Editorial: Weighted Logics for Artificial Intelligence - An Introductory Discussion -

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Abstract

Before presenting the contents of the special issue, we propose a structured introductory overview of a landscape of the weighted logics (in a general sense) that can be found in the Artificial Intelligence literature, highlighting their fundamental differences and their application areas.

Keywords: weighted logics, graded truth, uncertainty, preferences, similarity

1. Introduction

In the last decades, and especially in Artificial Intelligence (AI), there has been an explosion of logical formalisms capable of dealing with a variety of reasoning tasks that require an explicit representation of quantitative or qualitative weights associated with classical or modal logical formulas (in one form or another). The semantics of the weights refer to a large variety of intended meanings: belief degrees, preference degrees, truth degrees, trust degrees, etc. Examples of such weighted formalisms include probabilistic or possibilistic uncertainty logics, preference logics, fuzzy description logics, different forms of weighted or fuzzy logic programs under various semantics, weighted argumentation systems, logics handling inconsistency with weights, logics for graded BDI agents, logics of trust and reputation, logics for handling graded emotions, etc.

The underlying logics range from fully compositional systems, like systems of many-valued or fuzzy logic, to non-compositional ones like modal-like epistemic logics for reasoning about uncertainty, probabilistic or possibilistic logics, or even some combination of them. Sometimes the weights are not explicit and the formalisms use total or partial orderings instead.

In this short paper we present an introductory discussion organizing a landscape of weighted logics (in a general sense) that can be found in the literature, highlighting

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their differences and application areas. In particular we overview the main approaches in AI to deal with graded notions of uncertainty, truth, preferences and similarity, we discuss what are the logical issues behind them, and finally we also point out new emerging areas for graded settings. This is done in Sections 2 to 4.

Our aim is not to provide a full and exhaustive overview of graded formalisms in AI, but rather an informed discussion of the main general issues and approaches, that may help the reader position the papers in this special issue, whose content is presented in Section 5.

2. Typical graded notions

One heavily entrenched tradition in AI, especially in knowledge representation and reasoning is to rely on Boolean logic. However, many epistemic notions in common-sense reasoning are perceived as gradual rather than all-or-nothing. Neglecting this aspect may lead to insufficiently expressive frameworks and may cause some confusion. Such naturally gradual (in our opinion) notions are reviewed below.

2.1. Graded Truth

Truth is a key notion in the philosophy of knowledge that is often viewed as Boolean in essence. Yet in the scope of information storage and management, this absolute view becomes questionable. When representing knowledge, one may choose a language whose primitives are Boolean, but it is also possible to use one whose primitives are many-valued entities. Indeed, as claimed quite early by De Finetti [29] commenting Łukasiewicz logic, deciding that a proposition is an entity that can only be true or false is a matter of convention, as it is a matter of choosing the range of a (propositional) variable. In this sense, truth is an ontic notion, as one participating to the definition of a proposition. One may take into account the idea that in some contexts the truth of a proposition (understood as its conformity with a precise description of the state of affairs) is a matter of degree. For instance, if the height of John is known one might consider that the proposition "John is tall" is not always modeled as being just true or false. This is the view held by fuzzy logic [10]. However this notion of graded truth is restricted to an information processing perspective and has nothing to contribute to the foundations of formal philosophy.

If the truth set contains intermediary truth degrees, one issue is whether or not we can keep the truth-functionality assumption that is the key feature of classical logic. Mathematically, the answer is yes as demonstrated by the large set of multiple-valued logics that are now available. However there are a lot of unresolved issues about many-valued logics and their applications to AI such as

- Why are there so few papers using multiple-valued logics as a representation of gradual properties in AI?
- How to choose among the many available truth functional many-valued systems?
- Does truth-functionality always make sense from an applied knowledge representation perspective?

- How does the notion of truth-functional many-valued truth articulates with studies of vagueness ?

As for the last question, see e.g. [22]. On the other hand, the most popular many-valued logics in AI seem to be those with 3 (Kleene [80]), 4 (Belnap [11]) or 5 (equilibrium logic [96]) truth values, with a view to handle epistemic notions such as ignorance, contradiction, negation as failure or default knowledge, following a long tradition dating back to Łukasiewicz and Kleene [57]. However, the systematic use of truth-tables for basic connectives in these approaches looks questionable as such notions have a strong epistemic flavor closer to ideas of uncertainty, while truth is an ontic notion. For instance, Kleene suggested that the third truth value could mean “unknown”, and this view has been taken for granted by many. However, “unknown” can be opposed to “certainly true” and “certainly false”, not to (ontologically) true and false. This has led to confusing debates that cannot be solved without letting the representation of uncertainty and modalities enter the picture [35]. However, usual epistemic logics with their relational semantics look far away from Kleene logic. Yet, a simple fragment of a KD logic, where modalities only appear in front of literals, can capture Kleene logic and other three-valued logics of partial knowledge [24]. This restriction on the expressive power (in Kleene logic, it is not possible to represent the fact that $p \vee q$ is unknown) is the price paid for the truth-functionality of logics of the unknown.

2.2. Uncertainty

Uncertainty modeling pertains to the representation of an agent’s beliefs. There are several sources of uncertainty

- The random variation of a class of repeatable events leaves an agent unable to predict the event that corresponds to the outcome of the next trial.
- The sheer lack of information may cause an agent being uncertain about the answer to a question.
- The simultaneous presence of inconsistent pieces of information due to too many sources may equally prevent an agent from asserting the truth or the falsity of a statement.

There are two traditions in AI for representing uncertainty, that still need to be reconciled to a large extent

- The non-graded Boolean tradition of (monotonic) epistemic logics that rely on the modal formalism, and includes some exception-tolerant non-monotonic logics.
- The graded tradition typically relying on degrees of probability. Measures of uncertainty aim at formalizing the strength of our beliefs in the truth (occurrence) of some propositions (events) by assigning to those propositions a degree of belief [72].

Whatever the tradition, it must be stressed that belief is a higher order notion w.r.t. truth, that is, a statement pertaining to belief encapsulates a Boolean proposition inside a belief qualifier: the truth degree of the statement "I believe p" does not refer to the truth of p, but to the belief itself: it is the degree of belief in p, irrespectively of p being true or not. From the mathematical point of view, a measure of uncertainty is a function that assigns to each proposition or event (understood here as a formula in a specific logical language) a value from a given scale, usually the real unit interval $[0, 1]$, under some suitable constraints. A well-known example is given by probability measures which try to capture our degree of confidence in the occurrence of events by additive [0, 1]-valued assignments. It can be shown that such an uncertainty calculus cannot be compositional with respect to all logical connectives without leading to some form of trivialisation of the construction [45]. For instance, probabilities are only compositional with respect to negation. In contrast, we have seen that degrees of truth can be truth-functional and this assumption still leads to sophisticated logical systems.

It is not easy to reconcile the probabilistic and the logical view of beliefs (see [27] for a special issue on this topic). At the most elementary level a set of Boolean formulas is often interpreted as a belief base containing propositions an agent believes. In the Bayesian probabilistic tradition, information is represented by a single probability distribution on possible worlds possibly encoded as a Bayes net. The two approaches are at odds, irrespectively of the choice of Boolean vs. gradual representations of belief:

- The logical approach leaves room for incomplete information.
- The Bayesian approach [97] seems to be very information demanding as the lack of belief in a proposition is always equated with the belief in its negation.

On the other hand, a natural graded extension of the elementary logical approach is captured by possibilistic logic [46, 47] where degrees of uncertainty may be captured by means of a mere total ordering of possible worlds and some propositions can be more believed than others. This plausibility ordering can be encoded as a numerical possibility distribution if needed. It preserves the property that if an agent believes two propositions (to any degree), this agent also believe their conjunction. This property (typical of epistemic logic) is violated in probability theory.

The use of a modal language enables the syntactic expression of partial ignorance, and explicit patterns of reasoning from it, but the semantics of epistemic logic in terms of accessibility relations does not fit with the numerical tradition of representing beliefs. However, specializing this semantics to mere epistemic states and restricting its language to epistemic sentences restores the link with uncertainty theories [7]. Putting together the probabilistic and such simplified epistemic logic approaches to belief leads to reasoning in the setting of imprecise probabilities [114] which explicitly attach degrees of belief and degrees of plausibility to propositions [91]. Such degrees are graded versions of necessity and possibility modalities; besides, possibility distributions can encode special cases of convex probability families [43].

A nice by-product of reconciling probability and epistemic or possibilistic logics is to offer a logical handling of conditioning: pushing probability down to the Boolean context, in de Finetti style, leads to the three-valued logic of conditional objects [44], that is also a semantics for non-monotonic logics; adding weights to this construction bridges the gap between logical representations of belief and both probability and possibility theories [13].

It remains the issue of choosing a proper scale for grading beliefs in a given application context, namely how much numerical is it useful to be? A unified framework for reasoning about uncertainty leaves us the choice between various graded representations: ordinal, qualitative (with a finite value scale), integer-based (as with Spohn kappa functions [106]) or real-valued.

2.3. Preferences

Preferences are clearly not Boolean most of the time. Artificial Intelligence has developed a Boolean framework for decision-making problems based on constraint propagation and satisfaction. While many problems are amenable to a constraint-based formulation, it makes little sense to ignore the gradual nature of preferences.

The tradition in preference modeling has been to use either order relations (total or partial) or numerical utility functions, albeit with little attention to the issue of preference representation in practice. On the contrary, Artificial Intelligence has focused on compact logical or graphical representations of preferences on multi-dimensional (often Boolean domains) [33]. In this situation, an interpretation represents an option described by Boolean attributes. For instance, CP-nets have exploited an analogy with Bayes nets to design a graphical structure encoding ordinal preferences when local decision variables are Boolean. However CP-nets are far from capturing all possible ordering relations between possible worlds and the notion of preference handled by CP-nets is all-or-nothing. More general logical languages where a preference relation between formulas appears in the language have been used, that are more expressive. However the question of the meaning of comparing two logical formulas in terms of its consequences on a preference ordering between interpretations is not obvious, and several proposals exist that are at odds with each other. For instance do we mean that all models of the preferred formula should be preferred to all models of the other formula? Or just their best models? See, e.g. [76].

One alternative is to use weights attached to Boolean formulas. Such a weight may reflect the imperativeness of the satisfaction of the associated proposition, then viewed as a goal to reach (a prioritized constraint). When this weight is used as a penalty for the interpretations that violate the formula, the weight can be viewed as a lower bound of a necessity measure [14]: this is just another use of possibilistic logic. However, other approaches exist, e.g., [52, 20], where the weight is a reward when satisfying the formula, and [82] where desires or preferences have a utilitarian semantics. More generally at the semantic level one may focus on the least preferred interpretations that violate the formula or on the best preferred ones that satisfy it. Or on the contrary, an interpretation may be considered all the better (resp. worse) as the sum of the rewards (resp. penalties) attached to formulas it satisfies (resp. violates) is higher (resp. smaller). An alternative to the use of weights is the introduction of a preference relation inside the representation language, as in, e.g., [113, 109, 15].

In the above preference representation framework, neither the presence of several criteria nor the possibility of uncertainty is considered. In this case, there may be two kinds of weights

- Weights expressing preferences of options over other options.
- Weights expressing the likelihood of events or importance of groups of criteria.

In the case of decision under uncertainty, one way out is to use two sets of formulas, one for the knowledge base and one for the preference base, and to articulate some kind of inference technique exploiting both bases so as to encode the optimization of a given criterion mixing uncertainty and utility [36]. This approach looks more problematic for multiple criteria decision-making, where value scales may not be commensurate, and mportance weights pertain to yet another dimension of the evaluation process. Still, a logical counterpart to Sugeno integrals, a family of qualitative multiple criteria aggregation operators, has been recently obtained [49].

2.4. Similarity

The use of the idea of similarity in reasoning may refer two different points of view: either one does not want to distinguish between objects that are found to be similar, thus building a partition, or one wants to take advantage of the existence of a metric structure on a set of objects and exploit it in some way. In the first case, we perform a granulation of the universe of discourse, while in the second case we are interested in extrapolation or interpolation.

Similarity is often a graded notion, especially when it is related to the idea of distance. It may refer to a physical space as in spatial reasoning, or to an abstract space used for describing situations, as e.g., in case-based reasoning.

The representation of spatial relations between regions often relies on first order theories based on a family of partial preorders between regions, called mereologies. Although the term "mereology" usually refers to the idea of parthood as a basic notion, the idea of connection may be also used as a primitive notion. There are eight basic relations between regions known as RCC relations (RCC stands for "Region Connection Calculus") [25]. One may even start with a fuzzy connection relation (which might be defined from a distance or a pointwise closeness relation), and then define a "part of", or an "overlap" fuzzy relation between regions for instance, and obtain a graded extension of the RCC calculus [102]. Modal logics are also used for representing spatial information. Spatial interpretations of modalities have been provided for capturing various spatial concepts qualitatively with a topological or geometric flavor such as nearness or distance, for example. See [53] for a review of the logic-based representations of mereotopologies in classical or modal logics, and in fuzzy and rough sets settings, as well as modal logic representations of geometries.

Rough sets [95] provide a formal setting for the "granulation" of a universe of discourse partitioned into equivalence classes of elements that are found indistinguishable (because they share exactly the same properties among the ones that are considered). They come from the idea of information systems where the available descriptions of objects via attributes are not sufficient to distinguish between them (more attributes would be needed). Only upper and lower approximations of subsets of objects can be defined. Such clustering between indistinguishable objects also lead to eliminate inessential attributes. Modal logics have been proposed for reasoning with lower and upper approximations of sets of models in such a setting [56, 32]. Clearly the equivalence relation may be turned into a fuzzy relation, giving birth to graded rough set calculi [42, 99]. In that case the partition becomes a fuzzy covering of the set of objects. There is to-date a large technical literature putting rough sets, fuzzy sets and fuzzy similarity relations together.

Extrapolation and interpolation reasoning are based on the idea of closeness between interpretations. Thus, for instance, if the set of models of a proposition p fails to be included in the set of models of a proposition q, but remains included in the set of interpretations that are close to models of q, one may say that $p \to q$ is "close to being true". This has been advocated by different authors [101, 79] (and contrasts with nonmonotonic reasoning where one requires that the preferred / normal models of p be included in the models of q). The closeness (or proximity) relation between the interpretations gives birth to a graded consequence relation, which is the basis for a logic of similarity dedicated to interpolation [37], and captures fuzzy logic-based approximate reasoning. In a similar spirit, a logic allowing to reason about the similarity with respect to specific sets of prototypes has been recently proposed [111].

In the above approaches, similarity is graded. More qualitative approaches have been proposed, using comparative relations. The logic $\mathcal{CSL}$ [105] is based on a modal binary operator that is used for denoting the set of interpretations that are closer to p than to q. The underlying distance-based semantics can be restated in terms of preferential structures using a ternary relation expressing that all the points in a region z are at least as close to region x than to region y [2].

Another qualitative approach, without any grade, relies at the semantic level on the conceptual spaces framework [63] where the possibility of expressing spatial-like localization such as "being in between" (by means of a ternary relation), or "parallelism" (by means of a quaternary relation) provides a basis for capturing interpolative and extrapolative reasoning respectively [103]. This proposal comes close to logical reasoning with analogical proportions [98], which are quaternary statements of the form "a is to b as c is to d" (involving Boolean, or graded, properties), but remains more cautious.

Lastly, let us also mention that apart from reasoning *about* similarity, what may be termed reasoning *with* similarity has been also proposed as a semantic basis in nonmonotonic reasoning [65], information updating (which relies on ternary comparative closeness relation) [78], or in distance-based information fusion, where, e.g., Hamming distances are computed with respect to set of interpretations [81]. Another view of information fusion, recently proposed in [104], relies on the idea that inconsistency can be often resolved by enlarging the sets of models of the information to be fused, thanks to similarity relations.

3. Logical Issues

In this section our aim is to lay bare the main contrasting features of logical formalisms dealing with graded uncertainty and graded truth.

3.1. Uncertainty semantics

Let us assume an agent is to reason about what the world is and assume that each of the possible states of the world is described by a complete Boolean truth evaluation of a given (finite) set of atomic propositions Var. So let Ω denote the set of Boolean truth-evaluations $w : \mathcal{L} \to \{0,1\}$ of formulas from a propositional language $\mathcal{L}$ built from the finite set of variables Var and with the usual connectives $\wedge$, $\vee$, $\to$ and $\neg$. It is well known that the connectives $\wedge$, $\vee$ and $\neg$ endow $\mathcal{L}$, modulo logical equivalence, with a structure of a Boolean algebra.

We start by considering the case where the agent has complete information about the world, so he knows that the actual worlds is $w_0 \in \Omega$. In this case there is no uncertainty at all, in fact, knowing what the world is, the agent is able to ascertain the truth status of every possible proposition. This corresponds to consider the agent's epistemic state as being represented by the pair (Ω, E) where Ω reflects the agent's language (that govern the precision in the description of possible states of the world) and $E = \{w_0\}$ is a singleton.

A first form of uncertainty appears when the agent's information only allows him to know for certain that the actual world w_0 is in some given subset $E \subseteq \Omega$. This is the typical case where the agent has a theory T (a set of formulas) describing what he knows about the world. The epistemic state of the agent is then represented as (Ω, E) where E is a non-empty subset of interpretations, indeed the set of models of T, and the agent is only able to determine the truth status of some propositions, but not for some others. Indeed, a proposition φ is known to be *true* if $E \models \varphi$ (or equiv. $T \vdash \varphi$), it is known to be *false* when $E \models \neg\varphi$ and it is *unknown* otherwise, i.e. when both $E \not\models \varphi$ and $E \not\models \neg\varphi$. Therefore in this setting, propositions may be in three different epistemic statuses.

A more refined representation is when the agent may associate weights to interpretations related to the likelihood of each interpretation of describing the actual world. In this case, the epistemic state can be represented as a pair (Ω, μ), where $\mu : \Omega \to [0,1]$ attaches a weight to each possible world. In the probabilistic model (see e.g. early works by Nilsson [94]), $\mu = p$ is a probability distribution on Ω, hence requiring $\sum\{p(w) \mid w \in \Omega\} = 1$, that allows to rank the likelihood of any proposition of being true according to its probability measure $P(\varphi) = \sum\{p(w) \mid w \in \Omega, w \models \varphi\}$. When $p(w) \in \{0,1\}$ for all $w \in \Omega$, then the epistemic state reduces to a singleton $E = \{w_0\}$ where w_0 is the unique world such that $p(w_0) = 1$.

In the possibilistic setting [39], $\mu = \pi$ is a possibility distribution, where $\pi(w) = 1$ means that is totally possible (plausible) that $w = w_0$, $\pi(w) = 0$ means that $w = w_0$ is totally disbelieved. Then propositions are weighted according to the corresponding conjugate necessity and possibility measures: $N(\varphi) = 1 - \min\{\pi(w) \mid w \models \neg\varphi\}$, and $\Pi(\varphi) = \max\{\pi(w) \mid w \models \varphi\}$. Again, when $\pi(w) \in \{0,1\}$ for all $w \in \Omega$, then the epistemic state defined by π corresponds to the previously considered three-valued setting with $E = \{w \mid \pi(w) = 1\}$. Other measures can also be used, like the guaranteed possibility $\Delta(\varphi) = \min\{\pi(w) \mid w \models \varphi\}$, which is equal to 1 whenever all models of φ are considered plausible. The case when $\Delta(\varphi) > \Pi(\neg\varphi)$ expresses a form of strong belief in φ [8].

More generally, one may consider graded epistemic states of the form (Ω, μ), where $\mu : 2^{\Omega} \to [0,1]$ is a monotonic set-function, in such a way that every proposition φ can be attached a likelihood or belief degree of being true as the evaluation w.r.t. μ of the set of models of φ, i.e. of $\mu(\{w \in \Omega \mid w(\varphi) = 1\})$. This representation generalizes the previous probabilistic or possibilistic models, to other more general ones like for instance those defined by belief functions, upper and lower probabilities or imprecise probabilities (see e.g. [70, 71] for a general approach encompassing many uncertainty models in a modal logic setting). The information cannot then be reduced to a distribution over interpretations. In the qualitative setting, a monotonic set-function can always be viewed as the lower bound of a set of possibility measures ($\mu(A) = \min_{i=1}^{n} \Pi_i(A)$) which comes down to representing epistemic states expressed by μ as a set of possibility distributions over Ω (e.g. corresponding to several agents) [48].

3.2. Syntactic formats for uncertainty

From a syntactical point of view, a number of formalisms coping with graded uncertainty have been proposed in the literature. Most of them use a modality, either explicit or implicit, referring to graded belief. For illustration purposes, we mention two kinds of languages: those based on Boolean uncertainty statements restricting degrees of uncertainty of formulas into a given subset; and those expressing gradual uncertainty statement using many-valued predicates or modalities[1].

An instance of the first approach is Halpern's probability logic [71], where formulas express constraints among the probabilities of $\varphi_1, \ldots, \varphi_k$ as linear inequalities of the form $a_1 P(\varphi_1) + ... + a_k P(\varphi_k) \geq b$, with $a_1, \ldots, a_k, b$ being real numbers and where $P(\varphi_i)$ stands for probability of φ_i. The same language is used to reason about other kinds of uncertainty measures like plausibility measures, belief functions, etc., see also [115] for a related general approach to reason under uncertainty using linear constraints. In the Serbian school (Marković, Ognjanović and colleagues) on probability logics [34, 93, 100], they use graded modalities of the form $P_{\geq a}\varphi$, declaring that the probability of φ is at least a, and then these probabilistic atoms are combined by means of classical connectives. A similar approach is Lukasiewicz's probabilistic logic [89] (see also [28] for a logic allowing to express independence), where the author uses expressions of the form $(\varphi)[l, u]$ to denote that the probability of φ lies in the interval $[l, u]$, as well as van der Hoek and Meyer's approach [110] on graded probabilistic modalities. Likewise, in Dubois-Prade's possibilistic logic [39, 46, 47], formulas are pairs (φ, α), stating that the necessity $N(\varphi)$ of φ is at least α. Recently, this language has also been generalized to deal with Boolean combinations of possibilistic atoms $N_{\geq\alpha}\varphi$, in a similar way to the previous probabilistic logic language [50].

A different approach by Hájek and colleagues [68, 67, 64] has also been proposed, where the modality B used to denote graded belief (probability, necessity, etc.) is graded itself, that is, even if φ is Boolean, the atomic modal expression $B\varphi$, read as "φ is believed", is graded in nature (φ can be more or less believed, probable, necessary, etc.). In this way, the truth-degree of $B\varphi$ can be taken as, e.g., the probability (or necessity) degree of φ, and then these graded atoms are combined using the rules of a suitable fuzzy logic.

[1]There is a third approach using statements like "φ is more likely than ψ", that relies on conditional logics in the style of Lewis [83], that takes another point of view on logics of uncertainty.

3.3. Logics of graded truth

Indeed, formalisms that cope with graded truth (fuzzy logics) radically departs from the formalisms for uncertainty reasoning [45, 38]. Giving up the bivalence principle and adopting a truth scale with intermediate degrees between 0 (false) and 1 (true) (usually the unit real interval $[0, 1]$) lead to a number of many-valued truth-functional systems with connectives extending the classical ones. The most relevant examples of formal fuzzy logic systems are the so-called t-norm based fuzzy logics [67]. These correspond to logical calculi with the real interval $[0, 1]$ as set of truth-values and defined by a conjunction $\&$ and an implication $\rightarrow$ interpreted respectively by a continuous t-norm $*$ and its residuum $\Rightarrow$, and where negation is defined as $\neg\varphi = \varphi \rightarrow \overline{0}$, with $\overline{0}$ being the truth-constant for falsity. In this framework, each continuous t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\&$ and $\rightarrow$ and truth-constant $\overline{0}$ (further connectives, like $\varphi \wedge \psi$ as $\varphi\&(\varphi \rightarrow \psi)$, can also be defined). Evaluations of propositional variables are mappings e assigning to each propositional variable p a truth-value $e(p) \in [0, 1]$, which extend univocally to compound formulas as follows:

$$
\begin{aligned}
e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\
e(\varphi\&\psi) &= e(\varphi) * e(\psi) \\
e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}
$$

A formula φ is a said to be a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e. Let the set of all 1-tautologies of $PC(*)$ be denoted as $TAUT(*)$. The main axiomatic systems of fuzzy logic, like Łukasiewicz logic (Ł), Gödel logic (G) or Product logic (Π), syntactically capture different sets of tautologies $TAUT(*)$, according to the choice of the t-norm $*$, see e.g. [69, 67, 66, 23]. Indeed one always has results of the form:

$$\varphi \text{ is provable in the logic L iff } \varphi \in TAUT(*_L) \tag{1}$$

where $*_L$ can be chosen as $x *_{\text{Ł}} y = \max(0, x+y-1)$ (Łukasiewicz logic Ł), $x *_G y = \min(x, y)$ (Gödel logic G) and $x *_\Pi y = x \cdot y$ (Product logic Π).

It is worth noticing that, in contrast with classical logic, the algebraic structures of the set of formulas modulo logical equivalence in these systems of fuzzy logic are no longer Boolean algebras but weaker structures like MV-algebras, prelinear Heyting algebras, etc. Strictly speaking, a formula in a many valued logic is interpreted as a function from the the Cartesian product of copies of the truth-set to the truth set. Each many-valued logic captures a class of functions at the ontic level, and there is no epistemic ingredient in such logics, which may explain why they are not much used in Artificial Intelligence.

Also, in all these systems, the implication captures the truth-ordering, since if $\Rightarrow$ is the residuum of a left-continuous t-norm $*$ (i.e., $x \Rightarrow y = \max\{z \in [0, 1] \mid x*z \leq y\}$), then $x \Rightarrow y = 1$ iff $x \leq y$, and hence $e(\varphi \rightarrow \psi) = 1$ iff $e(\varphi) \leq e(\psi)$. Therefore, a formula $\varphi \rightarrow \psi$ actually represents that ψ is at least as true as φ, this is to say, t-norm based fuzzy logics are logics of *comparative* truth. To explicitly deal with truth-degrees in the reasoning, one may introduce truth-constants $\overline{r}$, e.g. for all rational values in $r \in [0,1]$. Then a formula $\overline{r} \to \varphi$ expresses that the truth-degree of φ is at least r (see e.g. [54]), which suggests that such many-valued logics, cast at the epistemic level, may account for weighted logics of uncertainty of the previous kind (using graded modalities).

From an epistemic point of view, in a fuzzy logic setting, the states of the world are described by complete $[0,1]$-evaluations of atomic formulas. Let Ω' be the set of these evaluations, i.e. $\Omega' = \{w \mid w : Var \to [0,1]\}$. Note that the set Ω of Boolean interpretations is indeed a subset of Ω'. Because of truth-functionality, a completely informed scenario thus corresponds now to a precise many-valued truth-value assignment $w_0 \in \Omega'$ to all propositional variables. Analogously to the Boolean case, different kinds of incomplete information about the world translates here to different many-valued generalizations of epistemic states. In general they are of the form (Ω', μ'), where $\mu' : [0,1]^{\Omega'} \to [0,1]$ is a generalized uncertainty measure over fuzzy sets of interpretations (see e.g. [59, 58, 60] for some logical formalisms that are able to deal with uncertainty over non-Boolean (fuzzy) events).

4. New areas for graded settings and conclusion of this overview

Beyond the handling of the ideas of truth, uncertainty, preference, and similarity that may require graded settings for a proper handling, the field of deontic reasoning is another area where one encounters basic notions whose strengths may be usefully compared, stratified into layers, or even graded: one may think of graded obligations, or permissions for instance [84, 30]. One may also distinguish between weak and strong permissions [77]. Priorities, or preference relations among worlds, aiming at ordering worlds from the most ideal ones to the least ideal ones, may help solving dilemmas [21].

Besides, there are several domains of active research in Artificial Intelligence nowadays where several of the basic notions mentioned above can be encountered, possibly combined with other notions that also need to be graded. Let us review them briefly.

- So called BDI agents [18] are supposed to have beliefs about the world, desires, from which they elicitate intentions that are feasible desires. Clearly beliefs may be pervaded with uncertainty, desires may be modeled as collections of goals with different priority levels, feasibility may be a matter of cost, leading to ordered / graded intentions, see, e.g. [19]. The modeling of desires in terms of guaranteed possibility measures, already at work in the previous reference, has been further discussed and investigated in [40, 41].

- Modeling the trust that can be associated with information sources or agents, as well as related notions such as distrust or reputation, is an important issue in practice. Many proposals exist - modal or numerical - where gradedness has been introduced in various ways [85, 88, 31, 112, 12]. Still, there is not yet a fully clear view of how graded trust may relate to beliefs and uncertainty.

- Argumentation is another area where the idea of strength seems naturally associated with arguments [5], as recently investigated [51, 61, 26, 6, 74]. However, it is likely that a uniform view of strength is not applicable here, since the strength of an argument may refer to the uncertainty pervading the pieces of information on which it is based [4, 73, 3] and on the reliability of the source(s) of the argument, as well as on the rhetoric form of the argument (e.g. its length); moreover the argument itself may refer to a gradual view of truth (when stating for instance that “the higher the fever the more certain the child should remain in bed”). The handling of such strengths may also depend on the kind of problem to which argumentation is applied: persuasion, negotiation, or deliberation, for instance.

- Emotions, such as surprise, fear etc. have been recently modeled by means of definitions in modal logic [1]. It seems natural here again, to have these modalities complemented with grades, leading to hybrid notions that might be a compound involving more basic notions such as uncertainty, preference, or similarity [107, 87, 86].

- In weighted social networks, the degrees of connection between actors may be described in a graded logical setting, such as many-valued modal logics [55].

In this introduction we have briefly discussed several issues in the main approaches at work to represent and reason with fundamental notions in AI, where the use of degrees appears to be natural, such as truth, uncertainty, preferences, or similarity (but also trust, permission, obligation, desires, etc.). Specifically, the basic difference between graded truth and other graded notions has been highlighted. While the former implies a change at the ontic level (from two truth values to multiple truth-values), without any reference to epistemic knowledge or ignorance, the latter relates to intensional notions that (usually) apply to Boolean propositions, like their epistemic (belief) status, or how they compare to other propositions in terms of preference, utility, similarity, etc. This is reflected on the kind of formal models that support these graded notions, many-valued truth-functional models in the former case, Kripke-like models and graded modalities in the latter case.

Actually, many-valued logics have been seriously criticized at the philosophical level because of the confusion between truth-values on the one hand and degrees of belief, or various forms of incomplete information, on the other hand, a confusion that even goes back to pioneers including Łukasiewicz (e.g., the idea of *possible* as a third truth-value). Actually, due to this issue and the numerical flavor of fuzzy logic, there is a long tradition of mutual distrust between Artificial Intelligence and fuzzy logic. There are two ways to fill this gap. One is to use fuzzy logics to encode graded epistemic modalities that can account for various uncertainty theories; these graded modalities may be applied to Boolean formulas, yielding a two-tiered logic. The other way is to show how reasoning about knowledge and uncertainty can also be defined on top of fuzzy/gradual propositions by augmenting fuzzy logic with epistemic modalities. Recent works along this line may be considered as first steps towards a reconciliation between possibility theory and other theories of belief as well with fuzzy logic (in the sense of a rigorous symbolic setting to reason about gradual notions), see e.g. [16, 59].

Finally, we would like to mention that, although we have mainly focused on logical issues, many of the concerns discussed here have also echoes in closely related areas like description logics, logic programming and answer set programming when they come to handle uncertainty, preferences or fuzziness, see for instance [90, 108, 62, 75, 92, 9, 17] for a variety of logic programming approaches coping with graded uncertainty and/or truth.

5. Contents of the Special Issue

This special issue organized by the last two authors of this introduction, originates in an international workshop on Weighted Logics for Artificial Intelligence (WL4AI'12), co-located with the 20th European Conference on Artificial Intelligence, and held in Montpellier on August 28, 2012. It includes expanded versions of papers presented at the workshop as well as papers submitted in response to a call for papers on the workshop topic published in this journal. All accepted contributions have largely benefited of useful reviewers' comments which helped the authors improving their papers substantially through several rounds of revision.

The topics of these papers illustrate several of the many concerns just surveyed. Indeed the special issue gathers 13 contributions which cover a broad variety of topics: probabilistic logics, three-valued logics, description logics, answer set programming, argumentation, deontic norms, causality, and information fusion.

The two first papers deals with probabilistic logics. The first one "Hierarchies of probabilistic logics" by N. Ikodinović, Z. Ognjanović, A. Perović, and M. Rasković, studies logics with probability operators expressing that the probability of a formula is at least s (where s is a rational number), or that the probability is in a set F ranging over different recursive families of recursive subsets of the set of rationals in the unit interval, leading to distinct logics in terms of expressivity. The second paper "Conditional p-adic probability logic" by A. Ilić Stepić, Z. Ognjanović, N. Ikodinović, presents a proof-theoretical approach to p-adic valued conditional probabilistic logics that extend classical propositional logic with conditional probability operators and where formulas are interpreted in Kripke-like models that are based on p-adic probability spaces.

The third paper, "Borderline vs. unknown - Comparing three-valued representations of imperfect information", deals with other basic concerns. Their authors, D. Ciucci, D. Dubois and J. Lawry compare the expressive power of elementary representation formats for vague, incomplete or conflicting information, namely Boolean valuation pairs, orthopairs of sets of variables, Boolean possibility and necessity measures, three-valued valuations, and supervaluations, making explicit their connections with strong Kleene logic and with Belnap logic of conflicting information. The paper aims at clarifying the similarities and differences between truth-functional and non-truth-functional approaches.

The three next papers deal with description logics, a family of knowledge representation formalisms for handling taxonomies. M. Cerami, A. García-Cerdaña and F. Esteva define fuzzy description logics on the basis of first-order many-valued fuzzy logics. Their paper "On finitely valued fuzzy description logics" is based on the fuzzy logic of a finite BL-chain in the sense of Hájek. The paper addresses the hierarchy of fuzzy attributive languages and discusses reasoning tasks and computational complexity. The second paper "Consistency reasoning in lattice-based fuzzy description logics" by S. Borgwardt and R. Peñaloza is also about fuzzy description logics with a semantics based on complete De Morgan lattices and one basic reasoning task, namely deciding whether an ontology representing a knowledge domain is consistent. The paper focuses on the fuzzy description logic L-SHI, providing a tableaux-based algorithm for deciding consistency when the underlying lattice is finite, and identifies decidable and undecidable classes of fuzzy description logics over infinite lattices. The last description logic paper "Completion-based generalization inferences for the description logic ELOR with subjective probabilities" by A. Ecke, R. Peñaloza and A.-Y. Turhan studies the logic Prob-ELOR that extends ELOR with probabilities, and presents a completion-based algorithm for polynomial time reasoning in a restricted version of Prob-ELOR. This algorithm is extended in order to provide tools for building ontologies automatically from examples by computing the most specific concept that generalizes a given individual into a concept description, and the least common subsumer that generalizes several concept descriptions into one. The feasibility of the approach is demonstrated empirically on a prototype.

The paper "Complexity of fuzzy answer set programming under Łukasiewicz semantics", by M. Blondeel, S. Schockaert, D. Vermeir and M. De Cock, illustrates another active area of research, namely answer set programming (ASP), here generalized to fuzzy ASP in which propositions may be graded. The paper analyzes the computational complexity of fuzzy ASP under Łukasiewicz semantics, and shows that the complexity of the main reasoning tasks is located at the first level of the polynomial hierarchy, even for disjunctive FASP programs for which reasoning is classically located at the second level, and a reduction from reasoning with such fuzzy ASP programs to bilevel linear programming is established.

The two next papers deal with argumentation systems, another hot topic in artificial intelligence nowadays. In their paper "Valued preference-based instantiation of argumentation frameworks with varied strength defeats", S. Kaci and Ch. Labreuche consider an extension of Dung's abstract argumentation framework where the degree to which an argument defeats another argument can be graded, and instantiate it by a preference-based argumentation framework with a valued preference relation (which may be derived from a weight function). They show under some conditions that there are less situations in which a defense between arguments holds with a valued preference relation compared to a Boolean preference relation. The paper "Postulates for logic-based argumentation systems", by L. Amgoud, focuses on argumentation systems that are based on deductive logics and that use Dung's semantics. It proposes and discusses five rationality postulates that such systems should satisfy: consistency and closure under the consequence operator of the underlying logic of the set of conclusions of arguments of each extension, closure under sub-arguments and exhaustiveness of the extensions, and a free precedence postulate ensuring that the free formulas of the knowledge base (i.e., the ones that are not involved in inconsistency) are conclusions of arguments in every extension. Postulates for weighted argumentation systems are finally addressed.

The paper "Reasoning about norms under uncertainty in dynamic environments" by N. Criado, E. Argente, P. Noriega, and V. Botti proposes a multi-context graded BDI architecture that models norm-autonomous agents able to deal with uncertainty in dynamic environments. A degree of saliency is attached to norms, and beliefs, desires and intentions are graded. This architecture has been experimentally evaluated on a fire-rescue scenario and results are reported in the paper.

In "A weighted causal theory for acquiring and utilizing open knowledge", J. Ji and X. Chen introduce a weighted counterpart to McCain and Turner's nonmonotonic causal theories, the action model of a robot/agent being specified as a causal theory. The authors solve an abduction problem for finding the most suitable set of causal rules from open-source knowledge resources, so that a plan for accomplishing the robot task can be computed using the action model and the acquired knowledge. The weights associated to hypothetical causal rules are used for comparing competing explanations which induce causal models satisfying the task. This is illustrated with a service robot example.

The last two papers discusses tools for information fusion. In "Sum-based weighted belief base merging: from commensurable to incommensurable framework", S. Benferhat, S. Lagrue, and J. Rossit provide a postulate-based analysis of the sum-based merging operator when sources to merge are incommensurable (i.e., they do not share the same meaning of uncertainty scales). They show that this operator can be characterized either in terms of an infinite set of compatible scales, or by a Pareto ordering on a set of propositional logic interpretations. Finally, T. Nakama and E. Ruspini, in "Combining dependent evidential bodies that share common knowledge", establish a formula for combining dependent bodies of evidence that are conditionally independent given their shared knowledge. They show that the Dempster-Shafer formula and the conjunctive rule of Smets' Transferable Belief Model can be recovered as special cases of their combination formula.

Hopefully this special issue, by providing an overview of weighted logics in AI and gathering a rich sampling of works illustrating the interest of introducing grades in many knowledge representation and reasoning problems, will contribute to a better understanding of these issues, and to the proper use of the different possible approaches.

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