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Generic intersection of orthogonal groups

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Abstract

Let V be a finite-dimensional complex vector space and let $g, h \colon V \times V \to \mathbb{C}$ be two non-degenerate symmetric bilinear forms. Let G, H be the groups of isometries of g, h, respectively. If the endomorphism $L \colon V \to V$ associated to g, h is diagonalizable, then $\dim(G \cap H) = \sum_{i=1}^{r} \binom{m_i}{2}$, where m_i, $i = 1, \ldots, r$, are the dimensions of the eigenspaces of L.

Key words: Bilinear form, Complex vector space, Diagonalizable endomorphism, Group of isometries.
MSC 2000: AMS codes (optional)

1 Introduction and preliminaries

Let V, W be two complex vector spaces of finite dimension and let $\mathcal{L}(V, W)$ be the space of $\mathbb{C}$-linear mappings from V into W. We write $\mathfrak{gl}(V) = \mathcal{L}(V, V)$ and we denote by $GL(V)$ the linear group of V, i.e., the group of invertible elements in $\mathfrak{gl}(V)$.

An element $A \in \mathfrak{gl}(V)$ is said to be an *isometry* of a symmetric bilinear form $g \colon V \times V \to \mathbb{C}$ if the following equation holds:

$$g\left(A(x), A(y)\right) = g(x, y), \quad \forall x, y \in V. \tag{1}$$

As a simple computation shows, we have

Lemma 1 *Let $g \colon V \times V \to \mathbb{C}$ be a symmetric bilinear form on an n-dimensional complex vector space V and let V', V'' be vector subspaces such that,* (1) *$g|_{V'}$ is non-degenerate,* (2) *$g(v, v'') = 0$, $\forall v \in V, \forall v'' \in V''$, and* (3) *$V = V' \oplus V''$.*

Then, every isometry $A \in \mathfrak{gl}(V)$ of g can be written as

$$A = \begin{pmatrix} A' & \mathrm{O} \\ B & C \end{pmatrix}, \quad B \in \mathcal{L}(V', V''),\ C \in \mathfrak{gl}(V''),$$

and A' is an isometry of $g|_{V'}$.

Consequently, the structure of the set of isometries of a degenerate symmetric bilinear form g can be recovered from the non-degenerate part of g. Because of this, below we confine ourselves to consider only non-degenerate symmetric bilinear forms. In this case, every isometry of g is invertible, as the equation (1) implies $\det A = \pm 1$, and the set of all isometries of g is a subgroup of $GL(V)$, which is denoted by G. By choosing a orthonormal basis in V, every element of G is represented by an orthogonal matrix and we have an isomorphism $G \cong O(n, \mathbb{C})$.

We also remark on the fact that G is a closed subgroup in $GL(V)$ and hence, G is a Lie subgroup of the linear group of V, whose Lie algebra will be denoted by $\mathfrak{g}$.

2 Main result

Theorem 2 *Let V be an n-dimensional complex vector space and let*

$$g, h \colon V \times V \to \mathbb{C}$$

be two symmetric bilinear forms, which are assumed to be non-degenerate. Let G, H be the groups of isometries of g, h, respectively and let $L \colon V \to V$ be the endomorphism associated to g, h, i.e., $g(x, L(y)) = h(x, y)$, $\forall x, y \in V$. If L is diagonalizable, then

$$\dim(G \cap H) = \sum_{i=1}^{r} \binom{m_i}{2},$$

where m_i, $i = 1, \dots, r$, are the dimensions of the eigenspaces of L.

Sketch of the proof. Let α_i, $i = 1, \dots, r$, be the distinct eigenvalues of L and let $E(\alpha_i)$ be the eigenspace attached to α_i. As L is diagonalizable, we have $V = \oplus_{i=1}^{r} E(\alpha_i)$ and $E(\alpha_i)$ and $E(\alpha_j)$ are orthogonal with respect to both metrics for $i \neq j$. There exist basis of every subspace $E(\alpha_i)$ to which the Gram-Schmidt process can be applied. Collecting all theses bases, we obtain a basis $(v_1, \dots, v_n)$ of eigenvectors for L which is also g-orthonormal and the matrices of g and h in this basis are,

$$M_g = I_n = n \times n \text{ identity matrix},$$
$$M_h = \text{diagonal}\left(\alpha_1, \overset{(m_1}{\ldots}, \alpha_1, \dots, \alpha_r, \overset{(m_r}{\ldots}, \alpha_r\right), \ m_1 + \ldots + m_r = n.$$

Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). As is known ([2, Theorem 3.31]) the exponential map $\exp \colon \mathfrak{g} \to G$ induces an diffeomorphism from an open neighbourhood of the origin in $\mathfrak{g}$ onto an open neighbourhood of the unit element in G. Hence

$\dim(G\cap H) = \dim(\mathfrak{g}\cap\mathfrak{h})$, and we are led to determine the Lie algebra of the intersection subgroup. As $\mathfrak{g} = \{A \in \mathfrak{gl}(V) : g\,(x, A(y)) + g\,(A(x), y) = 0, \forall x, y \in V\}$, and similarly for $\mathfrak{h}$, we conclude that $\mathfrak{g}\cap\mathfrak{h}$ can be identified to the subspace of $n \times n$ skew-symmetric matrices $A = (a_{ij})$ such that, $A^t M_h + M_h A = 0$. By decomposing A in blocks,

$$A = \begin{pmatrix} A_{11} & \dots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \dots & A_{rr} \end{pmatrix},$$

each A_{ij} being a $m_i \times m_j$ matrix for $i, j = 1, \dots, r$, we obtain $A_{ij} = 0, i \neq j$, and the submatrices $A_{11}, \dots, A_{rr}$ are arbitrary. As $\dim \mathfrak{o}(m, \mathbb{C}) = \binom{m}{2}$, we can conclude. □

Taking [1, Chapter 7,Theorem 1] into account, we also obtain

Corollary 3 *Let $\mathcal{U} \subset S^2V^*$ be the subset of non-degenerate bilinear forms. The pairs $(g, h) \in \mathcal{U} \times \mathcal{U}$ for which the conclusion of the theorem above holds is a dense subset in $\mathcal{U} \times \mathcal{U}$.*

3 Concluding remarks

Remark 4 *According to the proof of the previous theorem, the matrices of the form*

$$\exp(\tilde{A}_{11}) \cdots \exp(\tilde{A}_{rr}), \quad A_{ii} \in \mathfrak{o}(m_i, \mathbb{C}),\ 1 \leq i \leq r,$$

$$\tilde{A}_{ii} = \begin{pmatrix} \mathrm{O}_{\mu_i,\mu_i} & \mathrm{O}_{\mu_i,m_i} & \mathrm{O}_{\mu_i,n-\mu_{i+1}} \\ \mathrm{O}_{m_i,\mu_i} & A_{ii} & \mathrm{O}_{m_i,n-\mu_{i+1}} \\ \mathrm{O}_{n-\mu_{i+1},\mu_i} & \mathrm{O}_{n-\mu_{i+1},m_i} & \mathrm{O}_{n-\mu_{i+1},n-\mu_{i+1}} \end{pmatrix},$$

where $\mu_i = m_1 + \ldots + m_{i-1}$, and $\mathrm{O}_{\mu,\nu}$ denotes the null $\mu\times\nu$ matrix, span the intersection group $G \cap H$. Hence the problem of computing the intersection group is feasible: In fact, it reduces to exponentiate skew-symmetric matrices of size $m_1, \dots, m_r$.

Remark 5 *The previous theorem is no longer true if the endomorphism L is not diagonalizable. For example, for the metrics g, h with matrices*

$$M_g = \begin{pmatrix} \overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(k} & \mathrm{O} \\ \mathrm{O} & \overbrace{\begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}}^{(n-k} \end{pmatrix},$$

$$M_h = \begin{pmatrix} \overbrace{\begin{pmatrix} 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & \dots & \alpha & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & \alpha & \dots & 0 & 0 \\ \alpha & 0 & \dots & 0 & 0 \end{pmatrix}}^{(k} & \text{O} \\ \text{O} & \overbrace{\begin{pmatrix} 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & \dots & \alpha & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & \alpha & \dots & 0 & 0 \\ \alpha & 0 & \dots & 0 & 0 \end{pmatrix}}^{(n-k} \end{pmatrix},$$

respectively, as a computation shows, we have $\dim(\mathfrak{g} \cap \mathfrak{h}) = \min(k, n-k)$*, whereas* α *is the only eigenvalue of* L *and* $\dim E(\alpha) = 2$*.*

4 An example

Assume $\dim V = n = 5$, and that L has two distinct eigenvalues α, β such that $\dim E(\alpha) = 2$, $\dim E(\beta) = 3$. In this case, $\mathfrak{g} \cap \mathfrak{h}$ is identified to the matrices of the form

$$A = \begin{pmatrix} A_{11} & O \\ O & A_{22} \end{pmatrix}, \qquad A_{11} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, A_{22} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

According to Remark 4, the intersection group is generated by $\exp \tilde{A}_{11} \exp \tilde{A}_{22}$. Exponentiating, we obtain

$$\exp \tilde{A}_{11} \exp \tilde{A}_{22} = \begin{pmatrix} \begin{pmatrix} \cos d & \sin d \\ -\sin d & \cos d \end{pmatrix} & O \\ O & \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \end{pmatrix},$$

where $v = (a, b, c)$, and

$$
\begin{aligned}
\lambda_{11} &= \frac{c^2 + \left(a^2 + b^2\right)\cos(|v|)}{|v|^2}, \\
\lambda_{12} &= \frac{a|v|\sin(|v|) + bc\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{13} &= \frac{b|v|\sin(|v|) - ac\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{21} &= -\frac{a|v|\sin(|v|) - bc\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{22} &= \frac{b^2 + \left(a^2 + c^2\right)\cos(|v|)}{|v|^2}, \\
\lambda_{23} &= \frac{c|v|\sin(|v|) + ab\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{31} &= -\frac{b|v|\sin(|v|) + ac\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{32} &= -\frac{c|v|\sin(|v|) - ab\,(\cos(|v|) - 1)}{|v|^2}, \\
\lambda_{33} &= \frac{a^2 + \left(b^2 + c^2\right)\cos(|v|)}{|v|^2}.
\end{aligned}
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References

[1] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[2] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman and Company, London, 1971.