Appendix S6 for
“Stochastic Amplification of Fluctuations in Cortical Up-states”

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Power-spectrum evaluation for Model B

To firmly establish the correspondence between the phenomenology described for Model B in the main text and stochastic amplification we need to write down a set of effective Langevin equations (analogous to equation 1 in the main text) for the network averaged variables and, from it, compute power-spectra. This turns out to be a non-trivial task.

Our starting point is equation (S2-2). Multiplying it by $V$ and integrating over all possible values of the membrane potential variable

$$\dot{v}(t) = \int_{V_r}^{\theta} V \frac{\partial P(V, t)}{\partial t} dV = V D \left. \frac{\partial P(V, t)}{\partial V} \right|_{V_r}^{\theta} - D \int_{V_r}^{\theta} \frac{\partial P(V, t)}{\partial V} dV$$

$$- V \nu_d(V) P(V, t) \left|_{V_r}^{\theta} + \int_{V_r}^{\theta} \nu_d(V) P(V, t) dV \right.$$

$$= \theta D \frac{\partial P(\theta, t)}{\partial V} - V_r D \left. \frac{\partial P(V_r, t)}{\partial V} \right|_{V_r}^{\theta} + \nu_d(V_r) P(V_r, t) + \int_{V_r}^{\theta} \nu_d(V) P(V, t) dV$$

$$= -(\theta - V_r) f(t) - \frac{v - V_r}{RC} + V_e f_e + KuV_{in} f(t) + DP(V_r, t), \quad (S6-1)$$

where boundary conditions have been imposed and $\tau_{rp}$ has been, for simplicity, neglected.

The self-consistent method used for solving equation (S2-2) together with equation (S2-4) provides $v^*, u^*, f^*$ and $P(V_r)^*$, computed via the mean, the slope in $\theta$ and the value in $V_r$ of the steady state solution $P(V)$. Results are shown in Table S6. Differences with simulation results ($\langle v \rangle_{up} = -61.67 \text{ mV}, \langle u \rangle_{up} = 0.2352; \langle v \rangle_{down} = -68.3 \text{ mV}, \langle u \rangle_{down} = 0.997$) stem from the relatively small value $n_r = 6$ used in simulations; as explained above, the Fokker-Planck approach is strictly valid for $n_r \to \infty$.

In the case of finite networks, $P(V_r, t)$ and $f(t)$ become fluctuating time-dependent variables. Fig. S6-1 illustrates the result of numerical simulations for network of various sizes for the firing rate: $f(t)$ is observed to be strongly correlated with $v(t)$; the larger the average potential the larger the fraction of neurons firing per unit time. The inset of Fig. S6-1, where $f$ is plotted as a function of $v$ and $u$, illustrates the existence of two well-defined branches, one for up-to-down transitions and another for down-to-up, when $f$ is considered a function of $v$. 

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Table S6. Results obtained from the Up and Down steady state distributions shown in Fig. S2.

For the forthcoming analytical calculations, $f$ can be well approximated by a threshold-linear (or “split”) function of $v$ plus a noise, and with this we describe its shape in both Up and Down states. The noise amplitude, as shown in Fig. S6-2 decreases with network-size as expected on the central limit theorem basis.

As for the probability density of neurons at the resting state, $P(V_{r},t)$, it is consider as a constant of value $P(V_{r})^{*}$ for simplicity.

![Figure S6-1. Main: Firing rate $f$ as a function of the mean membrane potential $v$ for various finite-size networks in the model of Millman et al. for (red) the Up-and-Down state $p_{r} = 0.3$ and $N = 10^{3}$, (blue) Down-state $p_{r} = 0.2$ and $N = 10^{3}$, and (green $N = 10^{3}$, magenta $10^{4}$, and yellow $N = 10^{5}$) Up-state $p_{r} = 0.5$. By increasing the system size the cloud of points converges to the steady state fixed point. The approximate linear fit is: $f_{up}(v) = (12.86 \pm 0.05 \text{Hz/mV})v + (850 \pm 3)\text{Hz}$ for the Up-state and $f(v) = 0$ for the Down-state. Even in the case of Up-and-Down states, $f(v)$ can be well approximated by a bi-valuated function with two branches: one for transitions Down-to-Up and other for Up-to-Down. Inset: $f$ as a function of both $v$ and $u$ illustrating the origin of the two branches above.](image-url)

Having an analytical approximation for $f(v)$ it is now possible to perform a lineal
stability analysis. Defining $x = v - v^*$ and $y = u - u^*$ as the linear deviations from the deterministic fixed points, the corresponding Jacobian matrix is specified as follows:

$$
\begin{align*}
    a_{vv} &= -\frac{1}{RC} + f' \left\{ n_s u \tilde{V}_{in} \left( 1 + \frac{1}{2} u \tilde{V}_{in} P(V_r) \right) - (\theta - V_r) \right\} \\
    &+ G \left\{ \tilde{V}_e f_e + n_s f u \tilde{V}_{in} + 2 D P(V_r) \right\} \\
    a_{vu} &= n_s \tilde{V}_{in} f \left\{ 1 + u \tilde{V}_{in} P(V_r) \right\} \\
    a_{uv} &= n_s f \left\{ 1 + u \tilde{V}_{in} P(V_r) \right\} \\
    a_{uu} &= -\frac{1}{\tau_R} - p_r f \\
\end{align*}
$$

where $G$ is the derivative of the re-scaling factor of the incoming currents (see S7) which depends on $f(v)$:

$$
G \equiv \frac{\tau_s f' \left\{ e^{-\frac{1}{\tau_s}} \left( 1 + \frac{1}{\tau_f} \right) - 1 \right\}}{1 - f \tau_s \left\{ 1 - e^{-\frac{1}{\tau_s}} \right\}}
$$

(S6-3)

giving a non-trivial correction.

At the Up-state fixed point this leads to $a_{vv} = -120.12 \text{ Hz}$, $a_{vu} = 10.4272 \text{ V-Hz}$, $a_{uv} = -1355.44 \text{ Hz/V}$, $a_{uu} = -47.4422 \text{ Hz}$ for the coefficients of the stability matrix, and hence a minimum at the denominator of $P(w)$ at $w_0 = 76.1 \text{ rad/s} \Rightarrow f_0 = 12.11 \text{ Hz}$. Instead, in the Down-state, the equation for $u$ becomes decoupled from that for $v$, resulting in the absence of a non-trivial peak in the spectrum (complex $\omega_0$), even for a small but non zero firing rate.