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«Łukasiewicz logics and transitive logic»

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ŁUKASIEWICZ LOGICS AND TRANSITIVE LOGIC

by Lorenzo Peña & Txetxu Ausín

Even though Łukasiewicz never developed a treatment of graduality and in fact was prompted to implement his many-valued logics by considerations quite different from those which underlye theories of degrees of truth, his logical calculi have nevertheless been taken to constitute the canonical framework for approaches to fuzzy set theory, ever since Lofti Zadeh inaugurated the fuzzy line of research in the middle 60's.

There are several reasons for that. Łukasiewicz logics are characterized by the following properties:

(1) Biimplicative formulae of the form $\ulcorner p\leftrightarrow q\urcorner$

(1.a) take a value 1 iff the value of $\ulcorner p\urcorner$ equals that of $\ulcorner q\urcorner$;

(1.b) take a value of 0 iff one of those values is 1 and the other 0;

(1.c) take an intermediate value otherwise, which is the higher the smaller the distance in truth-value between $\ulcorner p\urcorner$ and $\ulcorner q\urcorner$; and

(1.d) there is no jump or discontinuity as regards the possible values of biimplicative formulae (i.e. it is not the case that there are three values $x \geq y \geq z$, such that $\ulcorner p\rightarrow q\urcorner$ can take value x, can take value z but never takes value y.

(2) Only value 1 is designated (i.e. true). Nothing can be asserted unless if it has as its truth value the value 1, i.e.. TRUTH.

(3) The principles of noncontradiction and excluded middle are abandoned, but that of self-implication is retained.

(4) The value of a disjunction is the higher (or truer) of the values of the two disjuncts.

(5) The principle of Factor obtains, namely $\ulcorner p\rightarrow q\rightarrow .p\rightarrow .p\wedge q\urcorner$, or — what within usual frameworks amounst to the same — $\ulcorner p\rightarrow q\leftrightarrow .p\leftrightarrow .p\wedge q\urcorner$, i.e. the principle of equivalence, namely that for a proposition to imply another is for the former to be equivalent to the conjunction of both.

If there are situations, *p*, which are in between complete truth and utter falseness, and excluded middle holds, $\ulcorner p\vee Np\urcorner$ ('N' being negation) will be true. But if being true entails being entirely true, i.e. having value 1, then $\ulcorner p\vee Np\urcorner$ will have value 1 while neither $\ulcorner p\urcorner$ nor $\ulcorner Np\urcorner$ has value 1, contrary to (4). Thus, (3) is — when (2) and (4) hold — a necessary condition for a logical treatment of intermediary situations.

The little-by-little degradation of implicative formulae, their sensitiveness to the degree of discrepancy between the protasis and the apodosis, seems to be in agreement with a principle of degrees: if $\ulcorner r\urcorner$ is more false than $\ulcorner q\urcorner$, then saying that **r if p** seems to be bound to be more false than saying **q if p** — at least for a suitable 'p' (e.g. when 'p' is completely true).

All those features seem to be very nicely adjusted to the informal consideration of degrees of something or other. The trouble is that there is a huge price to pay for those would-be advantages, and one we think mustn't be paid. Accordingly we argue that Łukasiewicz logics are better equipped for Łukasiewicz's own original purposes (such as implementing a logical account of future contingents) than for a satisfactory approach to graduality.

We feel deeply distressed by the absence of all the following principles in Łukasiewicz logics (our use of dots is à la Church): Clavius: p→Np→Np; Importation: p→(q→r)→.p∧q→r; Contraction: p→(p→q)→.p→q; Conjunctive Assertion: p→q∧p→q; **IF** (Implicational Funnel): p→q∨.p→q→r.

No less worried are we by the presence of three obnoxious principles (obnoxious for a gradualistic interpretation):

VEQ (*Verum e quolibet*): p→.q→p

Exported Cornubia: p→.Np→q;

Permutation: p→(q→r)→.q→.p→r.

Suppose ⌜p→q⌝ is read 'To the extent that p [at least], q'. Then VEQ means that, to the extent that p, then, to the extent that q, p; i.e., to the extent that p, p is at least as true as q is: to the extent that Bryan is clever, he is as least as clever as Maria is beautiful. That seems to thwart any useful treatment of comparatives, if 'x is more so-and-so than z is this-or-that' is to be some sort of negation of 'To the extent [at least] that x is so-or-so, z is this-or-that', the latter being equivalent to 'z is this-or-that at least to the extent that x is so-or-so'.

But VEQ cannot be given up within Łukasiewicz's framework. The very same apparently winsome feature (1) entails VEQ. (The proof can be made formally.)

Permutation also seems to us extreemely irksome if we are going to view Łukasiewicz logic as a logic of graduality (as Łukasiewicz himself didn't), since it entails what follows: that to the extent that, in so much as Mary is kind, Peter is at least as clever as Cecilia, to that extent at least it is the case that, in so much as Cecilia is clever, Peter is at at least as clever as Mary is kind.

While Clavius, Importation, Contraction, Conjunctive Assertion and **IF** must hold within a satisfactory approach to graduality, VEQ, Exported Cornubia and Permutation are undesirable.

Let us try to display some of the evidence supporting our claim. First, Clavius. As is well known, Clavius holds in mainstream systems of entailment and relevance logic — such as Anderson & Belnap's systems **E** and **R**, respectively. Although Clavius has been rejected by deep relevant logicians (such as the late Richard Sylvan), its desirability on gradualistic grounds seems to us very clear. What in effect Clavius boils down to is that, to the extent that a statement implies its own negation, it is false; or, what — in virtue of involutivity and contraposition — amounts to the same, that, to the extent that a proposition is at least as true as false, it is true. Conversely, to the extent that a proposition is true, it does not imply its own negation. Thus, to the extent that a man is either as tall as he is short or more tall than short, he is tall.

What could stand in the way of Clavius being true? From a gradualistic viewpoint the only reason could be for a certain sentence 'p' to be less true than the implication of 'p' by 'not-p'. In such a case, the truth-value of 'p', |p|, would be, let us say, $u < 1$, while |Np→p| = $v > u$ (|Np| would be at most as high as |p|). But in such a case $|p| \neq 1$ and hence $|Np| \neq 0$. For, when $|p|=1$, obviously $|q\to p|=1$ for any ‘q’ (within Łukasiewicz’s approach). Yet, surely for values of ‘p’ sufficiently close to 1, with $|Np| < |p|$, ‘p’ seems to be true enough for $\ulcorner Np\to p\to p\urcorner$ to hold. For, if an entirely true ‘p’ is implied by its being implied by its own negation, what could prevent the same to apply to a ‘p’ with a degree of truth of 0.999? But then no cutting-off must prevent the same to apply to all statements ‘p’ such that $|Np| \leq |p|$. Thus Clavius is vindicated: $\ulcorner Np\to p\to p\urcorner$ is bound to hold.

While in Łukasiewicz’s many-valued logics all theorems are implied by their own negations, in Anderson & Belnap’s entailment system **E** there is a formula schema which is implied by its own negation, namely $\ulcorner N(N(p\to p)\to(p\to p)\to(p\to p)\to N(p\to p))\urcorner$. Such a schema turns out to be valid in system **E**. The proof yields as a result the formula’s theoremhood. No such proof would be available should we be forced to give up Clavius. (Łukasiewicz secures the same result in the case of that particular formula thanks to VEQ, i.e. $\ulcorner p\to.q\to p\urcorner$, which is surely unsuitable from a gradualistic viewpoint.)

If Clavius thus emerges as a clearly correct principle, importation also seems entirely in order: $\ulcorner p\to(q\to r)\to.p\wedge q\to r\urcorner$: suppose that, to the extent that Peter is clever, Myriam is at least as clever as John; then, Myriam is at least as clever as both Peter and John are; should she be less clever than both of them, surely her being at least as clever as John would be utterly false; such a complete falsity could be implied by Peter’s cleverness only in case Peter was not clever at all — which runs against the very hypothesis of Myriam being less clever than Peter.

Of course a Łukasiewiczian would not accept that, whenever $\ulcorner p\urcorner$ is less true than $\ulcorner q\urcorner$, $\ulcorner q\to p\urcorner$ should necessarily be wholly false. Yet, without such a constraint — let us call it ‘the watershed principle’ — it seems hard to make good sense of comparative sentences or to see how issues of more and less could be settled at all; without the watershed principle, ‘Tegucigalpa is at least as populous as New York’ would remain partly true!

As for **IF**, $\ulcorner p\to q\to r\vee.p\to q\urcorner$, the watershed principle is again what is at issue. If $|q| < |p|$, $\ulcorner p\to q\urcorner$ is bound to be wholly false and so to imply everything and anything; else $\ulcorner p\to q\urcorner$ is bound to be true.

We thus propose an alternative, namely a chain of strengthenings of Andernon & Belnap’s system **E** of entailment. Our proposal we call ‘**Transitive logic[s]**’ since they are logics of transition — which aim at capturing the existence of transition-stages inbetween complete truth and complete falseness.

The resulting systems of transitive logic are intermediary between classical and entailmental logic. Some of them contain all of classical logic, and are indeed conservative extensions of classical logic. Only, the classical logic negation ‘$\neg$’ is given a different reading from what is customary. It is now read ‘not…at all’. Classical negation is complete negation — the functor which maps anything true, to whatever extent, into complete falsehood, and the utterly false into complete truth. All those systems have the laws of non-contradiction and excluded middle. They are paraconsistent copulative systems.

SETTING UP THE CHAIN:

System P0 = **E**

System P1 = P0 + factor

System P2 = P1 + Linearization ($\ulcorner p\to q\vee .q\to p\urcorner$)

System P3 = P2 + the Self-Self Principle ($\ulcorner N(p\to p\to N(p\to p))\to .p\to p\urcorner$)

System P4 = P3 + **IF** ($\ulcorner p\to q\vee .p\to q\to r\urcorner$)

System P4.5 = P4 + Mingle = Classical Logic (CL). Mingle is $\ulcorner p\to .p\to p\urcorner$

System P5 = P4 + one of these: Aristotle ($\ulcorner N(p\to Np)\urcorner$), Boethius ($\ulcorner p\to q\to N(p\to Nq)\urcorner$) or contradiction (namely, for some particular sentential constant j: $\ulcorner j\wedge Nj\urcorner$)

System P6 = P4 + both these principles: $\ulcorner p\to q\to .NHNHp\to Hq\urcorner$ and $\ulcorner Hp\to q\vee .Np\to r\urcorner$ (‘H’ is read: ‘It is altogether true that’)

System P3.5 = P3 + those two principles involving ‘H’ = P6 minus **IF**

System P8 = P3.5 + these two: $\ulcorner \alpha\urcorner$ and $\ulcorner \alpha\to p\vee .p\to q\urcorner$ (‘α’ is a sentential constant meaning the conjunction of all truths)

System P8.5 = P8 + $\ulcorner H\alpha\urcorner$ = CL

System P9 = P8 + $\ulcorner NH(\alpha\to\alpha)\urcorner$

System P10 = P8 + $\ulcorner HN(N\alpha\to\alpha)\urcorner$

System P7 = P6 + $\ulcorner Hp\vee Np\urcorner$

P8 contains P7; P10 contains P9; P9 contains P5; P8 is a conservative extension of classical logic (if classical negation, ‘$\neg$’, is defined as ‘HN’); P5 and such systems as contain P5 are not just paraconsistent but contradictorial (they have the theorem of Heracleitus: $\ulcorner N(p\to p)\urcorner$).

Among all those systems, P5 and P10 are by far the most important ones. They are the most stable. The main motivation in going from P4 to P6 and up is to encompass **CL** within fuzzy or gradualistic logic, as the extreme case wherein it is a question of either complete falseness or not (alternatively of either complete truth or not). System P8 and those above it allow to implement classical conditional (which can be defined in the usual way with strong negation, ‘$\neg$’, and disjunction, ‘$\vee$’) in the relevantly recommended way: $p\supset q$ iff there is some truth, namely α, such that $p\wedge\alpha\to q$.

A Hilbert style axiomatization for P5 is provided below, which is more elegant than the genetic presentation just described. This axiomatization contains ten axioms and one primitive inference rule:

Primitive symbols: ‘$\wedge$’, ‘N’, ‘$\to$’. ‘p’, ‘q’ etc are used as schematic letters.

Definitions:

$\ulcorner p\vee q\urcorner$ abbr $\ulcorner N(Np\wedge Nq)\urcorner$

$\ulcorner \blacktriangledown p\urcorner$ abbr $\ulcorner N(p\to Np)\urcorner$

Axioms:

P5a01 $p \to q \to r \land (q \to p \to r) \to r$

P5a02 $p \to q \land (q \to r) \to .p \to r$

P5a03 $p \land q \land r \to .r \land p \land q$

P5a04 $p \land q \to p$

P5a05 $p \to q \to r \to s \land (\blacktriangledown(p \to p) \to (p \to q) \to s) \to s$

P5a06 $p \to q \to .r \to s \to .p \to q \land .r \to s$

P5a07 $p \to q \to .p \to .p \land q$

P5a08 $Np \to q \to N(p \to q)$

P5a09 $p \to Nq \to .q \to Np$

P5a10 $NNp \to p$

Inference rule: **DMP** (i.e. disjunctive *modus ponens*): for $n \geq 1$:

$p^1 \to q \lor (p^2 \to q) \lor \ldots \lor .p^n \to q,\ p^1,\ \ldots,\ p^n \vdash q$

MP [*Modus Ponens*] is a particular case of the rule — the one wherein n=1. *Adj* is a derived inference-rule.

The significance of **DMP** is that to deduce $\ulcorner q \urcorner$ from a number of premises is to show that $\ulcorner q \urcorner$ is deduced from at least one of them. Which does not mean that necessarily there is a proof from one of the premises alone, since the whole proof of the conclusion from the premises consists in showing that it either follows from the first premise, or from the second premise, and so on. Those deductions are not complete proofs, but proof-branches or **alternative** sub-proofs.

Upon the basis of P4 all the follwing are equivalent (and all of them bring about a collapse into **CL**):

$p \leftrightarrow q \lor .q \leftrightarrow r \lor .p \leftrightarrow r$ (i.e. the Dugundji formula in three schematic letters)

$p \to .p \to q \to q$ (Exported Assertion)

$p \to (q \to r) \to .q \to .p \to r$ (Permutation)

$p \land q \to r \to .p \to .q \to r$ (Exportation)

$p \land q \to r \to .p \land Nr \to Nq$ (Antilogism)

$p \to .q \to .p \land q$ (the Adjunction Principle)

$p \to .p \to p$ (Mingle)

$p \land Np \to q$ (Cornubia)

$p \to .q \to p$ (VEQ)

$p \to q \lor .q \to r$

$N(p \to p) \to q$

$N(p \to p) \to .q \to .p \to p$

$N(p\rightarrow q)\rightarrow.p\rightarrow p$

$p\rightarrow.q\rightarrow q$

Now we present an axiomatisation for P10 by adding to P5 the following axiom schemata (‘H’ is a one-place primitive functor, read as ‘entirely the case’, while $\ulcorner\neg p\urcorner$ abbr $\ulcorner HNp\urcorner$ and $\ulcorner p\supset q\urcorner$ abbr $\ulcorner\neg p\lor q\urcorner$):

P10a1 $\neg p\land p\rightarrow q$

P10a2 $p\rightarrow q\rightarrow.Hp\rightarrow Hq$

P10a3 $Hp\rightarrow p$

P10a4 $NHp\rightarrow\neg Hp$

P10a5 α

P10a6 $p\supset.\alpha\rightarrow p$

P10a7 $H\blacktriangledown N\alpha$

Since in P10 we can prove Funnel (namely $\ulcorner p\rightarrow q\lor p\urcorner$), **IF** (as it stands in all the foregoing formulations of P5 — whether as such or as implicational Peirce, i.e. $\ulcorner p\rightarrow q\rightarrow p\rightarrow p\urcorner$ provided $\ulcorner p\urcorner$ is implicational) becomes redundant. Thus a more elegant axiomatisation of P10 has to be found.

SOME OUTSTANDING RESULTS

Here is a short list of theorem-schemata and rules which can be proved and derived, respectively, within P10:

$p\supset q\ ,\ p\ \vdash q$

$p\rightarrow q\supset.p\supset q$

$p\supset q,\ \neg q\ \vdash\neg p$

$\neg p\land\neg q\leftrightarrow\neg(p\lor q)$

$\neg p\lor\neg q\leftrightarrow\neg(p\land q)$

$p\supset q\ \vdash p\lor r\supset.q\lor r$

$p\supset.q\supset.p\land q$

$p\rightarrow.q\supset p$

$p\rightarrow.\neg p\supset q$

It can be easily proved that the fragment of P10 in $<\land,\lor,\supset,\neg>$ is exactly **CL** — of which, therefore, P10 is a **conservative** extension. We had rather look upon P10 as the result of enlarging **CL** with new functors, a nonstrong negation, ‘N’, and an implication, ‘$\rightarrow$’. The intended sense is that $\ulcorner p\rightarrow q\urcorner$ is true iff the degree of truth of $\ulcorner p\urcorner$ is smaller than or equal to that of $\ulcorner q\urcorner$, whereas $\ulcorner Np\urcorner$ is as true {false} as $\ulcorner p\urcorner$ is false {true} — in other words, simple or natural negation reverses the order expressed by ‘$\rightarrow$,’ and that is all it does. In the present context we’d better read classical negation, ‘$\neg$’, as ‘*not…at all*’.

The existence of two conditionals (the mere conditional '$\supset$' and implication, '$\rightarrow$') entails that there are in P10 two different deduction relations. We can keep '$\vdash$' as expressing mere inference, while using '$\Vdash$' as expressing a stronger inference relation, namely: $p^1, \ldots, p^n \Vdash_{P10} q$ iff $\ulcorner p^1 \ldots p^n \rightarrow q \urcorner$ is a theorem of P10; whereas for '$\vdash$' we do have the classical deduction metatheorem, viz. $p^1, \ldots, p^n, r \vdash_{P10} q$ iff $p^1, \ldots, p^n \vdash_{P10} r \supset q$ — theorems being such formulae as are $\vdash$-inferred from an empty antecedent. Clearly if $p \Vdash q$ then $p \vdash q$. Our proof theory has to be implemented in such a way that those differences are taken into account.

Entailment, duly strengthened, is a functor sensitive to degree differences, whereas the mere conditional, '$\supset$', only takes into account whether formulae are completely false or not. **CL** is a poor system, not a wrong one. Its flaw consists in the fact that, by lacking an implication and a negation which are degree sensitive, **CL** compels people to think, legislate and act in terms of all or nothing — since for all **CL** knows 'wholly' and 'just a little' are merely stylistic variations, with no semantic difference between *its being altogether true that* and *its being [just] true*.

Here is a semantic treatment of P5 and P10. By a **P5-matrix** we mean an algebra $\mathcal{A}=<\mathcal{A},\mathcal{O},\mathcal{D}>$ where $\mathcal{A}$ is a well-ordered set of entities, $\mathcal{D}\subseteq\mathcal{A}$, and $\mathcal{O}$ is an ordered set of operations $<N,\vee,\wedge,\rightarrow>$, N being unary and the other binary and such that there are three elements ½, 0, 1 satisfying these postulates: 0 is the minimum; 1, the maximum; $x\vee z = max(x,z)$; $x\wedge z = min(x,z)$; $x\leq z$ iff $Nx\geq Nz$; N ½=½; $\mathcal{D}$ is a proper filter such that $½\in\mathcal{D}$; $x\rightarrow z$ =: ½ if $x\leq z$, and else 0; N0=1; NNx=x. Let $\mathcal{T}$ be an extension of P5. A valuation, v is a mapping from $\mathcal{T}$ into a P5-matrix, $\mathcal{A}$, such that for any two formulae, $\ulcorner p\urcorner$, $\ulcorner q\urcorner$, the usual conditions are met: $v(p\wedge q) = v(p)\wedge v(q)$, and the like for the remaining functors and operations. A formula $\ulcorner p\urcorner$ of $\mathcal{T}$ is **valid** iff every valuation, v, is such that $v(p)\in\mathcal{D}$ (the respective set of designated entities within its range). An inference (i.e. an ordered pair $<\mathcal{C},\ulcorner p\urcorner>$ where $\mathcal{C}$ is a set of formulae) is valid iff every valuation v such that, for every $\ulcorner q\urcorner\in\mathcal{C}$, $v(q)\in\mathcal{D}$ is such that $v(p)\in\mathcal{D}$. Clearly, P5-matrices are Kleene algebras (i.e. Quasi-Boolean algebras satisfying the Kleene postulate that $x\wedge Nx \leq z\vee Nz$). In fact it is easily proved that each finite P5-matrix is isomorphic to one whose carrier is a finite subset of the set of the integers such that for every positive integer therein its negative also belongs, and conversely: the algebraic ½ will be the numeric zero, while the algebraic 1 will be the greatest number in the set. Soundness and completeness are straightforwardly proved. Notice, though, that none of those [finite] matrices is characteristic. But let us form an infinite P5-matrix, $\mathcal{A}_\infty$, whose carrier comprises any infinite subset of the natural numbers, including zero, and their respective negatives, plus ∞ and $-\infty$. Let $\mathcal{D}$ be $[0,\infty]$. Then $\mathcal{A}_\infty$ is characteristic in the sense that only all P5 theorems are mapped into designated elements, but it is not **strongly characteristic** in that there are valid inferences (with respect to $\mathcal{A}_\infty$) which are not derivable in the system P5 (e.g. $\{\ulcorner p\urcorner,\ulcorner Np\urcorner,\ulcorner q\urcorner,\ulcorner Nq\urcorner\} \vdash \ulcorner p\rightarrow q\urcorner$). On the other hand, if we let the set of designated values to comprise all elements except $-\infty$, there will be valid formulae which are not theorems of P5, e.g. $\ulcorner p\rightarrow q\vee p\urcorner$ (Funnel).

Let us form a **P10-matrix** by adding to a P5-matrix an additional unary operation, H, and additional elements, ω and α, such that: $\omega=N\alpha$; $x\geq\alpha$ if $x\neq 0$; Hx=: 1 if x=1, and else 0 (1 and 0 are the algebraic 1 and 0 respectively). A valuation is a mapping from a P10 theory $\mathcal{T}$ into a P10-matrix (we lay down that $v(\alpha)=\alpha$ and so on). Now, not only are soundness and completeness attained but, more importantly, a strongly characteristic matrix becomes available: the set of all integers plus four additional elements, $\infty > \omega \geq$ every integer; and of course every integer $\geq -\omega = \alpha > -\infty$, with $\mathcal{D}$ comprising all elements except $-\infty$. That matrix is the **canonic** P10 matrix.

Thus P10 has the finite model property, namely that every [nondeliquescent, i.e. Post-consistent] non-conservative extension thereof has a finite characteristic matrix. For let $\mathcal{L}$ be such an extension of P10. There is a formula or a schema ⌜p⌝ within the vocabulary of P10 which is a theorem in $\mathcal{L}$ but not in P10. Let $\mathcal{B}$ and $\mathcal{v}$ be a P10 algebra and a valuation, respectively, such that $\mathcal{v}$(p) is designated. The reason cannot be that the set of designated elements has been enlarged, of course. If the carrier of $\mathcal{B}$ is infinite, then $\mathcal{B}$ is isomorphic to the canonic matrix of P10. Therefore, $\mathcal{B}$ is finite. Which entails that for some natural number n $\mathcal{L}$ is P10 plus the Dugundji formula in n disjuncts, namely: $\ulcorner p^1 \leftrightarrow p^2 \vee \ldots . \vee . p^1 \leftrightarrow p^n \vee \ldots . \vee . p^{n-1} \leftrightarrow p^n \urcorner$. None of those formulae is a theorem of P10.

Most interestingly, when propositional quantifiers are used with the usual postulates for them (see A.R. Anderson, N.D. Belnap, Jr. & J.M. Dunn, *Entailment: The Logic of Relevance and Necessity*, vol. II. Princeton U.P., 1992., pp. 19ff), — using ‘j’ as a propositional variable — we define ⌜~p⌝ as ⌜p→∀jj⌝, and ⌜p‣q⌝ as ⌜∃j(j∧p→q∧j)⌝; the fragment of the ensuing system, $P10^{\exists\forall j}$, in <∧,∨,‣,~> is exactly **CL**. In that sense, P10 is to **CL** precisely as **E** is to intuitionistic logic.

While those results do not conclusively show our approach to be superior to Łukasiewicz's as an implementation of fuzzy logic, it seems to us that they go some way in that direction. Of course, Łukasiewicz himself never tried to develop a system of fuzzy logic, since he did not believe in degrees of truth. However, his many-valued systems have become extremely popular among fuzzy logicicans. This paper has tried to show that more adequate alternatives must be implemented in order to accommodate the intuitions underlying fuzzy logic.