer a translation, from the Latin-German, 18\textsuperscript{th} century into contemporary English by means of the NT-PA FS&L, of the statements conforming the family of conjectures known as Goldbach’s, and formally express and evoke at the same time a logical and mathematical truth, if and only if, the following four NT-FS&L theorems referred to the natural number and to the prime natural number concepts are at once well formed and well symbolically representation statements about the natural number concepts involved:

0 = (0 + 0); 1 = (0 + 1); 2 = ((0 + 2) = (1 + 1)); 3 = ((0 + 3) = (2 + 1) = (1 + 1 + 1))
1. -(GC-I)- Goldbach-Euler Statement:

“Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units”.

(Translated into contemporary English, from Latin-German, 18th century a.C.)

“All odd numbers greater than 7 are the sum of three odd primes”

(NT-FS&L theorem in contemporary English)

2. -(GC-IV)-Goldbach-Euler Statement:

“Every even integer greater than 2 can be written as the sum of two primes”.

(Translated into contemporary English, from Latin-German, 18th century a. C.)

“Every even “natural number non-zero-successor” of the natural number “two-successor of natural number zero” of prime natural number one” ( “two-successor of natural number zero” hereinafter briefly represented in turn in decimal system by both: i.- the decimal symbol 2 representing the natural number two concept, and ii.- by the well symbolic representation formula and true statements referred to natural number zero (0+1)+(0+1) = (1·(1+1) and/or (1·2+0) and/or (1+1), can be covariant-wise represented and identify as the addition of two prime natural numbers referred to natural number zero”

(NT-FS&L theorem in contemporary English)

3. -(GC-II), (GC-III) and -(GC-V) Goldbach-Euler Statements, respectively:

“Every integer greater than 2 can be written as the sum of three primes” and “Every integer greater than 5 can be written as the sum of three primes” and “All odd numbers greater than 7 are the sum of three odd primes”

(Translated into contemporary English, from Latin-German, 18th century a. C.)

“Every odd “natural non-zero-successor” of the natural number “one-successor of natural number zero” hereinafter briefly represented in turn in decimal system by both: i.- the decimal symbol 1 representing the natural number one concept, and, ii.- the well symbolic representation formula and true statements (0+1) = (1·(0+1) and/or (1·1+0)), can be covariant-wise represented as the sum of tree prime natural numbers referred to natural number zero and being “three” the “three-successor of natural number zero” (briefly represented in turn in decimal system by both: i.- the decimal symbol 3 representing the natural number three concept, and, ii.- the well symbolic representation formula and true statements 3 = (0+1)+(0+1)+(0+1) and/or (3·(1+0) and/or (1·(1+1)+ 1) and/or (1·2+ 1) and/or (1+1+1))”

(NT-FS&L theorem in contemporary English)
Hence, we have described that the new system model and formal language, which provide a new signature for the set of the natural numbers, the family of conjectures known as Goldbach’s holds entailment and truth and therefore has been proved (proof).

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Finally, we apologize to many who have not been mentioned today, but to whom we are grateful. Finally, let us point out that we specially apologize to many who have been mentioned herein for any possible misunderstanding regarding the sense and intension of their philosophic, scientific and/or technical hard work and milestone ideas; we hope that at least Goldbach, Euler and Feymann do not belong to this last human’s collectivity.

1 First contribution to the NT Intellectual Creations & Invention Reports set titled NT-Principia of Human Reasoning and Knowledge (P. Noheda and N. Tabarés, Copyright-NT, 2016).

NT-I-Intellectual Creation, Part I. On the Noheda-Tabarés Formal System and Language (NT-FS&L) and Part II. “The first set of Noheda-Tabarés (NT) Theorems. A primordial, mathematical, logical and computable, demonstration (proof) of the family of conjectures known as Goldbach’s”, P. Noheda and
Please, let us point out that a more detailed version of NT-I will follow, having due respect the processes the authors are required to follow in order to secure authorship, image, and industrial and intellectual property of all persons and institutions involved.


3 Leonhard Euler’s original correspondence, Internet access: http://eulerarchive.maa.org/.


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1. "Goldbach's conjecture is not just one of the most difficult unresolved problems in number theory, but of all mathematics." (Lecture, Society of Mathematics. Copenhagen. 1921; "A Mathematician's Apology" 1940 (First Electronic Edition, Version 1.0. University of Alberta Mathematical Sciences Society. (Internet access: http://www.wmath.ualberta.ca/mss. G. H. Hardy's quotation.).) 2. "On the other hand, it is documented that Descartes also was aware of the two prime version of the Goldbach's conjecture before Goldbach was. So is it named? Erdős said that "It is better that the conjecture be named after Goldbach because, mathematically speaking, Descartes was infinitely rich and Goldbach was very poor." (Internet access: http://www.daviddarling.info/encyclopedia/G/Goldbach_conjecture.html and https://primes.utm.edu/ and http://primes.utm.edu/glossary/page.php?sort=GoldbachConjecture C. Caldwell. "The Prime Glossary: Goldbach's conjecture. The Prime Pages". Tennessee University.)


17. Joseph (Giuseppe) Peano, (1889). 1."Arithmetices Principia. Nova methodo exposita". Augustae Taurinorum, Ediderunt Fratres Bocca (Internet Archive. Frates Bocca. Facsimile treatise in Latin. Peano would publish later works in his own artificial language, Latino sine flexione, which is nowadays considered a grammatically simplified version of Latin. (https://archive.org/details/arithmeticesprir00peanoog. See also Segre, M. (1994) "Peano's Axioms in their Historical Context". Archive for History of Exact Sciences. 48 (3/4): 201–342). Please, take into account that Peano’s document is currently globally accepted as a “paradigmatic document” introducing what is nowadays considered both; the first logical and mathematical axiomatization of a reference of the set of the natural numbers; as well as the most “pervasive and customary notation, nomenclature and symbolic formulation”. Peano introduces for the first time the customary symbols for the basic set operations $\cap$, $\cup$, $\subseteq$, which will be used in the present work (sense and intension). Additionally, in Peano’s document it is formally intended suggested that the identity concept has to be, not only semantically, but also symbolically, correlated with the equality concept. On the other hand, the equals sign (equality sign $\Rightarrow$ or “double hyphen”); in Unicode and ASCII, it is U+003D = EQUALS SIGN (HTML &amp;#61); initially proposed by Robert Recorde (1557) and which is a ubiquitous mathematical symbol that when is “placed” usually, in an equation, with the meaning that two expressions have the same value” (also, “magnitude

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of truth”); In Peano’s, in an equation, the equals sign will be “placed” between two expressions that are referring to any of their “conventional accepted identities” and to “their conventional accepted symbolic-identities representations” (Please, see Jeff Miller, “Earliest Uses of Symbols of Set Theory and Logic”. Internet access http://jeff560.tripod.com/set.html).


25 Please, let us introduce the following eight general NT-considerations about nomenclature, formulation and notation for the set of natural numbers \( \mathbb{N} \):

1. \([P]\), \(P\) and \(P\); **Peano axiomatic** of the natural number concept and any collectivity of them.

2. \([PA]\), \((PA)\) and \(PA\); **Peano Arithmetic axiomatic** of natural number concept and any collectivity of them.

3. In formula, \(\mathbb{N}\ [0\in\mathbb{N}]\), subindex \("[0\in\mathbb{N}]\) means that “natural number zero concept” is included in **axiomatic** briefly named PA “; the statement “ \(0 \in \mathbb{N}\)" means “natural number zero belongs natural numbers set \(\mathbb{N}\) described by set of axioms briefly named P".

4.
0 \in \mathcal{N}[P] \Rightarrow \mathcal{N}[\neg \mathcal{P}] := \mathcal{N}[P \subset 2.0.0-2.1.5]
\land
\mathcal{N}[\neg \mathcal{P}] := \{\text{successors of } 0\}0 \subset \mathcal{N}[\neg \mathcal{P}]
\text{(briefly, } \{\text{suc}(0)\}0 \subset \mathcal{N}[\neg \mathcal{P}]
\text{and/or } \{0\}0 \in \mathcal{N}[\neg \mathcal{P}]).

5.
0 \notin \mathcal{N}[P] \Rightarrow \mathcal{N}[\neg \mathcal{P}] := \mathcal{N}[P \subset 2.1.0-2.0.5]
\land
\mathcal{N}[\neg \mathcal{P}] := \{\text{successors of } 1\}0 \notin \mathcal{N}[\neg \mathcal{P}]
\text{(briefly, } \{\text{suc}(1)\}0 \notin \mathcal{N}[\neg \mathcal{P}]
\text{and/or } \{1\}0 \in \mathcal{N}[\neg \mathcal{P}]).

6.
\{0\}0 \in \mathcal{N}[\neg \mathcal{P}] := \{\text{suc}(0)\}0 \subset \mathcal{N}[\neg \mathcal{P}] \land \{\text{suc}(1)\}0 \notin \mathcal{N}[\neg \mathcal{P}]
\land\lor
\{0\}0 \in \mathcal{N}[\neg \mathcal{P}] := \{\{\text{suc}(0)\}0 \subset \mathcal{N}[\neg \mathcal{P}] \land \{\text{suc}(1)\}1 \subset \mathcal{N}[\neg \mathcal{P}]
\land\lor
\{1\}0 \in \mathcal{N}[\neg \mathcal{P}] := \{\{\text{suc}(0)\}0 \subset \mathcal{N}[\neg \mathcal{P}] \land \{\text{suc}(1)\}1 \subset \mathcal{N}[\neg \mathcal{P}]
\land\lor
\mathcal{N}[P] := \{\{\text{suc}(0)\}0 \subset \mathcal{N}[\neg \mathcal{P}] \land \{\text{suc}(1)\}1 \subset \mathcal{N}[\neg \mathcal{P}]
\land\lor
\mathcal{N}[P] := \{\{\text{suc}(1)\}1 \subset \mathcal{N}[\neg \mathcal{P}] \land \{\text{suc}(0)\}0 \notin \mathcal{N}[\neg \mathcal{P}]
\land\lor
\Rightarrow \mathcal{N}[P] \subseteq \mathcal{N}[\neg \mathcal{P}] \land \mathcal{N}[\neg \mathcal{P}]
\land\lor
\mathcal{N}[P] \subseteq \mathcal{N}[\mathcal{P}].
integer that is neither a natural number concept.

Please, let us observe that the singleton collectivity of natural numbers (set and subsets) herein will be symbolically represented by the natural number two (natural number one (magnitude (“amount of unities” quantifying a magnitude in Physics) of a singleton will be the amount of unities quantifying a magnitude in Physics) of a singleton will be the symbol “#” ("hash" in English; "almohadilla", in Spanish). Please, let us remember to the reader the concept of “mol” in the International System of Measuring as well as the universal constant named Avogadro’s number)

Hereinafter the natural number concept representing “the amount of elements (also: 1. members and, “entities” (Chemistry, Biology and Statistic), 2. “measurement and/or value of a magnitude” (“amount of unities” quantifying a magnitude in Physics) of a singleton will be the natural number one (1); the natural number concept representing the amount of a doubleton is the natural number two (2) and so on. The natural number identity of the amount of elements of a collectivity of natural numbers (set and subsets) herein will be symbolically represented by the symbol “#” (“hash” in English; "almohadilla", in Spanish). Please, let us remember to the reader the concept of “mol” in the International System of Measuring as well as the universal constant named Avogadro’s number)

Please, let us observe that the singleton \( \{ 0 \} \in \mathbb{N}[0 \subset PA] \) is conformed by one element, the natural number zero (0). Hereinafter, the “empty set” (\( \emptyset \)) defined by the next statement and formula:

\[
\emptyset := \{ \emptyset \mid \emptyset \notin \mathbb{N}[0 \subset PA] \land \mathbb{N}[0 \subset PA] \}, \forall n \in \mathbb{N}[0 \subset PA] \lor \mathbb{N}[0 \subset PA] ; \{ \emptyset \} \in \emptyset
\]

it is not “a countable set” as a direct consequence (sense and intension) that their elements are not a natural number concept.

\[
\forall n \in \mathbb{N}[0 \subset PA] \lor \mathbb{N}[0 \subset PA] \land \emptyset \notin \mathbb{N}[0 \subset PA] \lor \mathbb{N}[0 \subset PA]
\]

Quotation’s Internet access: http://mathworld.wolfram.com/Zero.html. Kaplan, Robert. (2000). The Nothing That Is: A Natural History of Zero. Oxford: Oxford University Press.: “Zero is the integer denoted \( 0 \) that, when used as a counting number, means that no objects are present ... It is the only integer that is neither (graded, categorized) as negative nor positive. A root of a function \( f \) is also customary graded and known as "a zero of \( f \)".
Please, let us remember two conceptually radical different perspectives (which are globally accepted per years) about the singleton concept and their “core-role” in the state of the art of the current accepted typed framework structures (the Set and Category Theories), from that described in this document (NT-FS-L): 1. (Please, take into account that “both, Zermelo-Frankel (briefly ZF) and Zermelo-Fraenkel axiomatic complemented with the “axiom of choice” (briefly ZFC) set theories, are focused and addressed -sense and intension- on a real number set (also, ℝ) axiomatic). Set Theory. " Within the framework of Zermelo–Fraenkel set theory, the “axiom of regularity” guarantees that no set is an element of itself. This implies that a singleton is necessarily distinct from the element it contains; thus 1 and {1} are not the same thing, and the empty set is distinct from the set containing only the empty set. A set such as \{1, 2, 3\} is a singleton as it contains a single element (which itself is a set, however, not a singleton). A set is a singleton if and only if its cardinality is 1. Additionally, in the standard set-theoretic construction of the natural numbers set (also, ℕ), the number n is defined as the singleton \{0\}. In axiomatic set theory, the existence of singletons is a consequence of the "axiom of pairing": for any set A, the axiom applied to A and A asserts the existence of \(A, A\), which is the same as the singleton \(A\) (since it contains A, and no other set, as an element). If A is any set and S is any singleton, then there exists precisely one function from A to S, the function sending every element of A to the single element of S. Thus, every singleton is a terminal object in the category of sets. A singleton has the property that every function from it to any arbitrary set is injective. The only non-singleton set with this property is the empty set". 2. Category Theory. “Typed structures built on singletons very often serve as “terminal objects” or “zero objects” of various categories: 1. The singleton sets are precisely the terminal objects in the category “set of sets”. No other sets are terminal. 2. Usually, any singleton admits a “unique topological space structure”. These singleton topological spaces are “terminal objects” in the category of topological spaces and continuous functions. No other spaces are terminal in that category. 3. Any singleton admits a unique group structure (the unique element serving as identity element). These singleton groups are zero objects in the category of groups and group homomorphisms. No other groups are terminal in that category. (Internet access to the quotation:

\[(\emptyset \in \mathcal{N}[0 \in \mathop{\text{CPA}}] \land \emptyset \notin \mathcal{N}[0 \in \mathop{\text{CPA}}]) \land (1 \in \mathcal{N}[0 \in \mathop{\text{CPA}}] \land 1 \in \mathcal{N}[0 \in \mathop{\text{CPA}}]) \Rightarrow
\]\
\[(\# \mathcal{N}[0 \in \mathop{\text{CPA}}] = \# \mathcal{N}[0 \in \mathop{\text{CPA}}] + 1, \{0\} \in \mathcal{N}[0 \in \mathop{\text{CPA}}] \land \{1\} \in \mathcal{N}[0 \in \mathop{\text{CPA}}] \land \mathcal{N} \subseteq \mathcal{N}[0 \in \mathop{\text{CPA}}])
\]


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1. **Robinson's Arithmetic** (briefly, "Q").  


36 Please, let us remember the next w.f.s statement: the infinity concept, which nowadays is the customary represented by symbol "∞", is not element of $\mathcal{M}(\ulcorner \text{FOF-PA} \urcorner)$ and/or $\mathcal{M}(\ulcorner \text{FOF-PA} \urcorner)$. Thus, in NT-FS&L first order language:

$$\left( \text{∞} \notin \mathcal{M}(\ulcorner \text{FOF-PA} \urcorner) \land \mathcal{M}(\ulcorner \text{FOF-PA} \urcorner) \right) \land \left( \text{∞} \notin \text{Alphabet} (\mathcal{M}(\ulcorner \text{PA} \urcorner)) \land \mathcal{M}(\ulcorner \text{FOF-PA} \urcorner) \right)$$
\[
(0 \in \mathcal{N}(\text{FP-PA}) \land 0 \notin \mathcal{N}(\text{FOL-PA}) ) \land (0 \in \text{Alphabet}(\mathcal{N}(\text{PA})) \land \mathcal{N}(\text{FOL-PA}))
\]


37 The “axiom of mathematical induction” (axiom 7 of PA’s, also nicknamed “induction schema” and onwards the “pillar” of logic induction-deductive apparatus of NT-FS&L) provides a unique methodology (mainly, by substitution) for reasoning about the set of all natural numbers and their Arithmet. Please, let us observe that:

\[
\mathcal{N}(0 \in \text{FOL-PA}) \subseteq \{0, S(0), S(S(0)), \ldots\} \ (\text{Entailment from NT-FS&L F.Th. 1})
\]

\[
\mathcal{N}(0 \in \text{FOL-PA}) \implies \{0, S(0), S(S(0)), \ldots\} \rightarrow \mathcal{N}(0 \in \text{FOL-PA}) \subseteq \{0, S(0), S(S(0)), \ldots\} \subseteq \{0, 1, 2, 3, 4, \ldots, x+0, x+1, x+2, \ldots\}
\]

38 Please, let us remember statement 2.1.3 of FOL-PA:

\[
\forall x (x \in \mathcal{N}(0 \in \text{FOL-PA})) (x + 0 = x)
\]

Please, take in account that in NT-FSL FOL-PA any statement about notation, formulation and nomenclature of succession operation (also function) of a natural number, zero for example, \( S_0(0) \), is accepted as a w.f.s-r statement, if and only if:

\[
\text{numerical index (also } \text{"x-fold" the operation \& "x-times" the operation)} \ni \\
\{ x \mid x \in \mathcal{N}(0 \subset \text{FOL-PA}) \land \text{Alphabet (} \mathcal{N}(0 \subset \text{FOL-PA}) \cap \mathcal{N}(0 \subset \text{FOL-PA}) \}) \forall x (x \in \mathcal{N}(0 \subset \text{FOL-PA})) \\
\text{&}
\]

\[
\text{numerical index concept (for example, } \text{"x-fold" the operation \& "x-times" the operation)} \subseteq \\
\{ x \mid x \in \mathcal{N}(0 \subset \text{FOL-PA}) \land \text{Alphabet (} \mathcal{N}(0 \subset \text{FOL-PA}) \cap \mathcal{N}(0 \subset \text{FOL-PA}) \}) \forall x (x \in \mathcal{N}(0 \subset \text{FOL-PA})) \\
\]


Please let us observe that: whenever \( \varphi \) is a well-formulated formula, the customary name of “the variables of the \( \varphi \) operation” referred into the induction schema axiom PA’s in first order language (FOL) are customary nicknamed as “occurrences” and/or “free variables”.

Así:

4. En el Estadística, usando el español del libro, son:

3. On the difficulty of counting (Estadística, using a Spanish desk), are symbolically represented. Wolfram Mathworld.

Además, por medio de las "World Herramientas operating (embed) in a Mac, the results (Estadística, using a Spanish desk), are symbolically represented by natural numbers (concepts) which are expressed by digits according to the locus-positional decimal numerical system (Cambia(Cuerpo: 10; normal)) are the following:

1. Sobre la dificultad de contar

(Páginas 1; Palabras 5; Caracteres(sin espacios) 25; Caracteres(con espacios) 30; Párrafos 1; Líneas 1)

2. About the difficulty of counting

(Páginas 1; Palabras 5; Caracteres(sin espacios) 28; Caracteres(con espacios) 32; Párrafos 1; Líneas 1)

3. On the difficulty of counting

(Páginas 1; Palabras 5; Caracteres(sin espacios) 25; Caracteres(con espacios) 29; Párrafos 1; Líneas 1)

4. "for the present complete (?) document", after (?) to insert these latter results, at this moment (!?), are: (Páginas 108; Palabras 37,464; Caracteres(sin espacios) 195,123; Caracteres(con espacios) 114,364; Párrafos 1,808; Líneas 4,512)

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This statement has to be considered a direct consequence of the uniarity of the formal representation above described. The customary process involved is call "counting", which can be understand as the customary meaning of "measuring the amount of any magnitude" (counting operation as before described) herein represented by symbol "#".

"Non-dense order property" of the natural number set holds for the concept that between two natural number concepts there is not another natural number concept. Please, let us point out that in NT-FS&L:

\[ \forall x (x \in \mathbb{N}(0 \subset PA)(S_{\ast}(0) = S_{\ast}(1)) \]


Moreover, the statement def.1 certifies the existence of countable, enumerable and complete ordered amount of “successors” of any element belonging to both \( \mathcal{M}(0\text{C}NT\text{-PA}) \) and \( \mathcal{M}(0\text{C}FOL\text{-PA}) \), which could be hereinafter re-presented by \( \mathcal{M}(0\text{FOL}\text{-PA}) \cap \mathcal{M}(0\text{C}FOL\text{-PA}) \).

Please, take in account that:

\[
\forall x \forall y \forall z \in \mathcal{N}(0\text{C}PA) \quad ((z = S_{x+y}(1) \land S_{x}(1)+1 \land S_{y}(1) + 1) \land (x = S_{0}(0)) \land (y = S_{0}(0)) \rightarrow
z = (x + 1) + y \land (y + 1) + x \land (x + y) + 1 \land (1 + (x + y))
\]

Please let us remember to the reader that natural number zero is not a member of \( \mathcal{N}(0\text{C}NT\text{-PA}) \).
Hence, it only has to be considered a symbol of its alphabet as well as the name of the set of all of their successors. Hence, both \( \text{Suc}_x(x) \) and \( \text{Suc}_x(0) \), \( \forall x (x \in \mathcal{N}(0\text{C}NT\text{-PA})) \) sets could be referred as two “infinite sets” in Cantor countability nomenclature (sense and intension), for which in NT-FS&L can be properly enumerated, numerated and can count every and each of their elements.

Please, let us remember that in NT-FS&L “one-membered set” is named “singleton”. Thus,

\[
\text{Suc}_x(0) := \{ 0 \in \mathcal{N}(0\text{C}NT\text{-PA}) \mid x = 0 \} \forall x (x \in \mathcal{N}(0\text{C}NT\text{-PA}))
\]

On the other hand, the next statements \( \forall x (x \in \mathcal{N}(0\text{C}NT\text{-PA})) \)

\[
\begin{align*}
\mathcal{N}(0\text{C}NT\text{-PA}) & \subseteq \mathcal{N}(0\text{C}NT\text{-PA}) \cup \text{Suc}_x(0) \\
& \quad \text{and} \\
\text{Suc}_x(0) & \subseteq \mathcal{N}(0\text{C}NT\text{-PA}) \cap \mathcal{N}(0\text{C}NT\text{-PA})
\end{align*}
\]

are both “false”. They are not w.f.s and w.f.s-r statements in any first order language, in which “equality” means “no-identity” and vice versa.

Please, take in account that:

\[
\forall x \forall y (x, y \in \mathcal{N}(0\text{C}NT\text{-PA})) \quad ((x = S_{0}(x) \land y = S_{0}(y) \land n = S_{0}(n) \rightarrow (x = n+y) \land (n \in \mathcal{N}(0\text{C}NT\text{-PA}))
\]

Please, let us considered and observe:

\[
((x = y+1 \rightarrow x = S_{y+1}(0)) \land ((x = y+1 \rightarrow x = P_{0}(x))
\]
, if and only if,
\[(x = S(y) + 1) \land (S_{y+1}(0) = P_y(0) + 1) \land (y = P_0(y) = P_0(x+1))\]


56 Please, see previous title Albert Einstein. Einstein summation convention. Tensorial Analysis, Differential Geometry and Calculus Concepts.

57 \(\text{Pre}_0(x), \text{Suc}_x(0)\) and \(\mathcal{N}(0 \subset \text{NT-PA})\) sets had to be grade as “enumerable, complete ordered, countable sets” (different sense and intension that for others authors as Cantor, Von Neumann, Gödel and Ramsey)

58 In NT-FS&L, there are two equivalent notations and/or formulations for “the accumulated union operation” of set (or subsets):

**Definition (Notation 1)**
\[
\bigcup \left[ \text{Pre}_0(x+1) \right]_{x \in \mathcal{N}(0 \subset \text{NT-PA})} := \{\text{Pre}_0(1) \cup \text{Pre}_0(2) \cup ... \cup \text{Pre}_0(x) \cup \text{Pre}_0(x+1)\}
\]

**Definition (Notation 2)**
\[
\bigcup \left[ \text{Pre}_1(x+1) \right]_{x \in \mathcal{N}(0 \subset \text{NT-PA})} := \{\text{Pre}_1(0) \cup \text{Pre}_1(1) \cup ... \cup \text{Pre}_1(x) \cup \text{Pre}_1(x+1)\}
\]

59 Please, let us indicate:

\[\forall x( x \in \mathcal{N}(0 \subset \text{NT-PA}) | \{P_0(x)\} \subseteq \{x\} \land \{x\} \nsubseteq \overrightarrow{\Theta} \land \{\overrightarrow{\Theta}\} \subseteq \Theta )\]

60 Please, let us remember to the reader the previous note title Albert Einstein. Einstein summation convention. Tensorial Analysis, Differential Geometry and Calculus Concepts.

61 Hereinafter, we will use the customary expression “presentation of” (for example, presentation of an abstract algebraic framework such as “Presentation of a Group” or “Presentation of a Monoid”) with the same sense and intension that it is used, for example, in the current “Representation Theory”.

62 Please, let us remember both, that every natural number has a unique direct successor identity, and a unique direct predecessor identity. The primordial of every natural number identity are 1-membered collectivities (subsets, also singletons) of the natural number set \(\mathcal{N}(0 \subset \text{NT-PA})\).

63 Please, let us remember that in NT-FS&L (NT-FOL-PA) are operating the \(\text{NT-FOL-de Morgan Laws}\) for the union and intersection of any collectivity of natural numbers \(x \in \mathcal{N}(0 \subset \text{NT-PA})\)

64 In NTFS&L, the word “lemma” is a synonymous of the word “theorem” referred to the meaning of “statement” that sustain entailment from the initial axiomatic set of wfs and wfs-r statements. They sustain “truth”.

65 Please let us consider that in customary notation the multiplication operation symbol \(\cdot\) is an element of the alphabet of NT-FOL first order language. Thus, \((x+x)\) can be “referred-represented” by the symbolic notations “2\(\cdot\)n” and/or “2n”; on the other hand, \((x+x+1)\) can be “referred-
represented” by symbolic notations by “$2\cdot n+1$” and/or “$2n+1$.”

Please, let us remember:

$$\forall x \forall y \left( x, y \in \text{Succ}^\alpha x(0) \land \text{Succ}^\beta y(0) \land \text{Pre}^\alpha_0(x) \land \text{Pre}^\beta_0(y) \right)$$

$$\left( x + y = (S_x(0) + S_y(0)) = P_0(x) + P_0(y) \right) \land \left( S_x(0) + P_0(y) = S_y(0) + P_0(y) \right) \rightarrow x = y; \quad \text{NT-th. 10}$$

Odd ($\text{Succ}_0(0)$) := \{0 | \Theta = \text{set } \{2\}\}

In NT-FS&L, “parity” and/or “imparity” induce a nicknamed “complete alternant-ordered two colors NT-partition” in $\mathcal{N}(0 \subset \text{NT-PA})$, which is in turn a non-dense and complete ordered set. Hence, please, let us remember the abstract algebraic structure of the group of the set-$\mathcal{N}(0, 1 \subset \text{NT-PA})$:=\{0, 1\}$\forall n \in \mathcal{N}(0 \subset \text{NT-PA})$ in NT-FS&L embed (sense and intension) both: first, a homomorph “complete alternant-ordered two colors partition”; second, a homomorph “complete ordered two colors-NT Logic and Algebra of Boole” extended field. (Please, take under your consideration the current customary “Minor field” concept)

Please, let us remember:

\[ i.\quad ((0, 1, 2, \ldots n \in \mathcal{N}(0 \subset \text{NT-PA})) \forall n \in \mathcal{N}(0 \subset \text{NT-PA}) \land (1, 2, \ldots n \in \mathcal{N}(0 \subset \text{NT-PA})) \cap \mathcal{N}(0 \subset \text{NT-PA})] \\
\[ ii.\quad \forall x \left( x \in \mathcal{N}(0 \subset \text{PA}) \right) \left( x \cdot 0 = 0 \right) \land \forall x \forall y \left( x \in \mathcal{N}(0 \subset \text{PA}) \right) \left( x \cdot S(y) = x \cdot y + x \right) \text{ (axioms 5 and 6 of NT-PA first order axiomatic).} \\
\[ iii.\quad \text{natural number 0 is the identity element for addition operation in } \mathcal{N}(0 \subset \text{NT-PA}) \text{ and, in turn, natural number 1 is the identity element for multiplication operation. Natural number 0 and every } S_n(0) \text{ as well that } P_0(x) \text{ are not elements of } \mathcal{N}(0 \subset \text{NT-PA}).} \]

Please, take in account that this fact involves an “infinite” collectivity of elements, which are all of the successors of a natural number but a “finitary collectivity of elements”, theirs corresponding predecessors. In NT-FS&L, the exchange operation successor of natural number identity for the predecessor of natural number (also, safeguards, confirms, certifies, warrants, make sure, make certain) both, the order (parity, logic and mathematical structure and the transformation of a “starting infinite no numerable collectivity (the successors of a natural number which are infinity) into a finitary numerable number (the predecessor of a natural number). Hence, in NT-FS&L, the processes of counting, enumeration, numeration and ordering are simultaneously all possible.

$\forall x \left( x \in \mathcal{N}(0 \subset \text{PA}) \right) \left( x \cdot 0 = 0 \right) \land \forall x \forall y \left( x \in \mathcal{N}(0 \subset \text{PA}) \right) \left( x \cdot S(y) = x \cdot y + x \right) \text{ (axioms 5 and 6 of NT-PA first order). Please, let us observe that the customary distributive property of the multiplication on addition, only is operating } \forall x \text{ (for each and every each natural number) if and only if } y = S_1(y) x = P_1(y)$

71 Please, take in account that:

\[
\begin{align*}
2 \cdot 1 + 2 + 3 + 4 + \ldots + (n) + (n+1) + (n+2) + (n+3) &= (n+3) - (n+4) = (n+3) - S_1(n+3) \quad \rightarrow \\
2 \cdot 1 + 2 + 3 + 4 + \ldots + (n) + (n+1) + (n+2) &= (n+2) - (n+3) = (n+2) - S_1(n+2) \quad \rightarrow \\
2 \cdot 1 + 2 + 3 + 4 + \ldots + (n) + (n+1) &= (n+1) - (n+2) = (n+1) - S_1(n+1) \quad \rightarrow \\
2 \cdot 1 + 2 + 3 + 4 + \ldots + (n) + (n+1) + (n+2) &= (n+2) - (n+3) = (n+2) - P_0(n+3) \quad \rightarrow \\
2 \cdot 1 + 2 + 3 + 4 + \ldots + (n) + (n+1) &= (n+1) - (n+2) = (n+1) - P_0(n+2) \quad \rightarrow
\end{align*}
\]
lo que Kirby y Paris demostraron de hecho, era que si P representa el procedimiento de inducción matemática (junto con las operaciones aritméticas y lógicas ordinarias), entonces podemos volver a expresar G(P) de la forma del teorema de Goodstein. Este nos dice que si creemos que el procedimiento de inducción matemática es digno de confianza (lo que difícilmente es una hipótesis dudosa), entonces debemos creer también la verdad del teorema de Goodstein, pese al hecho de que no es demostrable por inducción matemática!* ("Eso fue demostrado por: L.A.S. Kirby y J.B. Paris en: "Accesible independence results for Peano". Bulletin of the London Mathematical Society, 14, 1982, págs 285-293)"

73 *In "ad hoc" substitution notation:

$$\left(x\cdot S_1(x)\right) \subseteq \left(\text{Even-}\mathcal{P}\mathcal{N}_0(x\cdot x+x) \cap \text{Even-}\mathcal{P}\mathcal{N}_0(x\cdot (x+1))\right)_{x}$$


75 In NTFS&L; the following statement

$$\left(\mathcal{N}(\text{FOL-SP}(0\subset PA)) \subseteq \mathcal{N}(\text{SPA}(0\subset PA)) \rightarrow \mathcal{N}(0\subset NT-PA) \subseteq \mathcal{N}(\text{FOL-SP}(0\subset PA))\right)$$

, which is a member (hence, also could be considered its "singleton" identity) of NT-Pr-th. 1 Set, should be read (send and intension):

"If the axiomatic $$\mathcal{N}(\text{FOL-SP}(0\subset PA))$$ set has been intended de-embed logically from $$\mathcal{N}(\text{SPA}(0\subset PA))$$, then $$\mathcal{N}(0\subset NT-PA)$$ has been intended de-embed logically from $$\mathcal{N}(\text{FOL-}$$"
SP(0⊂PA) and, in turn, from $\mathcal{N}(\mathcal{SPA}(0\subset PA))$.

Additionally, let us observe that parity referred as the addition $(x+x)$ operation in NT-PA’s, is able to reinforce (and/or to “regenerate order” and/or to reestablish) complete order of every disordered subset of natural numbers by means of an inductive-type process into the non-dense but complete ordered set of the natural numbers ($\mathcal{N}(0\subset \text{NT-PA})$). This is achieved (also evinced) operationally by means of the following inductive process; renaming successor of zero as zero and successor of even $(0+0)$ by even $(1+1)$ and in general the natural number $x$ by the even (addition of itself) of its odd-direct successor.

Hence, operating with the elements of $\mathcal{N}(0\subset \text{NT-PA})$:

If:

\[
\begin{align*}
((0) = (0+0)) & \rightarrow ((0) = (0+0) +0) \quad \land \quad ((1) = (0+0)) \rightarrow ((2) = (1+1) +1)) \quad \land \\
((0) = (0+1)) & \rightarrow ((0) = (1+1) +0)) \quad \land \quad ((1) = (1+1)) \rightarrow ((3) = (1+1) +1)) \quad \land \\
((0) = (0+2)) & \rightarrow ((0) = (2+2) +0)) \quad \land \quad ((1) = (2+2)) \rightarrow ((4) = (2+2) +1)) \quad \land \\
((0) = (0+3)) & \rightarrow ((0) = (3+3) +0)) \quad \land \quad ((1) = (3+3)) \rightarrow ((7) = (3+3) +1)) \quad \land
\end{align*}
\]

\[
\begin{align*}
\ldots & \quad \quad \ldots \quad \quad \ldots \quad \quad \ldots
\end{align*}
\]

\[
\text{then:}
\]

\[
\begin{align*}
((0) = (S_0(0+x))) & \rightarrow ((0) = (S_0(x+x) +0)) \quad \land \quad ((1) = (S_0(x+x) \rightarrow (2\cdot x) = S_0(2\cdot x+1)))
\end{align*}
\]

In NT-FS&L it is enough to fix (also index) “cero as reference”, which in turn is the unique natural number, to enable us to index both;

i. a complete order and distribution of the complete $\mathcal{N}(0\subset \text{PA})$; and

ii. the identity of every and any natural number. Hence, both of their identities, “elements” and “part and/or parts of a collectivity”, of every “successor” and/or “predecessor” are transformed in “invariants elements” in a Projective Space, $\mathcal{N}(0\subset \text{NT-PA})$ generated in the NT-FS&L first order logic language which is free from any topological and geometrical (mathematical) context.

Please, let us note that we use $\mathcal{N}(0\subset \text{PA})$ in state of $\mathcal{N}(0\subset \text{PA})$ because we are indicated that “natural number zero concept” and “the statements about this concept” is a element according to PA’s axiomatic (In NT-FS&L, holds the identity of a set of statements, which in turns is a type of collectivity)

The former axiom 6 (PA’s) on multiplication operation, which is referred to the multiplication operation:

\[
\forall x \forall y \ (x, y \in \mathcal{N}(0\subset \text{PA})) \ (x\cdot S(y) = x\cdot y + x)
\]

(axiome 6; PA’s expressed in current first order logic)

\[
\forall x \forall y \ (x, y \in \mathcal{N}(0\subset \text{PA})) \ (x\cdot S(y) = x\cdot (y + 1)) \ \forall x \forall y \ (x, y \in \mathcal{N}(0\subset \text{PA})) \ (y\cdot S(x) = y\cdot (x + 1))
\]

\[
\rightarrow
\]

\[
(y\cdot (x + 1)) = (x\cdot (y + 1)) \ \Rightarrow \ (y + 1 = x+1) \ \rightarrow \ (x) = (y)
\]

(axiome 6; PA’s expressed by first order logic “ad hoc” substitution operation)

Please, let us observe and remember that “precessor operation” have to be considered in NT-FS&L as a “unintended embedded” (sense and intention) identity in the original “PA’s axiomatic” of G. Peano.

Covariance and contravariance concepts have to be considered as above described (sense and intension, succession and precession); thus we should introduce: and, or, and and/or logic operations from notation. The induction schema is expressed by “replacement notation”.

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Please, let us remember that:

\[ \forall x (x \in \mathbb{N} (0 \in \text{NT-PA})) ((\text{Suc}^\prime(x) \subseteq \text{Suc}_x(0) \cap \text{Suc}_{x+1}(0)) \rightarrow ((\text{Suc}_{x+1}(0) \land \text{Suc}_1(0) = 1) \land (\text{Suc}_x(0) \land \text{Pre}_1(1) \land \text{Pre}_x(x) = 0) \rightarrow (x = 0)) \]

Please, take in account that NT-FS&L F.Th 1-2 and NT-FS&L F.Th 3 are embed in the previous item natural number (NT-PA)- Pythagorean Successors & Predecessors identities.

**NT-FS&L PRESENTATION OF SET OF THE PRIMES NATURAL NUMBER SET:**

\[ \text{Primes}(p-P^0_0(x)) := \{1, 2, 3, 5, 7, 11, 13, 17, \ldots \} \]

Please, let us highlight that "Christian Goldbach listed natural number 1 as the first prime in his previously referred correspondence with Leonhard Euler; however, Euler himself did not consider 1 to be a prime number. Even more, by the early 20th century, mathematicians began to arrive at the consensus that positive integer number 1 is not a prime number". Nowadays, it is globally accepted that number 1 is not a prime number rooted in the argument from authority (Latin: argumentum ad verecundiam), which has been rejected for example by Henri León Legendre in the development during the last century of his innovative, efficient and effective Lebesgue Integral Theory). Hence, for a very long period of time a plethora of scholar and academic and professional books (and e-books), computer programs, applications and Computational Knowledge Engines globally reinforce low reasoning procedures rooted on conventions instead of arguments (Sagan, Carl. The Demon-Haunted World: Science as a Candle in the Dark. Ballantine Books. (2011) ISBN 9780307801043). For example:

1. The On-Line Encyclopedia of Integers Sequences. Founded in 1964 by N.J.A. Sloane https://oeis.org/A000040. Additionally, please see the sequences related with the named Mersenne primes (sequence A000043 (p) and A000668 (M_p) in OEIS).

2.2. Wolfram Mathematica 10. (10.4.0.0): 2.2.1. Input: PrimeQ[1] \( \text{¿ primo? Result: False; 2.2.2. Input: Select [table [K, [K, 10]], PrimeQ] Result: \{2,3,5,7\}; 2.2.3. Input: \text{a} = \{1,2,3,4,5,6,7\}; \text{Select[a, PrimeQ]} Result: \{2,3,5,7\}.


(https://www.wolframalpha.com/input/?i=ls+1+prime%3F).


\[82 \text{ Please, note that for more than twenty-five centuries this is the first time that in a first order system and language, the NT-FS&L, which has been built by means of (sense and intension) a primordial intended inductive FOL de-embedding methodology and from a topological-, algebraic- and geometrical-free context referred only to "the natural number concept" (linguistic, logic and mathematical) has been proven, the next set of NT-FS&L F.Th 1-3 statement:} \]
If and only if, 0 belongs to $\mathcal{N}(0 \subset \text{NT-PA})$ is the even “conserving” non-covariant/non-precessor parity/umparity neutral addition element referred to a natural number non-prime concept, then: 1 as element belonging to $\mathcal{N}(0 \subset \text{NT-PA})$ is both: first, the odd “breaking” covariant universal addition element representative of a natural number concept; second, the odd “conserving” non-covariant/(successor of natural number 0) /contravariant predecessor of natural number 2) parity/umparity neutral multiplicative element referred to natural number prime concept.

Hence, in NT-FS&L, if 0 is a natural number, then 0 is even then 1 is odd and then 1 has to be prime, which can be represented by the following theorems: (Einstein-like tensorial addition convention)

$$0 = 0 + 0; 0 + 1 = 1; 0 + 1 = 1 + 0; (0+1) + (1+0) = 1 + 1; 1 + 1 = 2; (0+1) + (1+0) = (0+0) + (1+1); 1+1+1= 3; (1+1)+1=3; 1+(1+1) = (1+1+1)$$

\[86\] Please, let us remember the following NT-“entailment-schema”:

[0-Entailment:]

$\mathcal{N}(0 \subset \text{NT-PA}) \subseteq \mathcal{N}(0 \subset \text{NT-PA}) \cup \mathcal{P} \mathcal{N}_0(0)$

\[(\text{NT-th. 6})\]

1-Entailment:

$\{ e \cdot \mathcal{P} \mathcal{N}_0(x) \} \cup \{ \sigma \cdot \mathcal{P} \mathcal{N}_0(x) \} \subseteq \mathcal{N}(0 \subset \text{NT-PA})$

$\mathcal{P} \mathcal{N}_0(x) = \mathcal{P} \mathcal{N}_{x-1}(0)$ has parity of $x$ ($x \in \mathcal{N}(0 \subset \text{CPA})$)

$\mathcal{P} \mathcal{N}_0(x+1) = \mathcal{P} \mathcal{N}_{x+1}(0)$ has parity of $x+1$,

\text{which is the 1-successor of } x \text{ (its direct successor) } (\forall x \in \mathcal{N}(0 \subset \text{NT-PA}))$

$\# \cdot \mathcal{P} \mathcal{R}_0(x) := x+1 \in \mathcal{N}(0 \subset \text{NT-PA}) \rightarrow x \in \mathcal{P} \mathcal{N}_0(x), (\forall x \in \mathcal{N}(0 \subset \text{NT-PA}))$

\[(\text{NT-th. 7})\]

$\{ \# \cdot \mathcal{P} \mathcal{R}_0(x) \} = \{ x+1 \in \mathcal{N}(0 \subset \text{NT-PA}) \cap \mathcal{P} \mathcal{N}_0(0) \} \cup \mathcal{P} \mathcal{N}_0(x)$

\%(\forall x \in \mathcal{N}(0 \subset \text{NT-PA}))$

\[(\text{NT-th. 7 bis})\]

2-Entailment:

$\mathcal{N}(0 \subset \text{PA}) = \mathcal{P} \mathcal{R}_0(x) \cup \text{Suc}_{x+1}(0)$

\[(\text{NT-th. 3})\]

$\mathcal{N}(0 \subset \text{PA}) = \mathcal{N}(0 \subset \text{NT-PA}) \cup \mathcal{P} \mathcal{N}_0(0)$

\%(\forall x \in \mathcal{N}(0 \subset \text{NT-PA}))$

\%(\text{NT-th. 6 & NT-th. 7})$

3-Entailment:

\text{HLParity-1-NT Theorem:} (\forall x \in \mathcal{N}(0 \subset \text{NT-PA}))

\[
\mathcal{P} \mathcal{N}_0(x) \cap \mathcal{P} \mathcal{N}_{x+1}(0) \subseteq \sigma \cdot \mathcal{P} \mathcal{N}_0(x+x) \quad \text{(always is an even primordial)}
\]

\[
\mathcal{P} \mathcal{N}_0(x) \cap \mathcal{P} \mathcal{N}_{x+2}(0) \subseteq \sigma \cdot \mathcal{P} \mathcal{N}_0((x+x)+1) \quad \text{(always is an odd primordial)}
\]

\[87\] \text{Please, it is very important indicate that natural number 2 is the unique natural number that :}

1. $2 = \mathcal{P}_0(2) \wedge \mathcal{P}_1(1+1) \wedge \mathcal{P}_2(2) \wedge \mathcal{P}_1(2+1+1) = \mathcal{P}_1(2+1+1)$

2. $0 = \mathcal{P}_0(2) \wedge \mathcal{P}_1(1+1) \wedge \mathcal{P}_0(0) \wedge \mathcal{P}_1(1+1+1)$

3. $\mathcal{P}_x(x) + 2 = (0+0) + (1+1) = ((0+1) + (0+1)) \wedge ((1+0) + (1+0)) \wedge ((0+0) + (1+1))$

4. $2(0+0)+(1+1) = 2(0+1) + (0+1) \wedge 2((1+0) + (1+0)) \rightarrow 2(0+0) \cdot 2(1+1) \wedge 2(1+1) \cdot 2(0+0)$

Entailment from (NT-th. 3) \( \forall x (x \in \mathcal{N}(0 \in \text{NT-PA})) \) \((\mathcal{P}r_0(x) \cup \text{Suc}_x(0) \subseteq \mathcal{N}(0 \in \text{NT-PA}))\)

Further entailment is ensured by taking in account the last result and the following two definitions:

\[
\text{Even}(\text{Suc}_0(x)) := \{ x+x \mid x+x \in \text{Suc}_0(x) \}, \forall x (x \in \mathcal{N}(0 \in \text{NT-PA})) \quad (\text{def}-\text{P. 1})
\]

\[
\text{Odd}(\text{Suc}_0(x)) := \{ (x+x)+1 \mid (x+x)+1 \in \text{Suc}_0(x) \} \mid x \in \mathcal{N}(0 \in \text{NT-PA}) \}, \forall x \quad (\text{def}-\text{P. 2})
\]

Additionally, please let us remember: \( \forall x (x \in \mathcal{N}(0 \in \text{NT-PA})) \)

\[
P^\alpha_0(0) = S^\alpha_0(0) = 0 \quad \&
0 + S^\alpha_0(0) = S^\alpha_0(0) + 0 \quad \&
0 + P^\alpha_0(0) = P^\alpha_0(0) + 0 \quad \&
0 + S^\alpha_0(2) = S^\alpha_2(0) + 0 \quad \&
0 + P^\alpha_2(0) = P^\alpha_0(1) + P^\alpha_0(0) \&
\]

\[
S^\alpha_1(x) + S^\alpha_1(x) = P^\alpha_{(1+1)+(1+1)}(x+x) \&
S^\alpha_1(x) + S^\alpha_1(x) = P^\alpha_{(2+2)}(x+x) \&
S^\alpha_1(x) + S^\alpha_1(x) = P^\alpha_{(2+2)}(x+x) \&
S^\alpha_1(x) + S^\alpha_1(x) = P^\alpha_{(4)}(x+x) \rightarrow
\]

\[
S^\alpha_{x+1}(0) + S^\alpha_0(x+1) = P^\alpha_0(x+1) + P^\alpha_0(x+1), \forall x (x \in \mathcal{N}(0 \in \text{NT-PA}))
\]

In NT-FS&L, tensorial nomenclature, formulation and notation holds for a topological analytic, algebraic and geometrical and classic-numerical free-context.

In NT-FS&L, these names of the natural numbers \( x, y, z, t \) (\( \forall x \forall y \forall z \forall t \) \( x, y, z, t \in \mathcal{N}^\alpha_0(1 \in \Theta, 0 \in \Theta, 0 \in \text{NT-PA}) \)) are intentionally (sence and intension) the nicknames referred-identities to the globally accepted and called as "dimensions" in the customary "tetra-dimensional space-time" concept. Please, take in account that NT-FS&L could be presented as a topological-, algebraic-, analytic -and geometrical free-context FOL language. However, take into account that, also according the authors, NT-FS&L never could be considered as a logical-mathematical-computational-cognitive free-context language.

Please let us observe the following two NT-theorems:

1. \( \mathcal{P}(0 \in \text{NT-PA}) \cup \mathcal{N}\mathcal{P}(0 \in \text{NT-PA}) \subseteq \mathcal{N}(0 \in \text{NT-PA}) \)

2. \( (\mathcal{Primes}(x)) \cup \mathcal{Primes}(p-P_0(x)) \subseteq \mathcal{P}(0 \in \text{NT-PA}) \rightarrow \)

\[
p \in \mathcal{Primes}(p-P_0(x)) \rightarrow p = x \) \( \forall x \forall p \) (\( x, p \in \mathcal{N}(0 \in \text{NT-PA}) \) )

NT-FS&L nomenclature, formulation and notation of NT-PA primordial sets (singletons):

\[
\mathcal{P}\mathcal{N}(1, p) := \mathcal{P}\mathcal{N}\left(\left(\{1\} \cup \{p\}\right)\right) \land \mathcal{P}\mathcal{N}(1, 2) := \mathcal{P}\mathcal{N}\left(\left(\{1\} \cup \{2\}\right)\right) \rightarrow \{0, 2\} := \mathcal{P}\mathcal{N}(0 \cup \{(1 \in 1)\})
\]

Translation from NT-FS&L into:

1.- The decimal numerical positional-based system language:
\(N^\alpha(0 \cap NT-PA)) = \{ e \cdot 0, o \cdot 1, e \cdot 2, o \cdot 3, \ldots, x, \ldots \} \ & \ N^\beta(0 \cap NT-PA)) = \{ \ldots, x, \ldots, o \cdot 3, e \cdot 2, o \cdot 1, e \cdot 0 \}; \)
\(Suc^\alpha(0) = \{ e \cdot 0, 1, 2, 3, \ldots, x, \ldots \} \ & \ Suc^\beta(0) = \{ \ldots, x, \ldots, 3, 2, 1, 0 \}; \)
\(Pre^\alpha_0(x) = \{ 0, 1, 2, 3, \ldots, x \} \ & \ Pre^\beta_0(x) = \{ \ldots, x, \ldots, 3, 2, 1, 0 \}; \)

2. The decimal numerical positional-based system language

\(\text{Natural Numbers Set} = \{ 0, 1, 2, 3, 4, 5, \ldots \}; \ \text{Prime Numbers Set} = \{ 1, 2, 3, 4, 5, \ldots \} \)

96 Briefly symbolized by next formula:
\[\{N^\alpha(0 \cap PA) \subseteq \{NT-\text{FS&L}\}\}\]

97 Expanded formal NT-\text{FS&L} formula:
\[
\{D^\alpha ( p \cdot p_n, z[x+t], z[x+x+t], z[x\cdot t], z[x\oplus t], z[x\otimes t], (0 \cap NT-PA)) \}
\]
\[p \cdot p_n, y, z, t, n = (S_0(x+1) \land P_0(x+1)) \forall n, n \in N^\alpha(1\otimes, 0\oplus, 0\cap PA) \]

98 Please, let us remember that “Quod erat demonstrandum” (Translating from the Latin into contemporary English yields "what was to be demonstrated") for centuries has been abbreviated by acronyms “Q.e.d” and Q.E.D; nowadays it could be misunderstood as the abbreviation of "Quantum ElectroDynamics" which has been proposed and globally “popularized” by the theoretical physicist Richard Feynman. Feynman, Richard 1. QED: The Strange Theory of Light and Matter (1985), Princeton University Press. ISBN 978-0-691-08388-9. 2. (1985) W. W. Norton & Company. ISBN 978-0-393-31604-9. Surely you’re joking, Mr. Feynman!