Quantum Gowdy T^3 model: A unitary description

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(Received 3 March 2006; published 19 April 2006)

The quantization of the family of linearly polarized Gowdy T^3 spacetimes is discussed in detail, starting with a canonical analysis in which the true degrees of freedom are described by a scalar field that satisfies a Klein-Gordon type equation in a fiducial time-dependent background. A time-dependent canonical transformation, which amounts to a change of the basic (scalar) field of the model, brings the system to a description in terms of a Klein-Gordon equation on a background that is now static, although subject to a time-dependent potential. The system is quantized by means of a natural choice of annihilation and creation operators. The quantum time evolution is considered and shown to be unitary, so that both the Schrödinger and Heisenberg pictures can be consistently constructed. This has to be contrasted with previous treatments for which time evolution failed to be implementable as a unitary transformation. Possible implications for both canonical quantum gravity and quantum field theory in curved spacetime are noted.

DOI: 10.1103/PhysRevD.73.084020 PACS numbers: 04.60.Ds, 04.60.Kz, 04.62.+v, 98.80.Qc

I. INTRODUCTION

In the search for a quantum theory of gravity within the canonical approach, it has always been useful to analyze symmetry reduced models. On the one hand, this allows us to discuss with specific examples conceptual and technical issues that arise when trying to conciliate gravity and quantum mechanics. On the other hand, these reduced models are usually of physical relevance in cosmology or in astrophysical situations. The most studied examples are mini-superspaces [1], where the infinite dimensional system is reduced by symmetry considerations to a model with a finite number of degrees of freedom. A more interesting and far-reaching class of reduced models, where the resulting system is still a field theory with an infinite number of degrees of freedom like general relativity, are known as midi-superspaces [2].

The simplest of all inhomogeneous midi-superspaces in pure general relativity with spatially closed spatial sections and cosmological solutions (expanding from a big-bang singularity) is the linearly polarized Gowdy T^3 model [3]. This explains the considerable attention that has been paid during the last 30 years to the problem of quantizing this model [4–11]. After the first preliminary attempts to construct a quantization and obtain physical predictions for the Gowdy T^3 cosmologies employing conventional (but not always rigorously implemented) canonical methods in quantum cosmology [4–6], the problem was revisited using Ashtekar variables in the context of a nonperturbative quantization [7,8]. Nonetheless, it is only recently that true progress has been achieved in the task of introducing a consistent quantization, at least for the (sub) model with linear polarization [9] for which the two spacelike Killing vector fields of the system are hypersurface orthogonal.

The quantization proposed in Ref. [9] for the linearly polarized Gowdy T^3 model is based on the equivalence that exists between the set of solutions for its spacetime metric and the classical solutions for a scalar field coupled to gravity in 2 + 1 dimensions, defined in a manifold whose topology is $\mathbb{R}^+ \times T^2$. In more detail, after a suitable (partial) gauge fixing of the Gowdy T^3 model, which includes the choice of an internal time, the linearly polarized Gowdy T^3 spacetimes are described (modulo a remaining global constraint) by a ‘‘point particle’’ degree of freedom and by a field ϕ that is subject to the same equation of motion as a massless, rotationally symmetric, free scalar field that propagates in a fictitious two-dimensional expanding torus. The quantization of the local degrees of freedom of the Gowdy model can hence be confronted by constructing a quantum theory for this scalar field. The quantum Gowdy T^3 model is defined by introducing a representation for the field ϕ on a fiducial Fock space and imposing on it the constraint that remains on the system as an operator condition, in order to finally obtain the Hilbert space of physical states.

However, there is an important drawback to the quantization presented in Ref. [9]. It can be proved that the quantum evolution admits no implementation as a unitary transformation. Moreover, this negative result applies to the implementation both on the kinematical Hilbert space [10] and on the physical Hilbert space of the model [11]. To make things even more subtle, it turns out that the dynamics can be approximated as close as one wants in terms of unitary transformations [12] but, nonetheless, the true evolution cannot be represented by a unitary operator. Owing to this failure of unitarity, we do not have at our disposal a

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Schrödinger picture with an evolution that conserves the standard notion of probability [11,13]. In a more pessimistic note, one might argue that such nonunitarity poses serious problems for a proper physical description of the system [10]. A careful analysis of this issue would require further discussion about the existence of the Heisenberg picture when the Schrödinger one is not available [14,15], clarifying its physical validity and elucidating whether one should or not abandon the concept of unitary evolution. We shall not pursue this avenue here but, considering the Gowdy cosmology as a particular arena in which one is addressing the issue of unitarity in cosmology, we will rather show that the problems with the quantum evolution can be solved, at least in this case, by adopting a different quantization for the model.

In the study of quantum cosmological models, a fundamental issue is the so-called problem of time. In general relativity there are no preferred foliations in spacetime and the dynamical evolution should consider all possible spacelike foliations. This is one of the main features of diffeomorphism invariance. Furthermore, in cases with compact Cauchy surfaces, dynamical evolution is pure gauge since there is no true Hamiltonian. Then, any interpretation of time evolution is normally obtained via a deparametrization which, in Hamiltonian language, is achieved by fixing the time gauge. Thus, the dynamics to be considered in these quantum cosmological models concerns the evolution of quantum states between Cauchy surfaces, defined by the particular choice of time gauge adopted. Different choices of time may lead to inequivalent quantizations. In the linearly polarized Gowdy T^3 model the system is partially gauge fixed at the classical level and, in particular, a time function t is chosen and interpreted as the time that defines "evolution." The surfaces of constant t for the quantum gravity model turn out to be Cauchy surfaces of the quantum scalar field in a fiducial background equipped with a foliation of preferred surfaces. Furthermore, even if there is no preferred time in the fundamental description of quantum gravity, as well as in the cosmological Gowdy models, and one would only expect a genuine notion of time to arise in a certain semiclassical regime, the introduction of a deparametrization allows us to introduce a family of true observables—the so-called *evolving constants of motion* [16] that can be associated with quantities "living at time t." A clean construction and interpretation of these observables turn out to be possible in our case, since both a Schrödinger and a Heisenberg picture will be shown to exist.

This work has several aims. First, as we have just commented, we will prove that it *is* indeed possible to achieve a unitary quantum dynamics in the linearly polarized Gowdy cosmology. Therefore, no fundamental obstruction exists to the standard probabilistic interpretation of quantum physics in this inhomogeneous cosmological framework. An outline of this result was presented in Ref. [17]. The second aim of the present paper is to systematically explore the canonical structure of the Gowdy model. As we will show through a detailed analysis of the implementation and consequences of a canonical transformation on phase space, one can arrive at a suitable field parametrization of the spacetime metric of the model (i.e., to adopt an adequate choice of basic field) which allows a fully consistent quantization. In addition, we also want to discuss some relevant physical phenomena that occur in the model, e.g. the production of particles by the vacuum of the cosmological system and the recovery of a time-translation invariance in the asymptotic region of infinite large times.

The rest of the paper is organized as follows: Section II reviews the standard formulation of the Gowdy T^3 model along the lines put forward in Refs. [9–12] (including the realization of its nonunitary character). In Sec. III we perform a time-dependent canonical transformation and find the corresponding new Hamiltonian for the model. The system is then recast in Sec. IV as a scalar field in a static background with a potential. In that section we also find the general classical solution for this scalar field and the finite symplectic transformation associated with the time evolution. This provides the starting point for Sec. V, where the quantum representation is defined and the Fock space is constructed. The issue of the unitarity of the evolution is analyzed in Sec. VI. It is shown that the symplectic transformations that define finite time evolution can indeed be implemented in a unitary way. Section VII studies the properties of the quantum theory, in particular, its relation with previous work and the issue of particle production. Finally, we present our conclusions and some further discussion in Sec. VIII. An appendix is added, where we show that the description of the model adopted in the main text corresponds to the same gauge fixing that had been imposed in previous treatments of the Gowdy T^3 cosmologies [9,11].

II. THE POLARIZED GOWDY T^3 MODEL

Let us briefly review the symmetry reduced model employed in Ref. [9] to introduce a quantum theory for the linearly polarized Gowdy T^3 cosmologies. These cosmologies are described by vacuum spacetimes with the spatial topology of a three-torus that possess two commuting, axial, and hypersurface orthogonal Killing vector fields [3]. They provide the simplest of all possible cosmological models with compact spatial sections and local degrees of freedom. After a gauge fixing procedure, in which all the gauge degrees of freedom are removed except for a homogenous one [12,18], the metric of the symmetry reduced model can be expressed in the form

$$ds^2 = e^{\gamma} e^{-\phi/\sqrt{p}}(-dt^2 + d\theta^2) + e^{-\phi/\sqrt{p}} t^2 p^2 d\sigma^2 + e^{\phi/\sqrt{p}} d\delta^2, \tag{2.1}$$

$$\gamma = -\frac{\bar{Q}}{2\pi p} - \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{2\pi n p} \oint d\bar{\theta} e^{in(\theta-\bar{\theta})} P_\phi \phi' + \frac{1}{4\pi p} \oint d\bar{\theta} [P_\phi \phi + P_\phi^2 + t^2(\phi')^2]. \quad (2.2)$$

Here, $t > 0$ is a positive time coordinate and the angular coordinates θ, σ, and δ belong to S^1. The two Killing vector fields are given by ∂_σ and ∂_δ, so that the metric functions are independent of these angles. The function ϕ and its canonical momentum P_ϕ may depend on both t and θ. Therefore, they describe a fieldlike degree of freedom of the metric. All cyclic integrals in Eq. (2.2) are performed on the corresponding angular dependence and the prime stands for the derivative with respect to θ. On the other hand, $\bar{Q}$ and p are homogenous variables [12]. Moreover, p is a constant of motion that we impose to be strictly positive. Actually, spacetimes with $p < 0$ can be related with those with $p > 0$ by means of a time reversal, whereas spacetimes with vanishing p can be consistently removed (since the considered sector of phase space is dynamically invariant). To avoid dealing with the positivity restriction, we introduce the definition

$$\bar{P} := \ln p. \quad (2.3)$$

The real constant of motion $\bar{P}$ provides the momentum canonically conjugate to $\bar{Q}$. The pair $(\bar{Q}, \bar{P})$ describes what we will call the point particle degree of freedom.

To be completely rigorous, the metric (2.1) should include a nonvanishing θ component of the shift vector equal to an arbitrary function of time [12]. Nonetheless, since this kind of shift can always be absorbed by means of a redefinition of the angular coordinate θ, we have obviated it. This freedom in the choice of shift appears because the gauge has not been totally fixed. There is still a global constraint remaining on the system, coming from the homogeneous part of the θ-momentum constraint:

$$C_0 := \frac{1}{\sqrt{2\pi}} \oint d\theta P_\phi \phi' = 0. \quad (2.4)$$

Starting with the Einstein-Hilbert action of general relativity with $4G/\pi = c = 1$ (G and c being Newton's constant and the speed of light, respectively) and after the above reduction process, one arrives at the following action for the model (modulo the constraint $C_0 = 0$ and spurious surface terms):

$$S_r = \int_{t_i}^{t_f} dt \Big(\bar{P} \dot{\bar{Q}} + \oint d\theta [P_\phi \dot{\phi} - \mathcal{H}_r] \Big), \qquad \mathcal{H}_r = \frac{1}{2t}[P_\phi^2 + t^2(\phi')^2], \quad (2.5)$$

where the dot denotes the derivative with respect to t. The reduced Hamiltonian is

$$H_r = \oint d\theta \mathcal{H}_r. \quad (2.6)$$

This Hamiltonian does not depend on the point particle degrees of freedom. Hence, $\bar{Q}$ and $\bar{P}$ are constants of motion and a nontrivial evolution occurs only in the field sector of the system. In addition note that, were not for the explicit time dependence of $\mathcal{H}_r$, the reduced Hamiltonian would be that of a massless scalar field with axial symmetry in a static background.

The (reduced) phase space of the system Γ_r can be decomposed as a direct sum, $\Gamma_r = \Gamma_0 \oplus \tilde{\Gamma}$, where Γ_0 and $\tilde{\Gamma}$ contain the point particle and the fieldlike degrees of freedom, respectively. They admit as coordinates the canonical pairs $(\bar{Q}, \bar{P})$ and (ϕ, P_ϕ). Owing to the presence of the global constraint $C_0 = 0$, the space of physical states does not really correspond to Γ_r, but rather to a submanifold of it. However, since this submanifold is nonlinear, the reduction by the constraint is postponed to the quantum theory, where it is imposed as an operator condition on the (kinematical) quantum states.

The Hamiltonian equations derived from H_r require that the field ϕ and its momentum satisfy

$$\dot{\phi} = \frac{P_\phi}{t}, \qquad \dot{P}_\phi = t\phi''. \quad (2.7)$$

Combining them, one concludes that ϕ is subject to a wave equation:

$$\ddot{\phi} + \frac{\dot{\phi}}{t} - \phi'' = 0. \quad (2.8)$$

We will call φ any smooth solution to this equation. Let us define $f_n(t, \theta) := \bar{f}_n(t) \exp[in\theta]$ for all $n \in \mathbb{Z}$ with

$$\bar{f}_0(t) := \frac{1 - i \ln t}{\sqrt{4\pi}}, \qquad \bar{f}_n(t) := \frac{H_0(|n|t)}{\sqrt{8}} \quad \text{if } n \neq 0. \quad (2.9)$$

Here, H_0 is the zeroth-order Hankel function of the second kind [19]. We can then express all solutions φ in the generic form

$$\varphi(t, \theta) = \sum_{n=-\infty}^{\infty} [A_n f_n(t, \theta) + A_n^* f_n^*(t, \theta)]. \quad (2.10)$$

The symbol $*$ represents complex conjugation, and the A_n's are complex constant coefficients. These coefficients must decrease faster than the inverse of any polynomial in n as $|n| \to \infty$ in order to guarantee the pointwise convergence of the series (2.10).

From action (2.5), one obtains the following symplectic structure on the space $\{\varphi\}$ of smooth solutions to Eq. (2.8):

$$\tilde{\Omega}(\varphi_1, \varphi_2) = \oint d\theta [\varphi_2 t \partial_t \varphi_1 - \varphi_1 t \partial_t \varphi_2], \quad (2.11)$$

where the integral is taken over any $t =$ constant slice.

The set of mode solutions $\{f_n(t,\theta), f_n^*(t,\theta)\}$ (with $n \in \mathbb{Z}$) is complete and "orthonormal" in the product $(f_l, f_n)_\varphi = -i\tilde{\Omega}(f_l^*, f_n)$, in the sense that $(f_l, f_n)_\varphi = \delta_{ln}$, $(f_l^*, f_n^*)_\varphi = -\delta_{ln}$, and $(f_l, f_n^*)_\varphi = 0$. As a consequence, it is not difficult to check that the complex conjugate constants $\{A_n, A_n^*\}$ behave as pairs of annihilationlike and creationlike variables under this symplectic structure.

Returning to Eq. (2.8), it is worth pointing out that it is formally identical to the Klein-Gordon equation of a free massless scalar field propagating in a fictitious three-dimensional background ($\mathcal{M} \simeq \mathbb{R}^+ \times T^2, g^{(B)}$), where the background metric is $g_{ab}^{(B)} = -dt_a dt_b + d\theta_a d\theta_b + t^2 d\sigma_a d\sigma_b$, again with $t \in \mathbb{R}^+$ and θ, $\sigma \in S^1$. We can then identify $\tilde{\Gamma}$ (the field part of phase space) with the canonical phase space of the scalar field in that background, while the space of smooth solutions can be considered as the covariant phase space of such a Klein-Gordon field. That is, ϕ and P_ϕ can be viewed as the configuration and momentum on the constant-time section $\Sigma_t \simeq T^2$ of the scalar field φ propagating in ($\mathcal{M}, g^{(B)}$).

A consequence of this equivalence between the gauge fixed Gowdy model and a Klein-Gordon field is that, in this description, the problem of quantization of the local degrees of freedom reduces to the construction of a quantum theory for the axially symmetric massless scalar field φ in the background ($\mathcal{M}, g^{(B)}$), which is an expanding torus. Employing this fact, the quantum Gowdy T^3 model was defined in Ref. [9] by introducing a Fock representation for φ and, on the fiducial Fock space obtained in this way, imposing the global constraint (2.4) as an operator condition in order to get the physical Hilbert space. More precisely, taking into account the field decomposition (2.10) [and remembering definition (2.9)], the symplectic vector space $\tilde{S} := (\tilde{\Omega}, \{\varphi\})$ can be endowed with the $\tilde{\Omega}$-compatible complex structure $\tilde{J}: \tilde{S} \to \tilde{S}$:

$$\tilde{J}[\bar{f}_n(t)] = i\bar{f}_n(t), \qquad \tilde{J}[\bar{f}_n^*(t)] = -i\bar{f}_n^*(t). \tag{2.12}$$

Using this complex structure, it is straightforward to construct from $\tilde{S}$ the "one-particle" Hilbert space of the theory, $\tilde{\mathcal{H}}$. This space allows to define in turn the (symmetric) Fock space $\mathcal{F}(\tilde{\mathcal{H}})$ on which one can introduce the formal field operator, expressed in terms of annihilation and creation operators that correspond to the positive and negative frequency parts determined by the complex structure $\tilde{J}$. Finally, one specifies the explicit operator that represents the constraint C_0 quantum mechanically on $\mathcal{F}(\tilde{\mathcal{H}})$. The physical Hilbert space $\tilde{\mathcal{F}}_{\text{phys}}$ is supplied by the kernel of this operator.

In the quantization sketched above, however, it is known that the dynamics dictated by the Hamiltonian H_r [see Eqs. (2.5) and (2.6)] cannot be implemented as a unitary transformation. This result applies not only to the kinematical Fock space $\mathcal{F}(\tilde{\mathcal{H}})$ [10], but also to the physical Hilbert space $\tilde{\mathcal{F}}_{\text{phys}}$ [11]. The antilinear part of the Bogoliubov transformation that implements the classical dynamics in the quantum theory, providing the relation between annihilation and creation operators at different times t_0 and t_f, has a single contribution for each of the field modes. The corresponding Bogoliubov coefficient can be deduced employing the "orthonormalization" of the set $\{f_n(t,\theta), f_n^*(t,\theta)\}$ (with $n \in \mathbb{Z}$) and formula (2.11). For the nonzero modes ($n \neq 0$), the coefficient is given by [10–12]:

$$D_n(t_f, t_0) = i2\pi[\bar{f}_n^*(t_0) t_f \partial_t \bar{f}_n^*(t_f) - \bar{f}_n^*(t_f) t_0 \partial_t \bar{f}_n^*(t_0)] \tag{2.13}$$

$$= \frac{i\pi|n|}{4}[H_0^*(|n|t_f) t_0 H_1^*(|n|t_0) - H_0^*(|n|t_0) t_f H_1^*(|n|t_f)], \tag{2.14}$$

where H_1 is the first-order Hankel function of the second kind [19]. Since the sequence $\{D_n(t_f, t_0)\}$ is not square summable for generic positive times $t_f \neq t_0$ [10], unitary implementability is impossible [20,21].

Actually, the failure of square summability can be derived easily by making use of Hankel's asymptotic expansions for the functions H_0 and H_1 [19]. Up to corrections of relative order $1/t$ in these expansions,

$$\begin{aligned} H_0(|nt|) &\approx \sqrt{\frac{2}{\pi|n|t}} e^{-i|n|t} e^{i\pi/4}, \\ H_1(|nt|) &\approx i\sqrt{\frac{2}{\pi|n|t}} e^{-i|n|t} e^{i\pi/4}. \end{aligned} \tag{2.15}$$

From these formulas and Eq. (2.14), one obtains at leading order

$$|D_n(t_f, t_0)|^2 \approx \frac{(t_f - t_0)^2}{4 t_f t_0}. \tag{2.16}$$

Thus, for asymptotically large values of $|n|$, $|D_n(t_f, t_0)|^2$ differs from zero if t_f and t_0 do not coincide, so that the sequence $\{D_n(t_f, t_0)\}$ is not square summable.

III. NEW FIELD PARAMETRIZATION

The above discussion shows that the failure of unitary implementability of the dynamics can be blamed on the inadequate asymptotic behavior of the Bogoliubov coefficient $D_n(t_f, t_0)$. Furthermore, a close look at Eq. (2.13) reveals that this problematic behavior can be traced back to the appearance of the factor t in the symplectic structure (2.11). The obvious way to get rid of this unwanted factor in the symplectic (two-)form is to absorb its square root, $\sqrt{t}$, in φ. Notice that such rescaled solutions are nothing but the classical smooth solutions for the rescaled field $\sqrt{t}\phi$. The proposed change results then in a rescaling of the complete set of mode solutions $\{f_n(t,\theta), f_n^*(t,\theta)\}$ ($n \in \mathbb{Z}$)

into the set $\{g_n(t,\theta), g_n^*(t,\theta)\} := \{\sqrt{t}f_n(t,\theta), \sqrt{t}f_n^*(t,\theta)\}$, which is again a complete set of solutions, but now for the classical equation of motion satisfied by the rescaled field.

Actually, the asymptotic behavior of the nonzero mode solutions $f_n(t,\theta)$ strongly suggests the proposed rescaling also from the following related perspective. Using Hankel's expansions for the nonzero modes we see that, both for asymptotically large wave numbers $|n|$ and for asymptotically large times t,

$$f_n(t,\theta) \approx \frac{e^{i\pi/4}}{\sqrt{4\pi|n|t}} e^{-i(|n|t-n\theta)}. \qquad (3.1)$$

Therefore, in these asymptotic regimes (and modulo negligible corrections compared with the unity) the functions $\sqrt{t}f_n(t,\theta)$ (with $n \neq 0$) behave like the standard nonzero mode solutions corresponding to an axially symmetric, free, and massless scalar field propagating in the static background $(\mathcal{M}, g^{(S)})$, with $\mathcal{M} \simeq \mathbb{R}^+ \times T^2$ and

$$g_{ab}^{(S)} = -dt_a dt_b + d\theta_a d\theta_b + d\sigma_a d\sigma_b. \qquad (3.2)$$

Thus, the change from φ to $\sqrt{t}\varphi$ should provide us with a description that corresponds asymptotically to a massless scalar field in Minkowski spacetime (except for the topology). We recall that time evolution between any two flat Cauchy surfaces is unitarily implementable for the latter system.

With these motivations, we will now proceed to reformulate the linearly polarized Gowdy model by considering as our new covariant phase space the set $\{\sqrt{t}\varphi\}$. We will also see that, for asymptotically large values of t, the dynamics of the (nonzero modes of the) system is indeed dictated by the Hamiltonian of a collection of harmonic oscillators with frequencies $\omega_n = |n|$.

Let us hence start by multiplying the field ϕ by the factor $\sqrt{t}$ and completing this rescaling into a time-dependent canonical transformation in the most straightforward way,[1] namely,

$$\chi := \sqrt{t}\phi, \qquad P_\chi := \frac{1}{\sqrt{t}} P_\phi. \qquad (3.3)$$

Obviously, the canonical variables $(\bar{Q}, \bar{P})$ for the point particle degrees of freedom need not be changed. The above transformation allows us to rewrite the Hamiltonian (2.6) as that of an axially symmetric free field,

$$H_r = \frac{1}{2} \oint d\theta [P_\chi^2 + (\chi')^2]. \qquad (3.4)$$

On the other hand, given the periodicity of the system in θ, we can expand χ and P_χ in Fourier series,

$$\chi = \sum_{n=-\infty}^{\infty} \chi_{(n)} \frac{e^{in\theta}}{\sqrt{2\pi}}, \qquad P_\chi = \sum_{n=-\infty}^{\infty} P_\chi^{(n)} \frac{e^{in\theta}}{\sqrt{2\pi}}. \qquad (3.5)$$

It is not difficult to check that these (implicitly time-dependent) Fourier coefficients form canonical pairs, with $P_\chi^{(-n)}$ being the momentum conjugate to $\chi_{(n)}$. In terms of them, the Hamiltonian H_r adopts the expression

$$H_r = \frac{1}{2} \sum_{n=-\infty}^{\infty} [P_\chi^{(n)} P_\chi^{(-n)} + n^2 \chi_{(n)} \chi_{(-n)}]. \qquad (3.6)$$

Then, H_r can be equivalently interpreted as describing a free particle (the zero mode $n = 0$) and a combination of harmonic oscillators with frequencies equal to $|n|$ [two for each value of $n \neq 0$, corresponding to the real and imaginary parts of $\chi_{(n)}$ and $P_\chi^{(-n)}$]. Our mode decomposition leads in this way to a natural choice of annihilationlike variables (up to trivial linear combinations),

$$a_n = \frac{|n|\chi_{(n)} + iP_\chi^{(n)}}{\sqrt{2|n|}} \qquad (3.7)$$

for all $n \neq 0$, with creationlike variables obtained by complex conjugation.

However, since the change (3.3) is a time-dependent canonical transformation, the dynamical evolution in the new field parametrization is not generated by H_r anymore. Instead, the new reduced Hamiltonian of the system is $\tilde{H}_r = H_r + \partial_t \tilde{F}$, where the partial derivative refers only to the explicit time dependence and $\tilde{F}$ is a generating functional for the canonical transformation. For instance, we can choose $\tilde{F} = -\oint P_\phi \chi/\sqrt{t}$. Then $\partial_t \tilde{F} = \oint P_\chi \chi/(2t)$ and

$$\tilde{H}_r = \frac{1}{2} \oint d\theta \left[P_\chi^2 + (\chi')^2 + \frac{P_\chi \chi}{t} \right]. \qquad (3.8)$$

One may decompose $\tilde{H}_r$ in Fourier modes in a similar way to what we did with H_r and express the result in terms of the annihilationlike and creationlike variables introduced in Eq. (3.7). Modulo a point particle, the new Hamiltonian corresponds to an infinite number of standard harmonic oscillators in the limit $t \to \infty$, as expected from Eq. (3.8).

In spite of this good feature, there is a serious problem that makes us disregard $\tilde{H}_r$ as a suitable Hamiltonian for the Gowdy model. In the Fock representation that (for the nonzero modes of the system) determines the choice (3.7) of annihilationlike and creationlike variables, one can easily check that the vacuum does not belong to the domain of the operator counterpart of $\tilde{H}_r$: the action of the operator on the vacuum has infinite norm. As a consequence, it follows that such an operator cannot be defined on the dense subspace of the (kinematical) Hilbert space formed by the states with a finite number of particles (indeed, none of these states has a normalizable image). Actually, this problem appears owing to the presence of the cross term $\oint P_\chi \chi/(2t)$ in the Hamiltonian. Nonetheless, this term is

[1]This kind of transformation was already considered in Ref. [5] although in a different and restricted context, namely, the study of the WKB regime.

negligible in the limit of infinite large times and is not required to arrive at the desired asymptotic behavior for the system.

In fact, the commented cross term can be eliminated by including a linear contribution of the field χ in its canonical momentum. As a result, the time-dependent canonical transformation (3.3) is replaced with the following one:

$$\xi := \sqrt{t}\phi, \qquad P_\xi := \frac{1}{\sqrt{t}}\left(P_\phi + \frac{\phi}{2}\right). \tag{3.9}$$

This transformation is generated by the functional

$$\bar{F} = -\oint d\theta \left(\frac{P_\phi \xi}{\sqrt{t}} + \frac{\xi^2}{4t}\right). \tag{3.10}$$

Then, the reduced action and Hamiltonian become

$$S_r = \int_{t_i}^{t_f} dt \left(\bar{P}\dot{\bar{Q}} + \oint d\theta [P_\xi \dot{\xi} - \bar{\mathcal{H}}_r]\right), \tag{3.11}$$

$$\bar{H}_r = \oint d\theta \bar{\mathcal{H}}_r = \frac{1}{2}\oint d\theta \left[P_\xi^2 + (\xi')^2 + \frac{\xi^2}{4t^2}\right]. \tag{3.12}$$

We notice that the new reduced Hamiltonian is just that of an axially symmetric Klein-Gordon field propagating in the fictitious static background ($\mathcal{M} \approx \mathbb{R}^+ \times T^2, g^{(S)}$) [see Eq. (3.2)], though now subject to a time-dependent potential that corresponds to an effective mass equal to $1/(2t)$. Note nevertheless that this potential vanishes in the limit of large times.

It is worth remarking that the time-dependent canonical transformation (3.9), introduced to recast the symmetry reduced model in terms of a new canonical set of variables (ξ, P_ξ), amounts just to a field reparametrization of the spacetime metric of the linearly polarized Gowdy model. The construction of the reduced model in terms of the canonical pair (ξ, P_ξ) is completely parallel to that explained in Ref. [12] for the variables (ϕ, P_ϕ) (which is essentially the description considered in Refs. [5,9–11]), the only difference being the distinct parametrization of the metric. In all other respects, the gauge fixing and reduction process is the same, including the conditions imposed to fix almost entirely the gauge freedom. We show this in detail in the Appendix.

IV. DYNAMICS IN THE NEW DESCRIPTION

Varying action (3.11) one recovers the result that the point particle degrees of freedom $(\bar{Q}, \bar{P})$ remain constant in the evolution, whereas the dynamics in the field sector is dictated by

$$\dot{\xi} = P_\xi, \qquad \dot{P}_\xi = \xi'' - \frac{\xi}{4t^2}. \tag{4.1}$$

The field ξ must then satisfy the second-order differential equation

$$\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0. \tag{4.2}$$

From Eqs. (2.10) and (3.9) we have that all smooth solutions ζ to Eq. (4.2) have the general form

$$\zeta(t,\theta) = \sum_{n=-\infty}^{\infty} [A_n g_n(t,\theta) + A_n^* g_n^*(t,\theta)]. \tag{4.3}$$

We recall that $g_n(t,\theta) := \sqrt{t} f_n(t,\theta)$. As anticipated at the beginning of Sec. III, the set of mode solutions $\{g_n(t,\theta), g_n^*(t,\theta)\}$ is complete.

On the other hand, the symplectic structure on the field sector of the canonical phase space is

$$\Lambda([\xi_1, P_{\xi_1}], [\xi_2, P_{\xi_2}]) = \oint d\theta(\xi_2 P_{\xi_1} - \xi_1 P_{\xi_2}). \tag{4.4}$$

Taking into account the first of Eqs. (4.1), this leads to the following symplectic structure on the space $\{\zeta\}$ of smooth solutions:

$$\Omega(\zeta_1, \zeta_2) = \oint d\theta(\zeta_2 \partial_t \zeta_1 - \zeta_1 \partial_t \zeta_2). \tag{4.5}$$

Comparing this with the symplectic structure (2.11) for the "old" field parametrization of the Gowdy model, we see that the problematic factor t has indeed disappeared. In addition, the set of mode solutions $\{g_n(t,\theta), g_n^*(t,\theta)\}$ is orthonormal in the product $(g_l, g_n)_\zeta = -i\Omega(g_l^*, g_n)$ (in the sense explained above), as required for consistency since the set $\{f_n(t,\theta), f_n^*(t,\theta)\}$ is orthonormal with respect to the corresponding inner product in $(\tilde{\Omega}, \{\varphi\})$.

Therefore, for the reformulated Gowdy model, the field sector of the covariant phase space is the symplectic vector space $S := (\Omega, \{\zeta\})$, which admits as coordinates the set $\{\zeta\}$ of smooth solutions to Eq. (4.2) or the complex constants of motion $(\mathsf{A}_0, \{\mathcal{A}_m\})$, where $\mathsf{A}_0 := (A_0, A_0^*)$ and $\mathcal{A}_m := (A_m, A_m^*, A_{-m}, A_{-m}^*)$ for all $m \in \mathbb{N} - \{0\}$. Remember that these constants are of the annihilation and creation type. Alternatively, one can consider the canonical phase space of the model, whose field sector is the symplectic vector space $\Gamma := (\Lambda, \{(\xi, P_\xi)\})$, with coordinates given by the configuration ξ and momentum P_ξ of the massive scalar field. Expanding in Fourier series our canonical variables ξ and P_ξ, as we did for χ and P_χ in Eq. (3.5), we can adopt equivalently as coordinates the set of (complex) canonical pairs $\{\xi_{(n)}, P_\xi^{(-n)}\}$, with $n \in \mathbb{Z}$. We emphasize that these Fourier coefficients depend implicitly on the time coordinate t.

Analogously to our definition in Eq. (3.7), we now introduce annihilationlike and creationlike variables for the nonzero modes $n \neq 0$,

$$b_n = \frac{|n|\xi_{(n)} + iP_\xi^{(n)}}{\sqrt{2|n|}}, \qquad b_{-n}^* = \frac{|n|\xi_{(n)} - iP_\xi^{(n)}}{\sqrt{2|n|}}. \tag{4.6}$$

For convenience, we introduce a similar change of variables for the zero mode, although the behavior of this mode corresponds to a free particle in the limit of asymptotically large times, rather than to an oscillator:

$$b_0 = \frac{\xi_{(0)} + iP_\xi^{(0)}}{\sqrt{2}}, \qquad b_0^* = \frac{\xi_{(0)} - iP_\xi^{(0)}}{\sqrt{2}}. \tag{4.7}$$

These transformations are canonical inasmuch as the pairs b_n and ib_n^* are canonically conjugate for all values of n. Hence, we have as alternative coordinates for Γ the complex variables $(\mathsf{B}_0, \{\mathcal{B}_m\})$, where $\mathsf{B}_0 := (b_0, b_0^*)$ and $\mathcal{B}_m := (b_m, b_m^*, b_{-m}, b_{-m}^*)$ for all $m \in \mathbb{N} - \{0\}$. The physical phase space consists then of those states $(\mathsf{B}_0, \{\mathcal{B}_m\})$ in Γ that satisfy the constraint C_0, which can be expressed in the form

$$C_0 = \sum_{m=1}^{\infty} m(b_m^* b_m - b_{-m}^* b_{-m}) = 0. \tag{4.8}$$

Since this constraint defines a nonlinear submanifold of Γ, we leave the corresponding reduction to the quantum theory.

The dynamics on Γ is dictated by the reduced Hamiltonian (3.12), which in terms of the new set of variables is given by

$$\bar{H}_r = \sum_{m=0}^{\infty} [\omega_{(m,t)}(b_m^* b_m + b_{-m}^* b_{-m}) + \rho_{(m,t)}(b_m^* b_{-m}^* + b_m b_{-m})], \tag{4.9}$$

where

$$\omega_{(0,t)} := \frac{1}{4} + \frac{1}{16t^2}, \qquad \rho_{(0,t)} := -\frac{1}{4} + \frac{1}{16t^2}, \tag{4.10}$$

$$\omega_{(m,t)} := m + \frac{1}{8mt^2}, \qquad \rho_{(m,t)} := \frac{1}{8mt^2}, \tag{4.11}$$

for the zero and nonzero modes, respectively. In coordinates $(\xi_{(0)}, P_\xi^{(0)})$, the zero mode part of this Hamiltonian $\bar{H}_r^{(0)}$ can be written

$$\bar{H}_r^{(0)} := \frac{(P_\xi^{(0)})^2}{2} + \frac{(\xi_{(0)})^2}{8t^2}. \tag{4.12}$$

In agreement with our above comments, this Hamiltonian describes a harmonic oscillator with unit mass and frequency $\omega = 1/(2t)$ that behaves asymptotically like a free particle. On the other hand, for the nonzero modes, we get $\lim_{t\to\infty}\omega_{(m,t)} = m$ and $\lim_{t\to\infty}\rho_{(m,t)} = 0$. Thus, asymptotically, the dynamics of the nonzero modes corresponds in fact to that of a collection of harmonic oscillators with frequency $\omega_n = |n|$. Besides, in contrast to the situation found in Sec. III with the field χ, the vacuum will be in the domain of the Hamiltonian operator since $\rho_{(m,t)}$ turns out to be square summable.

The map from the covariant phase space S to the canonical phase space Γ is given in the case of the zero mode by $\mathsf{B}_0(t) = W_0(t)\mathsf{A}_0$ (treating B_0 and A_0 as column vectors), where

$$W_0(t) = \begin{pmatrix} r_0(t) & s_0(t) \\ s_0^*(t) & r_0^*(t) \end{pmatrix} \tag{4.13}$$

with

$$r_0(t) := \sqrt{\pi} g_0(t,\theta)\left(1 + \frac{i}{2t}\right) + \frac{1}{2\sqrt{t}}, \qquad s_0(t) := \sqrt{\pi} g_0^*(t,\theta)\left(1 + \frac{i}{2t}\right) - \frac{1}{2\sqrt{t}}. \tag{4.14}$$

For the remaining modes ($m \in \mathbb{N} - \{0\}$) the map is $\mathcal{B}_m(t) = W(x_m)\mathcal{A}_m$, where we have defined $x_m := mt$,

$$W(x_m) = \begin{pmatrix} c(x_m) & 0 & 0 & d(x_m) \\ 0 & c^*(x_m) & d^*(x_m) & 0 \\ 0 & d(x_m) & c(x_m) & 0 \\ d^*(x_m) & 0 & 0 & c^*(x_m) \end{pmatrix}, \tag{4.15}$$

and

$$c(x_m) := \sqrt{\frac{\pi x_m}{8}}\left[\left(1 + \frac{i}{2x_m}\right)H_0(x_m) - iH_1(x_m)\right], \tag{4.16}$$

$$d(x_m) := \sqrt{\frac{\pi x_m}{8}}\left[\left(1 + \frac{i}{2x_m}\right)H_0^*(x_m) - iH_1^*(x_m)\right]. \tag{4.17}$$

Since

$$|r_0(t)|^2 - |s_0(t)|^2 = 1, \qquad |c(x_m)|^2 - |d(x_m)|^2 = 1, \tag{4.18}$$

for all $t > 0$ and $m \in \mathbb{N} - \{0\}$, the maps $W_0(t)$ and $W(x_m)$ are Bogoliubov transformations. Hence, the map from S to Γ is a time-dependent canonical transformation. A generating functional for this transformation (that depends on some appropriately chosen complete sets of compatible components—under Poisson brackets—both for S and Γ) is $F = \sum_{m\in\mathbb{N}} F_m$ with

$$F_0(t) = -\frac{i}{2r_0(t)}[s_0^*(t)b_0b_0 - s_0(t)A_0^*A_0^* + 2b_0A_0^*], \tag{4.19}$$

$$F_m(t) = ib_{-m}^*[c(x_m)A_{-m} + d(x_m)A_m^*] - ib_m[d^*(x_m)A_{-m} + c^*(x_m)A_m^*], \qquad m \neq 0. \tag{4.20}$$

A straightforward calculation shows that $\partial_t F = \bar{H}_r$. Since the evolution in S is frozen, an initial state $(\mathsf{B}_0(t_0), \{\mathcal{B}_m(t_0)\})$ in Γ at time t_0 will evolve to a state $(\mathsf{B}_0(t), \{\mathcal{B}_m(t)\})$ at time t according to

$$\mathsf{B}_0(t) = W_0(t)W_0(t_0)^{-1}\mathsf{B}_0(t_0), \qquad \mathcal{B}_m(t) = W(x_m)W(x_m^0)^{-1}\mathcal{B}_m(t_0), \tag{4.21}$$

where $x_m^0 := mt_0$. In other words, the transformation (4.21) is the integral curve of the Hamiltonian vector field $X^A = \Lambda^{AB}\nabla_B \bar{H}_r$ on Γ, with end points at $(\mathsf{B}_0(t_0), \{\mathcal{B}_m(t_0)\})$ and $(\mathsf{B}_0(t), \{\mathcal{B}_m(t)\})$. Alternatively, it also can be viewed as the map that relates copies of Γ at different times, e.g. $\{(\mathsf{B}_0(t_0), \{\mathcal{B}_m(t_0)\})\}$ at t_0 with $\{(\mathsf{B}_0(t), \{\mathcal{B}_m(t)\})\}$ at t.

V. QUANTUM TIME EVOLUTION

Given a Cauchy surface in $(\mathcal{M} \simeq \mathbb{R}^+ \times T^2, g^{(S)})$, for instance the surface $t = t_0$, one obtains a one-to-one correspondence between the spaces S and Γ by means of the maps $W_0(t_0)$ and $W(mt_0)$ [see Eqs. (4.13) and (4.15)]. Employing the inverse of these maps at $t = t_0$, one can then rewrite expression (4.3) in terms of a new set of orthonormal mode solutions $\{G_n(t, \theta), G_n^*(t, \theta)\}$,

$$\zeta(t, \theta) = \sum_{n=-\infty}^{\infty} [G_n(t, \theta) b_n(t_0) + G_n^*(t, \theta) b_n^*(t_0)]. \quad (5.1)$$

For the zero and nonzero ($n \neq 0$) modes, respectively, these new mode solutions are given by

$$G_0(t, \theta) = \sqrt{t}[r_0^*(t_0) f_0(t, \theta) - s_0^*(t_0) f_0^*(t, \theta)], \quad (5.2)$$

$$G_n(t, \theta) = \sqrt{\frac{t}{8}}[c^*(x_{|n|}^0) H_0(x_{|n|}) - d^*(x_{|n|}^0) H_0^*(x_{|n|})] e^{in\theta}. \quad (5.3)$$

Here, $x_{|n|} = |n|t$ and $x_{|n|}^0 = |n|t_0$.

Instead of $t = t_0$, we could have considered a different Cauchy surface $t = T$. In such a case, the field $\zeta(t, \theta)$ would have adopted an expression similar to (5.1), but now in terms of the set of coefficients $\{b_n(T), b_n^*(T)\}$ and the orthonormal mode solutions, $\{G_n^{(T)}(t, \theta), G_n^{(T)*}(t, \theta)\}$ that are obtained by replacing t_0 with T in Eqs. (5.2) and (5.3). Therefore, associated with an uniparametric family (UF) of Cauchy surfaces $T \in [t_0, t_f] \subset \mathbb{R}^+$, there exists an UF of orthonormal mode solutions $\{G_n^{(T)}(t, \theta), G_n^{(T)*}(t, \theta)\}_{T\in[t_0,t_f]}$, as well as an UF of copies of Γ, namely, $\{(\mathsf{B}_0(T), \{\mathcal{B}_m(T)\})\}_{T\in[t_0,t_f]}$, which are related via the evolution map (4.21).

Denoting $\bar{G}_n(t) := G_n(t, \theta) \exp[-in\theta]$, the explicit decomposition of the solutions in complex conjugate pairs provided by Eq. (5.1) allows one to introduce the following Ω-compatible complex structure:

$$J[\bar{G}_n(t)] = i\bar{G}_n(t), \qquad J[\bar{G}_n^*(t)] = -i\bar{G}_n^*(t). \quad (5.4)$$

Given an UF of Cauchy surfaces, we will also get an UF of Ω-compatible complex structures J_T, namely, those defined by the UF of orthonormal mode solutions $\{G_n^{(T)}(t, \theta), G_n^{(T)*}(t, \theta)\}_{T\in[t_0,t_f]}$:

$$J_T[\bar{G}_n^{(T)}(t)] = i\bar{G}_n^{(T)}(t), \qquad J_T[\bar{G}_n^{(T)*}(t)] = -i\bar{G}_n^{(T)*}(t), \quad (5.5)$$

where again $\bar{G}_n^{(T)}(t) := G_n^{(T)}(t, \theta) \exp[-in\theta]$. Thus, for each copy $\{(\mathsf{B}_0(T), \{\mathcal{B}_m(T)\})\}$ of Γ, we obtain a natural complex structure $J_T: S \to S$. Since the copies of Γ are related by the evolution map (4.21), the UF $\{J_T\}_{T\in[t_0,t_f]}$ is just the set of complex structures induced by time evolution. Obviously, $J = J_T|_{T=t_0}$ and $G_n(t, \theta) = G_n^{(T)}(t, \theta)|_{T=t_0}$.

Starting with (S, J), we can construct the one-particle Hilbert space $\mathcal{H}$. It is the (Cauchy) completion of the space of "positive frequency" solutions $S^+ := \{\zeta^+ = (\zeta - iJ\zeta)/2\}$ with respect to the norm $||\zeta^+|| = \sqrt{\langle\zeta^+, \zeta^+\rangle}$. Here, $\langle\cdot, \cdot\rangle$ denotes the Klein-Gordon inner product: $\langle\zeta^+, \mathsf{s}^+\rangle = -i\Omega(\zeta^-, \mathsf{s}^+)$ where $\zeta^- \in \bar{\mathcal{H}}$ (i.e., the complex conjugate space of $\mathcal{H}$). The (kinematical) Hilbert space of the quantum theory is the symmetric Fock space $\mathcal{F}(\mathcal{H})$ constructed from the one-particle Hilbert space. That is,

$$\mathcal{F}(\mathcal{H}) = \oplus_{k=0}^{\infty}(\otimes_{(s)}^k \mathcal{H}), \quad (5.6)$$

where $\otimes_{(s)}^k \mathcal{H}$ is the Hilbert space of all kth rank symmetric tensors over $\mathcal{H}$. Following this prescription, we can write the formal field operator $\hat{\zeta}$ in terms of annihilation and creation operators corresponding to the positive and negative frequency decomposition defined by the complex structure J:

$$\hat{\zeta}(t; \theta) = \sum_{n=-\infty}^{\infty} [G_n(t, \theta)\hat{b}_n + G_n^*(t, \theta)\hat{b}_n^\dagger]. \quad (5.7)$$

Remembering expression (5.1) for the classical solutions, we see that we might have obtained this field operator by a straightforward assignation of operators to constants of motion. This is the Schrödinger picture, where the complex constant coefficients $\{b_n(t_0), b_n^*(t_0)\}$ are promoted to annihilation and creation operators $\{\hat{b}_n(t_0) = \hat{b}_n, \hat{b}_n^\dagger(t_0) = \hat{b}_n^\dagger\}$.

We note that $\{(S, J_T)\}_{T\in\mathbb{R}^+}$ leads to the UF of Fock representations $\{(\mathcal{F}(\mathcal{H}_T), \{\hat{b}_n(T), \hat{b}_n^\dagger(T)\})\}_{T\in\mathbb{R}^+}$. Clearly, the Fock representation constructed from (S, J) belongs to this family and corresponds to $T = t_0$.

On the other hand, in the Heisenberg picture, time evolution for operators is determined by the Bogoliubov transformation (4.21). Introducing then the notation $\hat{b}_n^{(H)}(t_0) := \hat{b}_n$, one obtains the following relation between the annihilation and creation operators at t_0 and a different time t:

$$\hat{b}_n^{(H)}(t) = \alpha_n(t, t_0)\hat{b}_n^{(H)}(t_0) + \beta_n(t, t_0)\hat{b}_{-n}^{(H)\dagger}(t_0), \quad (5.8)$$

where the Bogoliubov coefficients for the zero modes are [see Eq. (4.14)]

$$\begin{aligned} \alpha_0(t, t_0) &= r_0(t) r_0^*(t_0) - s_0(t) s_0^*(t_0), \\ \beta_0(t, t_0) &= s_0(t) r_0(t_0) - r_0(t) s_0(t_0), \end{aligned} \quad (5.9)$$

while for the rest of modes with $n \in \mathbb{Z} - \{0\}$ one gets [see

Eqs. (4.16) and (4.17)]

$$\begin{aligned}\alpha_n(t,t_0) &= c(x_{|n|})c^*(x^0_{|n|}) - d(x_{|n|})d^*(x^0_{|n|}),\\ \beta_n(t,t_0) &= d(x_{|n|})c(x^0_{|n|}) - c(x_{|n|})d(x^0_{|n|}).\end{aligned} \tag{5.10}$$

Interchanging the roles of t and t_0 in Eq. (5.8), we can write the annihilation operator at time t_0 in terms of the annihilation and creation operators at time t. By substituting the result in Eq. (5.8), and using the relations $|\alpha|^2 - |\beta|^2 = 1$, $\alpha_n(t,t_0) = \alpha_n^*(t_0,t)$, and $\beta_n(t,t_0) = -\beta_n(t_0,t)$, we then get that the creation operator at time t_0 is equal to $\beta_n^*(t_0,t)\hat{b}^{(H)}_{-n}(t) + \alpha_n^*(t_0,t)\hat{b}^{(H)\dagger}_n(t)$. Hence, the field operator can be expressed fully in terms of operators in the Heisenberg picture:

$$\hat{\zeta}(t;\theta) = \frac{1}{\sqrt{4\pi}}(\hat{b}^{(H)}_0(t) + \hat{b}^{(H)\dagger}_0(t)) + \sum_{n=-\infty,n\neq 0}^{\infty} \frac{1}{\sqrt{4\pi|n|}}(e^{in\theta}\hat{b}^{(H)}_n(t) + e^{-in\theta}\hat{b}^{(H)\dagger}_n(t)). \tag{5.11}$$

VI. UNITARITY OF THE EVOLUTION

The time evolution in the Heisenberg picture described in the previous section is unitarily implementable on the (kinematical) Fock space $\mathcal{F}(\mathcal{H})$ constructed from (S, J) if and only if the sequence $\{\beta_n(t,t_0)\}$ that appears in relation (5.8) is square summable [20] (see also Ref. [21]). Let us remark that unitary implementability amounts to unitary equivalence between all the Fock representations in the UF under consideration. Since two complex structures J_{T_1} and J_{T_2} lead to unitary equivalent representations of the canonical commutation relations if and only if their difference $(J_{T_1} - J_{T_2})$ defines a Hilbert-Schmidt (HS) operator, either on $\mathcal{H}_{T_1}$ or $\mathcal{H}_{T_2}$ (see e.g. [22]), we have unitary implementability if and only if $\mathcal{J}_T := (J - J_T)$ is HS for every $T \in \mathbb{R}^+$. In fact, it is not difficult to see that $\mathcal{J}_T$ is HS if and only if the sequence $\{\beta_n(T,t_0)\}$ is square summable (as it should be, because the requirement of unitary equivalence between Fock representations is just a reformulation of the unitary implementability condition).

Let us discuss then the square summability of the sequence $\{\beta_n(t,t_0)\}$. Since $\beta_n(t,t_0) = \beta_{-n}(t,t_0)$ and, in addition, summability does not depend on the contribution of a single term (e.g. $n = 0$), it suffices to analyze the sequence $\{\beta_m(t,t_0)\}$ with $m \in \mathbb{N} - \{0\}$. We start by showing that the sequence $\{d(mt)\}$ $(m \in \mathbb{N} - \{0\})$ is square summable for all $t > 0$. From the asymptotic expansions of the Hankel functions for large positive arguments, we know that [19]

$$H_0(x) = \sqrt{\frac{2}{\pi x}}[P(0,x) - iQ(0,x)]e^{-ix}e^{i\pi/4}, \tag{6.1}$$

$$H_1(x) = \sqrt{\frac{2}{\pi x}}[Q(1,x) + iP(1,x)]e^{-ix}e^{i\pi/4}, \tag{6.2}$$

where

$$P(\nu,x) := 1 + p(\nu,x) \sim 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(\nu,2k)}{(2x)^{2k}}, \tag{6.3}$$

$$Q(\nu,x) \sim \sum_{k=0}^{\infty}(-1)^k\frac{(\nu,2k+1)}{(2x)^{2k+1}}. \tag{6.4}$$

Here, $\nu = 0, 1$ and (ν, m) are Hankel's symbols, which in terms of the gamma function are

$$(\nu,k) = \frac{\Gamma(\nu + k + \frac{1}{2})}{k!\Gamma(\nu - k + \frac{1}{2})}. \tag{6.5}$$

Substituting Hankel's expansions in Eq. (4.17), the square modulus of $d(x_m)$ in the asymptotic region $x_m \gg 1$ becomes

$$|d(x_m)|^2 = \left(\frac{p(0,x_m) - p(1,x_m)}{2} - \frac{Q(0,x_m)}{4x_m}\right)^2 + \left(\frac{Q(0,x_m) - Q(1,x_m)}{2} + \frac{1 + p(0,x_m)}{4x_m}\right)^2. \tag{6.6}$$

Employing the expressions of the functions $p(\nu,x)$ and $Q(\nu,x)$, it is a simple exercise to check that the last term in round brackets is $o(1/x_m^5)$ at infinity,[2] whereas the first one presents the behavior $1/(4x_m)^4 + o(1/x_m^5)$. So, for $x_m \gg 1$ one gets

$$|d(x_m)|^2 = \frac{1}{(4x_m)^4} + o\left(\frac{1}{x_m^5}\right). \tag{6.7}$$

This implies, for instance, that one can find a positive constant C so that $|d(x_m)|^2 \leq 1/(3x_m)^4$ if $x_m > C$ [because $1/(3)^4 > 1/(4)^4$]. For every given value of $t > 0$, let us call $M_0 := \mathrm{int}(C/t) < \infty$, where $\mathrm{int}(x)$ denotes the integer part of x. We then obtain

$$\sum_{m=1}^{\infty}|d(mt)|^2 \leq \sum_{m=1}^{M_0}|d(mt)|^2 + \frac{1}{(3t)^4}\sum_{m=M_0+1}^{\infty}\frac{1}{m^4} < \infty. \tag{6.8}$$

In the last inequality we have used that the first sum involves only a finite number of (well-defined and bounded) terms and that the sequence $\{1/m^4\}$ is summable. This proves that the sequence $\{d(mt)\}$ with $m \in \mathbb{N} - \{0\}$ is square summable for all positive times $t > 0$.

Given two values of the time coordinate, e.g. $t > 0$ and $t_0 > 0$, we know from the square summability of the sequences $\{d(mt)\}$ and $\{d(mt_0)\}$ [and without appealing to the

[2] We say that a function $f(x)$ is $o(1/x^n)$ when $x \to \infty$ if the product $x^n f(x)$ tends to zero in this limit.

explicit form of $d(x)$] that there exist two integers m_0 and $\tilde{m}_0$ such that $|d(mt)| < 1$ for all $m > m_0$ and $|d(mt_0)| < 1$ for all $m > \tilde{m}_0$. Remembering that $|c|^2 = 1 + |d|^2$, one also has that $|c(mt)|$ and $|c(mt_0)|$ are smaller than $\sqrt{2}$ for all $m > M_1 := \max(m_0, \tilde{m}_0)$. In this case, we obtain from Eq. (5.10)

$$|\beta_m(t, t_0)| \leq \sqrt{2}(|d(x_m)| + |d(x_m^0)|). \qquad (6.9)$$

Using the inequality

$$(|d(x_m)| + |d(x_m^0)|)^2 \leq 2(|d(x_m)|^2 + |d(x_m^0)|^2), \qquad (6.10)$$

we then conclude that

$$\sum_{m=1}^{\infty} |\beta_m(t, t_0)|^2 \leq \sum_{m=1}^{M_1} |\beta_m(t, t_0)|^2 + 4 \sum_{m=M_1+1}^{\infty} (|d(x_m)|^2 + |d(x_m^0)|^2). \qquad (6.11)$$

Provided that $|\beta_m(t, t_0)|$ is finite for all t, $t_0 > 0$ and $m \in \mathbb{N} - \{0\}$, the first sum in the right-hand side of Eq. (6.11) is bounded since it contains only a finite number of contributions. The second sum is bounded from above as well, because both sequences $\{d(x_m)\}$ and $\{d(x_m^0)\}$ are square summable. Therefore, the sequence $\{\beta_m(t, t_0)\}$ is square summable for all t, $t_0 > 0$.

As an aside, let us comment that Hankel's expansions (6.1) and (6.2) provide the following asymptotic behavior of the Bogoliubov coefficient $\beta_m(t, t_0)$ for large m:

$$\beta_m(t, t_0) = \frac{1}{16m^2}\left[\left(\frac{1}{(t-\Delta t)^2} - \frac{1}{t^2}\right)\cos(m\Delta t) - i\left(\frac{1}{(t-\Delta t)^2} + \frac{1}{t^2}\right)\sin(m\Delta t)\right] + o\left(\frac{1}{m^2}\right), \qquad (6.12)$$

where $\Delta t := t - t_0$. Obviously, in agreement with our above comments and in contrast to the situation found in Eq. (2.16), the dominant term in the asymptotic regime is square summable. Moreover, it is not difficult to see that, when $t \gg 1$ and $\Delta t/t \ll 1$, the above expression continues to be valid for all $m > 0$ if one merely replaces $o(1/m^2)$ with $o(1/t^2)$. Approximating to the same order the rest of the terms in the expression, we arrive at a square modulus for $\beta_m(t, t_0)$ with the asymptotic form

$$|\beta_m(t, t_0)|^2 = \frac{\sin^2(m\Delta t)}{64(mt)^4} + o\left(\frac{1}{t^4}\right). \qquad (6.13)$$

For asymptotic large times we see that, if we keep Δt fixed, the square modulus of $\beta_m(t, t_0)$ decreases to zero as $1/t^4$. Since the number of "particles" produced by the vacuum in the nonzero modes is given by the sum of the sequence $\{|\beta_m(t, t_0)|^2\}$, in a fixed lapse of time Δt the particle production will be attenuated as time increases. In particular, regardless of the fixed value of Δt, one gets the bound

$$\sum_{m=1}^{\infty} |\beta_m(t, t_0)|^2 \leq \sum_{m=1}^{\infty} \frac{1}{64m^4t^4} + o\left(\frac{1}{t^4}\right) = \frac{1}{64t^4} Z(4) + o\left(\frac{1}{t^4}\right), \qquad (6.14)$$

where the Riemann function $Z(4)$ is equal to $\pi^4/90$. We will discuss the issue of particle production in more detail in the next section.

The proved square summability of the sequence $\{\beta_m(t, t_0)\}$ for all positive times t and t_0 ensures that the time evolution is unitarily implementable on the (kinematical) Fock space $\mathcal{F}(\mathcal{H})$ (and that the Fock representations of the introduced UF are all unitarily equivalent), so that probability is preserved. Moreover, one can check that the evolution (5.8) leaves invariant the constraint

$$\hat{C}_0 = \sum_{m=1}^{\infty} m(\hat{b}_m^\dagger \hat{b}_m - \hat{b}_{-m}^\dagger \hat{b}_{-m}), \qquad (6.15)$$

which imposes the condition that the total momentum of the field ξ in the θ direction be equal to zero [see Eq. (4.8)]. This invariance guarantees that the dynamics is unitarily implementable not just on $\mathcal{F}(\mathcal{H})$, but furthermore on the Hilbert space $\mathcal{F}_{\text{phys}}(\mathcal{H})$ of physical states, which are the states that belong to the kernel of the constraint $\hat{C}_0$. In conclusion, we have shown that the quantization put forward for the polarized Gowdy T^3 model is such that the physical evolution is unitary.

VII. FEATURES AND CONSEQUENCES OF THE QUANTUM EVOLUTION

In this section, we want to clarify certain mathematical aspects of the quantization and evolution proposed for the Gowdy model and discuss some of their physical consequences, including the cosmological production of particles by the vacuum of the theory. We divide this analysis in several parts.

A. The ($\mathsf{A}_0, \{\mathcal{A}_m\}$) description

Associated with the field decomposition (4.3) we have the Ω-compatible complex structure

$$J_g[\bar{g}_n(t)] = i\bar{g}_n(t), \qquad J_g[\bar{g}_n^*(t)] = -i\bar{g}_n^*(t), \qquad (7.1)$$

where $\bar{g}_n(t) := g_n(t, \theta)\exp[-in\theta]$. Starting with $(\Omega, \{\zeta\}, J_g)$ but adopting $(\mathsf{A}_0, \{\mathcal{A}_m\})$ as coordinates for $\{\zeta\}$ instead of $(\mathsf{B}_0(t_0), \{\mathcal{B}_m(t_0)\})$, one can construct the one-particle Hilbert space $\mathcal{H}_g$ as well as the corresponding symmetric Fock space $\mathcal{F}(\mathcal{H}_g)$. This Fock space would now provide the (kinematical) Hilbert space of the quantum theory. Defining $\mathcal{J} := J - J_g$, we have that

$$\frac{1}{4}\sum_{n=-\infty}^{\infty}\langle G_n, \mathcal{J}^\dagger\mathcal{J}[G_n]\rangle_{\mathcal{H}} = |s_0(t_0)|^2 + 2\sum_{m=1}^{\infty}|d(x_m^0)|^2 < \infty. \tag{7.2}$$

We can therefore assure that $(\mathcal{F}(\mathcal{H}_g), \{\hat{A}_n, \hat{A}_n^\dagger\})$ and $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n, \hat{b}_n^\dagger\})$ are unitarily equivalent Fock representations. Besides, since the sequence $\{d(x_m)\}$ is square summable for all $t > 0$, the unitary equivalence holds regardless of the value chosen for the instant t_0 in the construction of the representation $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n = \hat{b}_n(t_0), \hat{b}_n^\dagger = \hat{b}_n^\dagger(t_0)\})$. In this sense, the role of the time of reference t_0 is irrelevant.

In the considered description, on the other hand, an initial state $(\mathsf{A}_0, \{\mathcal{A}_m\})$ at time t_0 evolves to the final state $(\bar{\mathsf{A}}_0, \{\bar{\mathcal{A}}_m\})$ at time t according to

$$\bar{\mathsf{A}}_0 = W_0^{-1}(t_0)W_0(t)\mathsf{A}_0, \qquad \bar{\mathcal{A}}_m = W^{-1}(x_m^0)W(x_m)\mathcal{A}_m. \tag{7.3}$$

The antilinear part of the map (7.3) is obviously square summable and, consequently, the classical dynamics is unitarily implementable with respect to the Fock representation $(\mathcal{F}(\mathcal{H}_g), \{\hat{A}_n, \hat{A}_n^\dagger\})$, as required for consistency with the unitary equivalence between this representation and $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n, \hat{b}_n^\dagger\})$.

However, it is worth pointing out that the transformation (7.3) does not represent the total change in time, which is actually dictated by $K := \bar{H}_r - \partial_t F = 0$ [see Eqs. (4.19) and (4.20)]. It is rather the relation between constants of motion that generates the Hamiltonian (4.9), written in coordinates $(\mathsf{A}_0, \{\mathcal{A}_m\})$. This situation contrasts with that described in Eq. (4.21), where the total Hamiltonian is indeed $\bar{H}_r$. This observation is one of the main motivations for the construction of the Fock representation $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n, \hat{b}_n^\dagger\})$ that we have presented; representation where the total dynamics provided by Eq. (4.21) is implemented in a natural way.

B. Asymptotic behavior for large times

Let us analyze now the regime of times $T \in [\tilde{T}, \infty)$, with $\tilde{T}$ large enough so that $d(|n|\tilde{T})$ can be neglected with respect to the unity in an asymptotic approximation. We will then have $d(|n|T) \approx 0$ and $c(|n|T) \approx \exp[i\delta_{|n|}(T)]$ where, for each nonzero integer $|n|$, the phase $\delta_{|n|}(T)$ is some smooth real function of T. Therefore, for the nonzero modes, the solutions $G_n^{(T)}(t, \theta)$ considered in Sec. V behave at leading order as

$$G_n^{(T)}(t, \theta) \approx \sqrt{\frac{t}{8}}e^{in\theta}e^{-i\delta_{|n|}(T)}H_0(|n|t) \tag{7.4}$$

for $T \geq \tilde{T}$. From Hankel's asymptotic expansion (6.1) of H_0, we also get that, for large values of t,

$$G_n^{(T)}(t, \theta) \approx \eta_{|n|}(T)e^{-i(|n|t - n\theta)}, \tag{7.5}$$

$$\eta_{|n|}(T) := \frac{1}{2\sqrt{\pi|n|}}e^{-i\delta_{|n|}(T)}e^{i\pi/4}. \tag{7.6}$$

Hence, in the asymptotic region of large times, solutions (7.5) approach the (nonzero) orthonormal mode solutions for a free massless scalar field propagating in the static background $(\mathcal{M} \simeq \mathbb{R}^+ \times T^2, g^{(S)})$ [see Eq. (3.2)]. From Eqs. (5.5) and (7.5) it follows that, in the limit in which the system becomes massless, J_T approaches the (counterpart of the) Poincaré-invariant complex structure of Minkowski spacetime, namely, $J_M = -(-\mathcal{L}_t\mathcal{L}_t)^{-1/2}\mathcal{L}_t$, where $\mathcal{L}_t$ is the Lie derivative along $t^a := (\partial/\partial t)^a$. In fact, this is not unexpected: for asymptotically large values of t, the Hamiltonian (4.9) describes (in the sector of nonzero modes) a collection of harmonic oscillators with frequency $\omega_n = |n|$, as we have seen already.

C. Evolution in the Schrödinger picture

In Secs. V and VI, we have formulated the quantum time evolution in the Heisenberg picture. For completeness, we will now discuss the Schrödinger picture. In this picture, the evolution is attained by implementing the Bogoliubov transformation (4.21) on the one-particle Hilbert space $\mathcal{H}$ [23]. Namely, the initial state $\zeta^+(t_0) = \sum_{n\in\mathbb{Z}}G_n(t, \theta)b_n(t_0)$ at $t = t_0$ will evolve to the final state $\zeta^+(t_f) = \sum_{n\in\mathbb{Z}}G_n(t, \theta)b_n(t_f)$ at $t = t_f$, with $b_n(t_0)$ and $b_n(t_f)$ related by Eq. (4.21). This transformation defines a pair of bounded linear maps $\alpha:\mathcal{H} \to \mathcal{H}$ and $\beta:\mathcal{H} \to \bar{\mathcal{H}}$ (recall that $\bar{\mathcal{H}}$ is the complex conjugate space),

$$\alpha \cdot \zeta^+(t_0) = \sum_{n=-\infty}^{\infty}G_n(t, \theta)\alpha_n(t, t_0)b_n(t_0), \tag{7.7}$$

$$\beta \cdot \zeta^+(t_0) = \sum_{n=-\infty}^{\infty}G_{-n}^*(t, \theta)\beta_n^*(t, t_0)b_n(t_0). \tag{7.8}$$

Unitary implementability is possible if and only if the operator β is Hilbert-Schmidt, that is, if and only if

$$\begin{aligned}\mathrm{tr}\,(\beta^\dagger\beta) &= \sum_{n=-\infty}^{\infty}\langle G_n, \beta^\dagger\beta \cdot G_n\rangle_{\mathcal{H}} \\ &= \sum_{n=-\infty}^{\infty}\langle \beta \cdot G_n, \beta \cdot G_n\rangle_{\bar{\mathcal{H}}} \\ &= \sum_{n=-\infty}^{\infty}\langle \beta_n^*G_{-n}^*, \beta_n^*G_{-n}^*\rangle_{\bar{\mathcal{H}}} = \sum_{n=-\infty}^{\infty}|\beta_n|^2 < \infty.\end{aligned} \tag{7.9}$$

As we have shown, this is in fact the case.

It should be stressed that the maps α and β define the unitary map $\mathcal{U}$ that implements the dynamics on $\mathcal{F}(\mathcal{H})$. Namely, considering the standard annihilation operator associated with $\zeta^+(t_0)$ [24], i.e. the smeared annihilation operator $\hat{\mathfrak{b}}(\overline{\zeta^+(t_0)})$, we know that $\mathcal{U}$ is defined—up to a phase—by [21,24]

$$\mathcal{U}(t_f, t_0)\hat{\mathfrak{b}}(\overline{\zeta^+(t_0)})\mathcal{U}^\dagger(t_f, t_0) = \hat{\mathfrak{b}}(\overline{\alpha \cdot \zeta^+(t_0)}) - \hat{\mathfrak{b}}^\dagger(\overline{\beta \cdot \zeta^+(t_0)}). \qquad (7.10)$$

Since the antilinear part β of the Bogoliubov transformation is not null, the vacuum [the state $|0, t_0) \in \mathcal{F}(\mathcal{H})$ annihilated by $\hat{\mathfrak{b}}(\bar{\eta})$ for all $\eta \in \mathcal{H}$] does not remain invariant under the action of $\mathcal{U}$. That is, $\mathcal{U}(t_f, t_0)|0, t_0)$ will not be annihilated by $\hat{\mathfrak{b}}(\bar{\eta})$ for all $\eta \in \mathcal{H}$. Note that Eq. (7.10) is just the smeared version of Eq. (5.8). In terms of the considered UF of Fock representations, the operator (7.10) can be viewed as the annihilation operator in the t_f Fock space (that associated with the Cauchy surface t_f). Of course, the vacuum state of the t_f Fock representation will not coincide with $|0, t_0)$ but rather be given by $|0, t_f) = \mathcal{U}(t_f, t_0)|0, t_0)$.

Some additional comments may be worth making at this stage. As we have said, the state $\mathcal{U}(t_f, t_0)|0, t_0) \in \mathcal{F}(\mathcal{H})$ can be identified as the vacuum of the t_f Fock space. Thus, the evolution map $\mathcal{U}(t_f, t_0)$ in $\mathcal{F}(\mathcal{H})$ can be viewed also as the unitary map relating the t_0 and t_f Fock representations. On the other hand, in order to determine the evolution in the dense subspace of states with a finite number of particles, one only needs to know how the "n-particle" states evolve, and this in turn becomes completely fixed by specifying how the vacuum and the creation operators change in time: given a n-particle state $|n_\zeta) = \hat{\mathfrak{b}}^\dagger(\zeta_1^+)\hat{\mathfrak{b}}^\dagger(\zeta_2^+)\ldots\hat{\mathfrak{b}}^\dagger(\zeta_n^+)|0, t_0)$ [and abbreviating $\mathcal{U}(t_f, t_0)$ to $\mathcal{U}$], one has

$$\mathcal{U}|n_\zeta) = \mathcal{U}\hat{\mathfrak{b}}^\dagger(\zeta_1^+)\mathcal{U}^\dagger\mathcal{U}\hat{\mathfrak{b}}^\dagger(\zeta_2^+) \times \mathcal{U}^\dagger\ldots\mathcal{U}\hat{\mathfrak{b}}^\dagger(\zeta_n^+)\mathcal{U}^\dagger\mathcal{U}|0, t_0). \qquad (7.11)$$

Therefore, the adjoint of Eq. (7.10) and the corresponding relation between vacua that provides $\mathcal{U}(t_f, t_0)|0, t_0)$ determines indeed the evolution in $\mathcal{F}(\mathcal{H})$.

Similar comments apply to the t_0 Fock representation constructed without smearing operators, i.e., the Fock representation $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n(t_0), \hat{b}_n^\dagger(t_0)\})$. One only has to replace the smeared operators $\hat{\mathfrak{b}}$ with the $\hat{b}_n$'s and notice that the unitary operator is now defined by Eq. (5.8). Let us explain this point in more detail, for the sake of clarity. In the Fock representation $(\mathcal{F}(\mathcal{H}), \{\hat{b}_n(t_0), \hat{b}_n^\dagger(t_0)\})$, the evolution operator is defined by

$$U(t_f, t_0)\hat{b}_n(t_0)U^\dagger(t_f, t_0) = \alpha_n(t_f, t_0)\hat{b}_n(t_0) + \beta_n(t_f, t_0)\hat{b}_{-n}^\dagger(t_0). \qquad (7.12)$$

The corresponding vacuum state $|0, t_0\rangle$ (characterized by the conditions $\hat{b}_n(t_0)|0, t_0\rangle = 0$ for all n) evolves to the state $|0, t_f\rangle = U(t_f, t_0)|0, t_0\rangle$, which corresponds in turn to the vacuum of the t_f Fock representation $(\mathcal{F}(\mathcal{H}_{t_f}), \{\hat{b}_n(t_f), \hat{b}_n^\dagger(t_f)\})$, where the annihilation operator $\hat{b}_n(t_f)$ is given by Eq. (7.12). Since $\beta_n(t_f, t_0)$ does not vanish for $t_f \neq t_0$, $|0, t_0\rangle$ and $|0, t_f\rangle$ do not simply differ by a phase. The explicit relation between these vacua will be presented below.

D. Particle production

Let us analyze now the issue of particle production, focusing on the nonzero modes. Among the UF of representations $\{(\mathcal{F}(\mathcal{H}_T), \{\hat{b}_n(T), \hat{b}_n^\dagger(T)\})\}_{T\in\mathbb{R}^+}$, let us consider the $T = t_0$ and $T = t_f$ Fock representations. The expectation value of the number operator at time t_f, namely, $\hat{N}(t_f) = \sum_{n\neq 0}\hat{b}_n^\dagger(t_f)\hat{b}_n(t_f)$, in the vacuum state at time t_0, $|0, t_0\rangle$, is given by

$$\langle 0, t_0|\hat{N}(t_f)|0, t_0\rangle = \sum_{n=-\infty, n\neq 0}^{\infty} |\beta_n(t_f, t_0)|^2 = 2\sum_{m=1}^{\infty}|\beta_m(t_f, t_0)|^2. \qquad (7.13)$$

This expectation value is different from zero, but also bounded from above, because the sequence $\{\beta_m(t_f, t_0)\}$ is square summable for all times $t_0, t_f > 0$. As we have seen, for asymptotically large values of t_0 and t_f we can neglect the value of $d(|n|t_0)$ and $d(|n|t_f)$, so that $\beta_n(t_f, t_0) \approx 0$ and $\alpha_n(t_f, t_0) \approx \exp(i[\delta_{|n|}(t_f) - \delta_{|n|}(t_0)])$. Then, in the asymptotic region, $\hat{b}_n(t_f)$ and $\hat{b}_n(t_0)$ differ only by the phase $\exp(i[\delta_{|n|}(t_f) - \delta_{|n|}(t_0)])$ [i.e., $(J_{t_f} - J) \approx 0$], and $\hat{N}(t_0) \approx \hat{N}(t_f)$. That is, the particle production decreases as t_0 and t_f grow and, consequently, $|0, t_f\rangle \approx |0, t_0\rangle$.

Actually, the evolution of the vacuum can be straightforwardly calculated by remembering that the vacuum at time T is characterized (up to a phase) as the unit state annihilated by all of the operators $\hat{b}_n(T)$. From the evolution of these operators, it is then not difficult to see that the relation between the studied vacua is

$$|0, t_f\rangle = F\exp\left[-\sum_{m=1}^{\infty}\lambda_m(t_f, t_0)\hat{b}_m^\dagger(t_0)\hat{b}_{-m}^\dagger(t_0)\right]|0, t_0\rangle, \qquad (7.14)$$

where $\lambda_m(t_f, t_0) := \beta_m(t_f, t_0)/\alpha_m(t_f, t_0)$ is the ratio of Bogoliubov coefficients and F is a normalization factor. Demanding that the vacua have unit norm, we obtain

$$|F| = \prod_{m=1}^{\infty}\sqrt{1 - |\lambda_m(t_f, t_0)|^2}. \qquad (7.15)$$

To the best of our knowledge expressions (of the form of) (7.14) and (7.15) were first derived, in a cosmological context, almost 40 years ago [25]. Notice that $|F|^2 = |\langle 0, t_0|0, t_f\rangle|^2$. That is, $|F|^2$ is the probability of finding no particles [corresponding to $\hat{b}_m(t_f)$] at time t_f, provided that the state contains no particles at time t_0. Therefore, $|\lambda_m(t_f, t_0)|^2$ gives the probability of observing a nonzero number of particles in the mode m (or $-m$) at time t_f. For asymptotically large values of t_0 and t_f, the latter probability tends to zero (so that $|F|^2$ approaches the unity) and the vacua $|0, t_0\rangle$ and $|0, t_f\rangle$ become indistinguishable. Hence, the vacuum tends asymptotically to be stable.

Let us remark that, as an immediate consequence of Eq. (7.14), particles are created in pairs. Besides, it should be emphasized that particle production is due to the fact that $\hat{N}(t)$ does not commute with the reduced Hamiltonian,

$$[:\hat{\tilde{H}}_r:, \hat{N}(t)] = \sum_{m=1}^{\infty} \rho_{(m,t)}(\hat{b}_m(t)\hat{b}_{-m}(t) - \hat{b}_m^\dagger(t)\hat{b}_{-m}^\dagger(t)). \tag{7.16}$$

However, since $\rho_{(m,t)} = 1/(8mt^2)$, the commutator vanishes in the asymptotic limit of large times when the system becomes massless, i.e. when the time-dependent potential equals zero. So, the theory becomes "free" and the quantum representation approaches (the counterpart of) the Poincaré-invariant one.

VIII. CONCLUSION AND FURTHER COMMENTS

Let us summarize our results, discuss some consequences of the introduced quantization and compare it with previous ones. The first observation that is worth emphasizing is that, even if the field $\hat{\xi}(t_0;\theta)$ evolves unitarily to $\hat{\xi}(t;\theta)$, the (explicitly time-dependent) formal operator $\hat{\phi}(t;\theta) := \hat{\xi}(t;\theta)/\sqrt{t}$, which was regarded as the basic field for the quantum model in Ref. [9], does not display a unitary evolution. In other words, $\hat{\phi}(t_0;\theta)$ and $\hat{\phi}(t;\theta)$ are not related by means of a unitary operator. The choice of fundamental field plays, therefore, a decisive role in the construction of a satisfactory quantization (together with the subsequent choice of annihilation and creation operators, i.e. the complex structure J). To arrive to the new field parametrization, we have benefitted from the freedom available to introduce a time-dependent canonical transformation and redistribute in that way the time dependence in an implicit part, whose evolution is generated by the corresponding reduced Hamiltonian of the model, and an explicit part (the factor $1/\sqrt{t}$ for ϕ in our case), whose variation does not necessarily have to be described by a unitary transformation. We notice that, in systems like the symmetry reduced Gowdy model where the Hamiltonian depends explicitly on time, it is natural to take into account the possibility of performing canonical transformations that vary with time. It is also worth pointing out that this system exhibits what seems to be a general feature of quantum field systems, in the sense that generic linear canonical transformations do not become unitarily implemented in the quantum theory (see for instance [20,26]).

On the other hand, we have seen that the vacuum of the quantum theory proposed for ξ is not left invariant by the time evolution. As a consequence, there is some particle production by the vacuum, which certainly attenuates as t becomes large but never becomes strictly zero *except* in the limit in which the time-dependent potential vanishes, i.e. at infinitely large times. However, we have seen in detail that this poses no problem for unitarity and that the quantum theory is perfectly consistent, with a well-defined evolution that is compatible with the standard probabilistic interpretation of quantum mechanics.

The fact that particles can be created (in pairs) in expanding universes was realized already in the late sixties [27]. Since then, particle creation has been extensively discussed and studied in diverse contexts in QFT (see e.g. [5,25,28]), leading to remarkable results as the so-called Hawking [29] and Unruh [30] effects. In general, these results rest on the analysis of the (specific) Bogoliubov transformation that relates the canonical operators between the *in* and *out* states. In the Gowdy T^3 cosmological model, as we have seen, the positive and negative frequency parts of the basic field ξ become mixed during evolution owing to the time-dependent potential $V(\xi) = \xi^2/(4t^2)$. The canonical annihilation and creation operators associated with *out* Fock states, at time T_{out}, are linear combinations of those associated with *in* Fock states, at an earlier time T_{in}. From the unitary implementability of the dynamics, it follows that every *in* state with a finite number of particles evolves to an *out* state which also has a finite (although possibly different) number of particles.

This result applies even when T_{out} tends to infinity. In that limit, a neat particle interpretation is available for the *out* Fock states. Indeed, finiteness in the number of particles is a simple consequence of the fact that $\lim_{T_{\text{out}}\to\infty}|\beta_n(T_{\text{out}}, T_{\text{in}})| = |d_n(T_{\text{in}})|$ and that $d_n(t)$ is square summable for all $t > 0$. In addition, the normalized *out* modes $G_n^{(\infty)}(t,\theta) := \lim_{T_{\text{out}}\to\infty}\exp[i\delta_{|n|}T_{\text{out}} - i\pi/4]G_n^{(T_{\text{out}})}(t,\theta)$ (see Sec. VII) behave like the positive frequency modes of Minkowski spacetime (except for the different background topology) in the limit where the system becomes free. Therefore, states in the *out* Fock space $\mathcal{F}_{\text{out}}(\mathcal{H}_\infty)$, which is constructed from the space of solutions and the natural complex structure defined by the modes $G_n^{(\infty)}$, admit a natural particle interpretation in the asymptotic future. That is, $\mathcal{F}_{\text{out}}(\mathcal{H}_\infty)$ can be asymptotically identified with the standard Fock representation of a free massless scalar field propagating in a Minkowski spacetime, so that well-defined asymptotic notions of vacuum and particles arise. Clearly, in the limit where the system is invariant under time translations, ambiguities in the particle interpretation are avoided. But, furthermore, an approximate adiabatic notion of particles [31] can be introduced for each finite, sufficiently large value of T, because in the asymptotic regime the potential V varies then very slowly in time. Thus, for large T_{in} and T_{out}, a notion of particles with a conventional interpretation is available. One should, of course, keep in mind that the notion of particle we refer to is in the sense of QFT on curved space, where the existence of a well-defined notion of particles refers to actual particles (as registered in detectors). In the present quantum gravity system, even when its degrees of freedom are captured by the scalar field, the particles associated with this field are far from having a clear interpretation in terms of geometrical objects or "quanta."

The Gowdy model was reduced in Ref. [9] to a free massless scalar field ϕ on a flat but time-dependent background, subject to a global constraint. We know that a conventional quantization of this field does not allow one to represent the evolution by means of a unitary transformation. By a field redefinition, which involves the time parameter, we have mapped the system into a scalar field ξ satisfying a "Klein-Gordon" type equation in a background which is flat and time independent (like three-dimensional Minkowski spacetime, apart from the topology), although in presence of a time-dependent potential. The natural quantization of this new field that we have presented here provides a theory with unitary dynamics. The relation between this theory and the natural *time-translation invariant* quantization that is available in this background (namely, the analog of the Poincaré invariant quantization in the considered spacetime) becomes manifest in the asymptotic regime where the time-translation symmetry is recovered and a preferred vacuum with that symmetry can be selected. More precisely, for asymptotically large values of T, we have shown that our complex structure J_T approaches the "Poincaré-invariant" one in the limit in which the system becomes massless and hence invariant under time translations. Although this result suggests the appealing possibility that there exists a connection between unitary implementability and asymptotic symmetries, further research is needed to elucidate what kind of physical requirements lead to acceptable quantizations (an equivalent of the Hadamard condition in QFT in curved space [24]).

In order to arrive at a quantum description of the linearly polarized Gowdy model, one needs inputs at both the classical and quantum levels. Classically, there are two important inputs. One is the choice of deparametrization, namely, the choice of a fictitious time. The other is the field parametrization adopted for the metric. We have seen that, once a choice of time gauge has been selected, the freedom in the choice of basic field can be understood as that in performing time-dependent canonical transformations in the system. Besides, the order in which the two previously mentioned choices are made in our case is irrelevant, because it does not affect the final outcome, as we show in the Appendix. In the quantum part, on the other hand, the cornerstone is the choice of a complex structure (still a classical construct on phase space) that determines the vacuum and the structure of the quantum theory. In this respect, our choice of complex structure J was in some sense natural, guided both by previous proposals for the quantization [9] and by the symmetries of the system in the asymptotic region of large times. It would be very interesting to determine all other possible quantum representations of the canonical commutation relations for our specific field parametrization (and time gauge) that permit a unitary implementation of the dynamics, together with some additional physical requirements, and elucidate whether they are all unitarily equivalent. If this were so, the quantum theory presented here would be essentially unique, once the choice of internal time and fundamental field has been fixed. This issue will be the subject of a future investigation [32].

Finally, let us present some general comments on the validity of our results in the more general context of quantum gravity. Quantum gravity, both in its full glory and in reduced (midi-superspace) models, suffers from the celebrated problem of time. Roughly speaking, this means that there is no fundamental notion of time (even classically) and that this has implications for the usual probabilistic interpretation in the quantum theory. The Gowdy models are not free from such a problem. There are basically two different approaches, resulting from the two different ways of quantizing constrained systems: quantize first and then reduce (Dirac) or reduce first an then quantize (reduced phase space). The procedure that has been followed here and in Refs. [9–11] for the Gowdy model, although closer to the second option, is a mixture of both approaches: one reduces the system classically via gauge fixing and deparametrization but keeps a global constraint, which is dealt with in the quantum theory. For the problem of time, one is choosing an internal time t, via the deparametrization procedure, that takes the role of an "external parameter" in ordinary quantum theory. Of course, the parameter t is not the physical time even classically, but it provides us with the familiar framework of quantum theory to answer real physical questions.

From the viewpoint of canonical quantum theory, where the theory is defined over an abstract 3-manifold Σ, the natural picture for the quantum description of gravity is the Heisenberg picture. A quantum state $|\Psi\rangle$ of the system is defined on Σ (that should *not* be thought of as a "constant-time slice" since there is no time and no spacetime to embed it), and observables are of the Heisenberg type, namely, evolving constants of motion [10,16]. To be precise, we have for the Gowdy model a family of operators $\{\hat{O}_i(t)\}$, one observable for each value of t. One can, of course, define these operators and relate them by means of the "evolution operator" $U(t, t_0)$ which is, as we have shown, perfectly well defined. The Heisenberg picture has to be contrasted with the Schrödinger one that in full quantum gravity (in the Dirac approach) is simply not defined, since there is no notion of embedding of the hypersurface Σ on a spacetime, much less the notion of "time evolution." In our case, however, since we have defined a notion of (internal) time, we are free to try and construct both the Schrödinger and the Heisenberg pictures. As we have shown, these two pictures are well defined in our model. Finally, let us end this note by pointing out that in order to make full justice to the quantum geometry description given by our choice of quantization, one would need to analyze the behavior of (quantum) metric objects that provide a description of the quantum geometry as in Ref. [9] and the observables recently introduced in Ref. [33]. We shall leave that analysis for future research.

ACKNOWLEDGMENTS

The authors are greatly thankful to J. M. Velhinho for enlightening conversations and fruitful interchange of ideas. This work was supported by the Spanish MEC Projects No. FIS2004-01912 and No. FIS2005-05736-C03-02 and by CONACyT (México) Grant No. U47857-F. J. Cortez was funded by the Spanish MEC, No./Ref. SB2003-0168.

APPENDIX: GAUGE FIXING IN THE NEW FIELD PARAMETRIZATION

In this appendix, we explicitly show that the canonical transformation (3.9) amounts to a field reparametrization of the metric of the Gowdy model which commutes with the process of gauge fixing [12]. In particular, the gauge choice is not modified.

Let us start with the 3 + 1 decomposition of the metric for the polarized Gowdy T^3 spacetimes after fixing the gauge corresponding to diffeomorphisms in the direction of the coordinates $\sigma \in S^1$ and $\delta \in S^1$, with the two axial Killing vector fields identified with ∂_σ and ∂_δ [12]:

$$ds^2 = -N^2dt^2 + h_{\theta\theta}[d\theta + N^\theta dt]^2 + h_{\sigma\sigma}d\sigma^2 + h_{\delta\delta}d\delta^2. \tag{A1}$$

Here, N is the lapse function, N^θ is the θ component of the shift vector, and h_{ij} is the induced spatial metric. To arrive at this expression, we have employed the fact that the two Killing vector fields are hypersurface orthogonal, so that $h_{\sigma\delta}$ must vanish. Besides, the presence of the Killing symmetries implies that all metric functions depend only on $\theta \in S^1$ and on the time coordinate t, which we choose to be positive.

Instead of adopting the same field parametrization as in Ref. [12] for the induced metric, namely,

$$h_{\theta\theta} = e^{\gamma-\psi}, \qquad h_{\sigma\sigma} = e^{-\psi}\tau^2, \qquad h_{\delta\delta} = e^{\psi}, \tag{A2}$$

we now introduce an alternative parametrization in terms of a new set of fields $\{Q^\alpha\} := \{\xi, \bar{\gamma}, \tau\}$,

$$h_{\theta\theta} = e^{\bar{\gamma}-(\xi/\sqrt{\tau})-\xi^2/(4\tau)}, \qquad h_{\sigma\sigma} = e^{-\xi/\sqrt{\tau}}\tau^2, \qquad h_{\delta\delta} = e^{\xi/\sqrt{\tau}}. \tag{A3}$$

With this new field parametrization, we obtain

$$ds^2 = e^{\bar{\gamma}-(\xi/\sqrt{\tau})-\xi^2/(4\tau)}(-\tau^2\underset{\sim}{N}^2dt^2 + [d\theta + N^\theta dt]^2) + e^{-\xi/\sqrt{\tau}}(\tau^2d\sigma^2 + e^{2\xi/\sqrt{\tau}}d\delta^2), \tag{A4}$$

where $\underset{\sim}{N} := N/\sqrt{h}$ is the densitized lapse function and h is the determinant of the induced metric.

Regarding the change from the components of h_{ij} to the set $\{Q^\alpha\}$ as a point transformation, it is straightforward to find momenta P_α canonically conjugate to Q^α in terms of those for the induced metric [12]. In this way, one arrives at the following Einstein-Hilbert action in Hamiltonian form:

$$S = \int_{t_i}^{t_f} dt \oint d\theta[P_\tau\dot{\tau} + P_{\bar{\gamma}}\dot{\bar{\gamma}} + P_\xi\dot{\xi} - (\underset{\sim}{N}\tilde{\mathcal{C}} + N^\theta\mathcal{C}_\theta)], \tag{A5}$$

where the momentum and (densitized) Hamiltonian constraints adopt the respective expressions

$$\mathcal{C}_\theta = P_\tau\tau' + P_{\bar{\gamma}}\bar{\gamma}' + P_\xi\xi' - 2P'_{\bar{\gamma}}, \tag{A6}$$

$$\tilde{\mathcal{C}} = \frac{\tau}{2}P_\xi^2 + \frac{\xi^2}{8\tau}P_{\bar{\gamma}}^2 - \tau P_\tau P_{\bar{\gamma}} + \frac{\tau}{2}\left[4\tau'' - 2\bar{\gamma}'\tau' - \left(\frac{\xi\tau'}{2\tau}\right)^2 + (\xi')^2\right]. \tag{A7}$$

On the other hand, a comparison between the field parametrizations (A2) and (A3) shows

$$\gamma = \bar{\gamma} - \frac{\xi^2}{4\tau}, \qquad \psi = \frac{\xi}{\sqrt{\tau}}, \tag{A8}$$

with the same field τ in both cases. This point transformation leads then to the following relations between the corresponding canonical momenta:

$$P_\xi = \frac{P_\psi}{\sqrt{\tau}} - \frac{P_\gamma \psi}{2\sqrt{\tau}}, \qquad P_\tau = \tilde{P}_\tau + \frac{P_\gamma \psi^2}{4\tau} - \frac{P_\psi \psi}{2\tau}, \qquad P_{\bar{\gamma}} = P_\gamma, \tag{A9}$$

where we have called $\tilde{P}_\tau$ the momentum conjugate to τ in the old parametrization, to distinguish it from the new one, P_τ.

In order to deparametrize the model and fix (almost all of) the remaining gauge freedom, we must impose additional conditions that, together with the constraints (A6) and (A7), form a set of second class constraints allowing the reduction of the system. In Ref. [12], the conditions imposed were $g_1 := P_\gamma + p = 0$ and $g_2 := \tau - tp = 0$, where $p = -\oint d\theta P_\gamma/(2\pi)$ is a constant of motion for the model. Note that the only canonical variables that appear in these conditions are P_γ and τ. With our change of metric fields, τ is not modified and P_γ becomes $P_{\bar{\gamma}}$. Therefore, the gauge fixing selected to arrive at the usual description of the Gowdy model, and that we choose to impose also in the new field parametrization, is

$$g_1 := P_{\bar{\gamma}} + p = 0, \qquad g_2 := \tau - tp = 0. \tag{A10}$$

An analysis along the lines discussed in Ref. [12] shows that the gauge fixing is well posed provided that $p \neq 0$. As explained in the main text, we restrict all considerations to the sector of positive p. The compatibility of the gauge fixing with the dynamics sets $\underset{\sim}{N} = 1/(pt)$, whereas N^θ can be any function of t. Although the shift vector is not entirely determined, the allowed functional form is such that its contribution to the metric can be absorbed by means of a redefinition of the coordinate θ.

The momentum constraint $\mathcal{C}_\theta = 0$ together with the gauge fixing conditions imply that

$$p\bar{\gamma}' = P_\xi \xi'. \tag{A11}$$

Since $p > 0$, this relation determines the function $\bar{\gamma}$, except for its zero mode. Given the periodicity of the system in θ, Eq. (A11) also supplies the homogenous constraint that remains on the system,

$$C_0 := \frac{1}{\sqrt{2\pi}} \oint d\theta P_\xi \xi' = 0. \tag{A12}$$

As a result of the commented gauge fixing, one finally obtains a reduced system with spacetime metric

$$ds^2 = e^{\bar{\gamma} - (\xi/\sqrt{pt}) - \xi^2/(4pt)}(-dt^2 + d\theta^2) + e^{-\xi/\sqrt{pt}} t^2 p^2 d\sigma^2 + e^{\xi/\sqrt{pt}} d\delta^2, \tag{A13}$$

$$\bar{\gamma} = -\frac{\bar{Q}}{2\pi p} - i \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{2\pi n p} \oint d\bar{\theta} e^{in(\theta - \bar{\theta})} P_\xi \xi' + \frac{t}{4\pi p} \oint d\bar{\theta} \left[P_\xi^2 + (\xi')^2 + \frac{1}{4t^2}\xi^2 \right], \tag{A14}$$

where $\bar{Q}$ is the configuration variable canonically conjugate to $\bar{P} = \ln p \in \mathbb{R}$. This metric coincides in fact with that obtained from Eq. (2.1) (expressed in terms of the field ϕ) when the time-dependent canonical transformation (3.9) is applied directly in the reduced model.

In addition, the action that one obtains after the gauge fixing procedure is (modulo a spurious boundary term)

$$S_r = \int_{t_i}^{t_f} dt \left\{ \bar{P} \dot{\bar{Q}} + \oint d\theta \left[P_\xi \dot{\xi} - \frac{1}{2}\left(P_\xi^2 + (\xi')^2 + \frac{1}{4t^2}\xi^2 \right) \right] \right\}. \tag{A15}$$

So, the reduced Hamiltonian is precisely that deduced in Eq. (3.12). Therefore, as we wanted to show, the field reparametrization commutes with the gauge fixing and reduction procedure.

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