Cosmological Perturbations in LQC: Mukhanov-Sasaki Equations in Different Approaches

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The model can generate inflation.

The most interesting case is flat topology.

We assume compact spatial sections.

We consider **perturbed** FRW universes filled with a **massive** scalar field, in the context of LQC.
In LQC, two types of approaches have been considered:

- **Effective** equations from the closure of the constraint algebra.
- **Direct** quantization:
  - Hybrid approach.
  - *Dressed metric*, with perturbations propagating on it as test fields.
Our strategy

- **Approximations**: As few as possible. Should be derived or at least checked for consistency.

- **LQC techniques**, with a quantum metric, and exploring potentially observable consequences.

- We want to include the **quantum nature** of spacetime, rather than treating perturbations as test fields in a generalized QFT.
Approximation: Truncation at \textit{quadratic} perturbative order in the action.

- Uses the \textit{modes} of the Laplace-Beltrami operator of the FRW spatial sections.
- Perturbations have \textbf{no zero modes}.
- Corrections to the action are \textit{quadratic}.
- Not necessarily the same \textit{truncation} order in the \textit{metric}.
- The system is \textit{symplectic} and \textit{constrained}. 

Perturbations of \textbf{compact} FRW
Approximation: Effects of (loop) quantum geometry are only accounted for in the background.

- Loop quantum corrections on matter d.o.f. and perturbations are ignored.
- Successfully applied in Gowdy cosmologies.
We use these **gauge invariants** for the perturbations.

Then, the primordial power spectrum is easy to derive.

Their field equations match criteria for the choice of a unique Fock quantization.

Their use facilitates comparison with other approaches.

Although we will fix the gauge, they can be part of a canonical set which includes the perturbative constraints.
Uniqueness of the Fock description

- The ambiguity in selecting a Fock representation in QFT is removed by:
  - appealing to background symmetries.
  - demanding the UNITARITY of the quantum evolution.

- There is additional ambiguity in the separation of the background and the matter field. This introduces time-dependent canonical field transformations.

- Our proposal selects a UNIQUE canonical pair and an EQUVALENCE CLASS of invariant Fock representations for their CCR's.

Other works DO NOT incorporate the same field scaling. This may affect the quantum description, and the effective approach.
Loop Quantum FRW Cosmology

- Avoids the Big Bang.
Classical system: FRW

Geometry:

Ashtekar-Barbero variables.

\[ A^i_a = \epsilon^i_a (2\pi)^{-1} ; \quad E^a_i = p \sqrt{0} \epsilon^0_a (2\pi)^{-2}. \]

\[ \{ c, p \} = 8\pi G \gamma / 3. \quad V = [p]^{3/2}. \]

Scale factor and its momentum.

\[ a^2 = e^{2\alpha} = [p](2\pi\sigma)^{-2} ; \quad \pi_\alpha = -pc(\gamma 8\pi^3\sigma^2)^{-1}. \]

\[ \sigma^2 = G(6\pi^2)^{-1}. \]

Matter:

\[ \varphi = (2\pi)^{3/2} \sigma \phi ; \quad \pi_\varphi = (2\pi)^{-3/2} \sigma^{-1} \pi_\phi. \]
We expand inhomogeneities in a (real) Fourier basis: \( \vec{n} \in \mathbb{Z}^3, \quad n_1 \geq 0 \).

\[
Q_{\vec{n},+} = \frac{1}{2\pi^{3/2}} \cos \vec{n} \cdot \vec{\vartheta}, \quad Q_{\vec{n},-} = \frac{1}{2\pi^{3/2}} \sin \vec{n} \cdot \vec{\vartheta}.
\]

The basis is orthonormal, and we exclude the zero mode in the expansions.

These functions are eigenmodes of the Laplace-Beltrami operator of the standard flat metric on the three-torus, with eigenvalue \( -\omega_n^2 = -\vec{n} \cdot \vec{n} \).

We only consider scalar perturbations.
Mode expansion of the inhomogeneities: metric and field.

\[ h_{ij} = (\sigma e^{\alpha})^2 \left[ h_{ij} + 2\epsilon(2\pi)^{3/2} \sum \left\{ a_{\vec{n},\pm}(t) Q_{\vec{n},\pm} 0 h_{ij} + b_{\vec{n},\pm}(t) \left( \frac{3}{\omega_n^2} (Q_{\vec{n};\pm})_{ij} + Q_{\vec{n},\pm} 0 h_{ij} \right) \right\} \right], \]

\[ N = \sigma N_0(t) \left[ 1 + \epsilon(2\pi)^{3/2} \sum g_{\vec{n},\pm}(t) Q_{\vec{n},\pm} \right], \quad N_i = \epsilon(2\pi)^{3/2} \sigma^2 e^{\alpha} \sum k_{\vec{n},\pm}(t) \frac{\omega_n^2}{g_n^2} (Q_{\vec{n},\pm}), \]

\[ \Phi = \frac{1}{\sigma} \left[ \frac{\varphi(t)}{(2\pi)^{3/2}} + \epsilon \sum f_{\vec{n},\pm}(t) Q_{\vec{n},\pm} \right]. \]

Truncating at quadratic order in perturbations:

\[ H = \frac{N_0 \sigma}{2} C_0 + \epsilon^2 \sum \left( N_0 H_{\vec{n},\pm}^2 + N_0 g_{\vec{n},\pm} H_{\vec{n},\pm}^1 + k_{\vec{n},\pm} \overline{H}_{\vec{n},\pm}^1 \right). \]
Longitudinal gauge
We adopt a longitudinal gauge.

After **REDUCTION**, the background variables are corrected with **quadratic perturbations** to form a **CANONICAL SET**.

The remaining **Hamiltonian constraint** reads:

\[
H = \frac{N_0 \sigma}{2} C_0 + \epsilon^2 N_0 \sum H^{\vec{n}, \pm}_2,
\]

\[
H^{\vec{n}, \pm}_2 = \bar{E}_{\vec{f}_f} \bar{f}_{\vec{n}, \pm}^2 + \bar{E}_{\vec{f}_\pi} \bar{f}_{\vec{n}, \pm} \pi_{\vec{n}, \pm} + \bar{E}_{\pi \pi} \pi_{\vec{n}, \pm}^2,
\]

that is, quadratic in the **RESCALED** perturbative field variables.
The Mukhanov-Sasaki gauge invariants are related to the perturbative variables by a linear transformation in the gauge-fixed system:

\[
\begin{align*}
\nu_{\vec{n}, \pm} &= A_n(\bar{q}_A, \pi_{\bar{q}_A}) \bar{f}_{\vec{n}, \pm} + B_n(\bar{q}_A, \pi_{\bar{q}_A}) \pi_{\bar{f}_{\vec{n}, \pm}}, \\
\pi_{\nu_{\vec{n}, \pm}} &= C_n(\bar{q}_A, \pi_{\bar{q}_A}) \bar{f}_{\vec{n}, \pm} + D_n(\bar{q}_A, \pi_{\bar{q}_A}) \pi_{\bar{f}_{\vec{n}, \pm}},
\end{align*}
\]

\(\bar{q}_A = \{\bar{\alpha}, \bar{\varphi}\}\) are the canonical homogeneous variables after correcting them with quadratic perturbations, and \(\pi_{\bar{q}_A}\) their momenta.

This change can be completed at our truncation order into a canonical transformation on the whole of the phase space:

\[
\begin{align*}
\tilde{q}_A &= \bar{q}_A + \frac{\epsilon}{2} \sum \left( \partial_{\pi_{\bar{q}_A}} \pi_{\bar{f}_{\vec{n}, \pm}} \right) \bar{f}_{\vec{n}, \pm} - \frac{\epsilon}{2} \sum \left( \partial_{\pi_{\bar{q}_A}} \bar{f}_{\vec{n}, \pm} \right) \pi_{\bar{f}_{\vec{n}, \pm}}, \\
\pi_{\tilde{q}_A} &= \pi_{\bar{q}_A} - \frac{\epsilon}{2} \sum \left( \partial_{\bar{q}_A} \pi_{\bar{f}_{\vec{n}, \pm}} \right) \bar{f}_{\vec{n}, \pm} + \frac{\epsilon}{2} \sum \left( \partial_{\bar{q}_A} \bar{f}_{\vec{n}, \pm} \right) \pi_{\bar{f}_{\vec{n}, \pm}}.
\end{align*}
\]
After this canonical transformation, the new Hamiltonian constraint (at our perturbative order) is:

$$
H = \frac{N_0 \sigma}{2} C_0 (\tilde{q}_A, \pi_{\tilde{q}_A}) + \epsilon^2 N_0 \sum \tilde{H}_2^{\tilde{n}, \pm} (\tilde{q}_A, \pi_{\tilde{q}_A}, \nu_{\tilde{n}, \pm}, \pi_{\nu_{\tilde{n}, \pm}}),
$$

$$
\epsilon^2 \sum_{\tilde{n}, \pm} \tilde{H}_2^{\tilde{n}, \pm} = \frac{\sigma}{2} \sum_A \left( [\bar{q}_A - \tilde{q}_A] \partial_{\bar{q}_A} C_0 + [\pi_{\bar{q}_A} - \pi_{\tilde{q}_A}] \partial_{\pi_{\bar{q}_A}} C_0 \right) + \epsilon^2 \sum_{\tilde{n}, \pm} H_2^{\tilde{n}, \pm}.
$$

The quadratic perturbative Hamiltonian is just the Mukhanov-Sasaki Hamiltonian in the rescaled variables.

$$
4 \pi e^{\tilde{\alpha}} \sum \tilde{H}_2^{\tilde{n}, \pm} = \pi_{\nu_{\tilde{n}, \pm}}^2 \left[ 4 \pi^2 \omega_n^2 + e^{-4\tilde{\alpha}} \left( 19 \pi_{\tilde{\phi}}^2 - 18 \frac{\pi_{\tilde{\phi}}^4}{\pi_{\tilde{\alpha}}^2} \right) + \sigma^2 m^2 e^{2\tilde{\alpha}} \left( 1 - 2 \tilde{\phi}^2 - 12 \tilde{\phi} \pi_{\tilde{\phi}} \pi_{\tilde{\alpha}} \right) \right] \nu_{\tilde{n}, \pm}^2.
$$

It has no crossed configuration-momentum term.
After this canonical transformation, the **new Hamiltonian constraint** (at our perturbative order) is:

\[
H = \frac{N_0 \sigma}{2} C_0(\tilde{q}_A, \pi_{\tilde{q}_A}) + \epsilon^2 N_0 \sum \tilde{H}^{\tilde{n}, \pm}_2(\tilde{q}_A, \pi_{\tilde{q}_A}, v_{\tilde{n}, \pm}, \pi_{v_{\tilde{n}, \pm}}),
\]

\[
\epsilon^2 \sum \tilde{n}, \pm \tilde{H}^{\tilde{n}, \pm}_2 = \frac{\sigma}{2} \sum_A \left( [\bar{q}_A - \tilde{q}_A] \partial_{\bar{q}_A} C_0 + [\pi_{\bar{q}_A} - \pi_{\tilde{q}_A}] \partial_{\pi_{\bar{q}_A}} C_0 \right) + \epsilon^2 \sum \tilde{n}, \pm \tilde{H}^{\tilde{n}, \pm}_2.
\]

The quadratic perturbative Hamiltonian is just the **Mukhanov-Sasaki Hamiltonian** in the rescaled variables.

\[
4 \pi e^{\tilde{\alpha}} \sum \tilde{H}^{\tilde{n}, \pm}_2 = \pi_{v_{\tilde{n}, \pm}}^2 + \left[ 4 \pi^2 \omega_n^2 + e^{-4 \tilde{\alpha}} \left( 19 \pi_\phi^2 - 18 \frac{\pi_\phi^4}{\pi^{2}_{\tilde{\alpha}}} \right) + \sigma^2 m^2 e^{2 \tilde{\alpha}} \left( 1 - 2 \tilde{\phi}^2 - 12 \tilde{\phi} \frac{\pi_\phi}{\pi_{\tilde{\alpha}}} \right) \right] v_{\tilde{n}, \pm}^2.
\]

It has **no crossed** configuration-momentum term.
We quantize the homogeneous sector with standard loop techniques.

We can adopt a basis of **volume** eigenstates \( \{ |v\rangle ; v \in \mathbb{R} \} \), with \( \hat{v} \propto |\hat{p}|^{3/2} \).

The inner product is **discrete**: \( \forall \, v_1, v_2 \in \mathbb{R}, \quad \langle v_1 | v_2 \rangle = \delta_{v_1 v_2} \).

On straight edges, holonomy elements are linear in \( N_{\bar{\mu}} := e^{j \bar{\mu} c/2} \).

We use the so-called improved dynamics. Then

\[
\hat{N}_{\bar{\mu}} |v\rangle := |v+1\rangle, \quad \hat{v} |v\rangle = v |v\rangle.
\]
The inverse volume is regularized as usual in LQC.

We decouple the zero-volume state and change the constraint densitization

\[ \hat{C}_0 = \left[ \frac{1}{V} \right]^{1/2} \hat{C}_0 \left[ \frac{1}{V} \right]^{1/2}. \]

\[ \hat{C}_0 = -\frac{3}{4 \pi G \gamma^2} \hat{\Omega}_0^2 + \hat{\pi}_\phi^2 + m^2 \hat{\phi}^2 \hat{V}^2. \]

With our proposal, the gravitational part is a difference operator:

\[ \hat{\Omega}_0^2 |v\rangle = f_+ (v) |v+4\rangle + f (v) |v\rangle + f_- (v) |v-4\rangle. \]

that acts on the superselection sectors \( \mathcal{L}_{\pm \epsilon}^{(4)} := \{ \pm (\epsilon + 4n), \ n \in \mathbb{N} \}, \epsilon \in (0,4] \).
Quantization: Homogeneous sector

- $\Omega_0 = pc$ has been approximated with holonomies by $\Omega_0 \approx 2\pi G \gamma v \sin b$,
  with $\{b, v\} = 2$.

- States evolve in the scalar field with the square root of

$$\hat{H}_0^2 = \frac{3}{4\pi G \gamma^2} \hat{\Omega}_0^2 - m^2 \hat{\phi}^2 \hat{V}^2.$$

(Or any alternate Hamiltonian for positive frequencies...)
We use annihilation and creation operators for the (rescaled) Mukhanov-Sasaki variables, constructed, e.g. with no mass.

We obtain a **Fock space** $\mathcal{F}$, with basis of $n$-particle states:

$$\{ \left| N \right> = \left| N_{(1,0,0), +}, N_{(1,0,0), -}, \ldots \right> ; \quad N_{\vec{n}, \pm} \in \mathbb{N}, \quad \sum N_{\vec{n}, \pm} < \infty \}.$$  

The Hilbert space of the hybrid quantization is $H_{\text{kin}}^{\text{FRW-LQC}} \otimes H_{\text{kin}}^{\text{matt}} \otimes \mathcal{F}$.

Any translational invariant Fock representation in the same equivalence class would be acceptable. Restrictions may come from demands on non-linear operators.
We substitute $\pi_\phi^2$ by $H_0^2$ in the quadratic perturbative contribution to the Hamiltonian.

This perturbative contribution $\tilde{H}_2 = \sum \tilde{H}_{2}^{\vec{n},\pm}$ becomes linear in the homogeneous field momentum:

$$\tilde{H}_{2}^{\vec{n},\pm} \equiv \frac{\sigma}{2V} C_{2}^{\vec{n},\pm};$$

$$C_{2}^{\vec{n},\pm} = -\Theta_{e}^{\vec{n},\pm} - \Theta_{o}^{\vec{n},\pm} \pi_\phi.$$
We quantize the quadratic contribution of the perturbations adapting the **proposals of the homogeneous sector** and using a symmetric factor ordering:

- **We symmetrize** products of the type $\hat{\phi}\hat{\pi}_\phi$.
- **We take a symmetric geometric** factor ordering $V^k A \rightarrow \hat{V}^{k/2} \hat{A} \hat{V}^{k/2}$.
- **We adopt the LQC representation** $(cp)^{2m} \rightarrow [\hat{\Omega}_0^2]^m$.
- **In order to preserve the FRW superselection sectors**, we adopt the prescription $(cp)^{2m+1} \rightarrow [\hat{\Omega}_0^2]^{m/2} \hat{\Lambda}_0 [\hat{\Omega}_0^2]^{m/2}$, where $\hat{\Lambda}_0$ is defined like $\hat{\Omega}_0$ but with double steps.

- The Hamiltonian constraint reads then

$$\hat{C}_0 - \sum \hat{\Theta}_e^{\vec{n}, \pm} - \sum (\Theta_o^{\vec{n}, \pm} \hat{\pi}_\phi)_{\text{sym}} = 0.$$
Consider states whose evolution in the inhomogeneities and FRW geometry split, with positive frequency in the homogeneous sector:

\[
\Psi = \chi_0(V, \phi) \psi(N, \phi), \quad \chi_0(V, \phi) = P \left[ \exp \left( i \int_{\phi_0}^{\phi} d \phi \, \hat{H}_0(\phi) \right) \right] \chi_0(V).
\]

The FRW state is peaked (semiclassical) and evolves unitarily.

Disregard nondiagonal elements for the FRW geometry sector in the constraint and call:

\[
d_\phi \hat{O} = \partial_\phi \hat{O} - i \left[ \hat{H}_0, \hat{O} \right].
\]
The diagonal FRW-geometry part of the constraint gives:

\[-\partial^2_\phi \psi - i (2 \langle \hat{H}_0 \rangle_\chi - \langle \hat{\Theta}_o \rangle_\chi) \partial_\phi \psi = \left[ \langle \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0)_{sym} \rangle_\chi + i \langle d_\phi \hat{H}_0 - \frac{1}{2} d_\phi \hat{\Theta}_o \rangle_\chi \right] \psi.\]

The term in cyan can be ignored if \( \langle \hat{H}_0 \rangle_\chi \) is not negligible small.

Besides, if we can neglect: a) The second derivative of \( \psi \),
  b) The total \( \phi \)-derivatives.

\[-i \partial_\phi \psi = \frac{\langle \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0)_{sym} \rangle_\chi}{2 \langle \hat{H}_0 \rangle_\chi} \psi.\]

Schrödinger-like equation; similar (but not so) to the dressed metric approach.
There are restrictions on the range of validity.

The extra terms are negligible if so are the $\phi$-derivatives of

$$\langle \hat{H}_0 \rangle_\chi, \langle \hat{\Theta}_e \rangle_\chi, \langle \hat{\Theta}_o \rangle_\chi, \langle \hat{H}_0 \hat{\Theta}_o \rangle_{\text{sym}} \rangle_\chi.$$  

These derivatives contain contributions arising from $[\hat{\Omega}_0^2, \hat{V}]$.

In the effective regime, these are proportional to $\sin 2b$.

These effects are also important in the closure of the constraint algebra.
At the truncation order, the constraint \( C = \pi_\phi^2 - H_0^2 - \Theta_e - \Theta_o \pi_\phi \) can be written:

\[
C = \left[ \pi_\phi + H_0 + \frac{1}{2} (\Theta_e + \Theta_o \pi_\phi) H_0^{-1} \right] \left[ \pi_\phi - H_0 - \frac{1}{2} (\Theta_e + \Theta_o \pi_\phi) H_0^{-1} \right].
\]

Hence, for perturbed solutions of *homogeneous* positive frequency, with a **Born-Oppenheimer** ansatz and ignoring nondiagonal elements:

\[
- i \partial_\phi \psi = \frac{1}{2} \left< \hat{H}_0^{-1/2} \left( \Theta_e + (\Theta_o \hat{H}_0)_{\text{sym}} \right) \hat{H}_0^{-1/2} \right> \chi \psi \quad \text{and} \quad - i \frac{1}{2} \left< \hat{H}_0^{-1/2} d_\phi (\Theta_o \hat{H}_0^{-1}) \hat{H}_0^{1/2} \right> \chi \psi.
\]
This Schrödinger equation is like the one obtained in the dressed metric approach.

The difference between the two factor orderings is a commutator.
Starting from the Born-Oppenheimer form of the constraint and assuming a direct effective counterpart for the inhomogeneities:

\[ d_{\eta_{\chi}}^2 v_{\bar{n}, \pm} = -v_{\bar{n}, \pm} \left[ 4 \pi^2 \omega_n^2 + \langle \hat{\Theta}_{e, (v)} + \hat{\Theta}_{o, (v)} \rangle_{\chi} \right], \]

where we have defined the state-dependent conformal time

\[ d \eta_{\chi} = \langle [1/V]^{-2/3} \rangle_{\chi} (dt/V). \]

The effective equations are of harmonic oscillator type, with no dissipative term, and hyperbolic in the ultraviolet regime.
Effective Mukhanov-Sasaki equations

In the alternate factor ordering, with the same assumptions:

\[ d_{\eta_{\chi}}^2 v_{\tilde{n}, \pm} = -v_{\tilde{n}, \pm} \left[ 4 \pi^2 \omega_n^2 + \langle \hat{\Theta}_{e, (v)}^{\text{dress}} + \hat{\Theta}_{o, (v)}^{\text{dress}} \rangle \chi \right], \]

\[ \langle \hat{\Theta}_{e, (v)}^{\text{dress}} + \hat{\Theta}_{o, (v)}^{\text{dress}} \rangle \chi v_{\tilde{n}, \pm}^2 = - \frac{\langle 2 \hat{H}_0^{-1/2} [ \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0)_{\text{sym}} ] \hat{H}_0^{-1/2} - i \hat{H}_0^{-1/2} d_\phi (\hat{\Theta}_o \hat{H}_0^{-1}) \hat{H}_0^{1/2} \rangle \chi}{2 \langle \hat{H}_0^{-1/2} [1/V]^{-2/3} \hat{H}_0^{-1/2} \rangle \chi}, \]

\[ -4 \pi^2 \omega_n^2 v_{\tilde{n}, \pm}^2 - \pi v_{\tilde{n}, \pm}^2. \]

where we have defined the \textbf{state-dependent conformal time}

\[ d \eta_{\chi}^{\text{dress}} = \langle \hat{H}_0^{-1/2} [1/V]^{-2/3} \hat{H}_0^{-1/2} \rangle \chi \langle \hat{H}_0 \rangle \chi (dt/V). \]

There is a \textbf{change} in the derivative contribution to the potential (usually negligible), and the FRW state is replaced with \( \hat{H}_0^{-1/2} \chi \).
Conclusions

- We have considered the **hybrid quantization** of a FRW universe with a massive scalar field perturbed at **quadratic** order in the action.

- The system is a **constrained symplectic manifold**. **Backreaction** is included at the considered perturbative order.

- The model has been described in terms of **Mukhanov-Sasaki** variables.

- A **Born-Oppenheimer** ansatz leads to a Schrödinger equation for the inhomogeneities. We have discussed the range of validity.

- An alternate factor ordering gives similar results to the **dressed metric**.

- We have derived the effective **Mukhanov-Sasaki equations**. The ultraviolet regime is **hyperbolic**.