Modeling effective FRW cosmologies with perfect fluids from states of the hybrid quantum Gowdy model

[in collaboration with B. Elizaga Navascués (UCM) and G. A. Mena Marugán (IEM-CSIC); arXiv:1409.2927]

Mercedes Martín-Benito
Radboud Universiteit Nijmegen

2nd i-Link Workshop Macro-from-Micro: Quantum Gravity and Cosmology
Madrid, 15–18 September 2014
Introduction

- **LQC** is a quantum approach for cosmological modes inspired by LQG that provides a satisfactory quantization leading to the resolution of singularities

  Big Bang $\rightarrow$ Big Bounce

- Aim: to study the effects of LQC phenomena in **inhomogeneous cosmological models**

- Proposal: **Hybrid quantization** combining LQC quantization of the homogeneous d.o.f. with a Fock quantization for the inhomogeneities

- We need to develop approximation methods to solve the quantum dynamics (the operator representing the Hamiltonian constraint is quite involved)
The Gowdy model with 3-torus topology and linear polarization is a most suitable arena to work with:

- Classical solutions well known. The subfamily of homogeneous solutions represent Bianchi I spacetimes

  > Gravitational waves over a (flat, homogeneous, anisotropic) Bianchi I background

- A Fock quantization of the deparametrized system has been achieved and shown to be essentially unique

• Inclusion of a massless scalar field

  > The homogeneous sector contains flat FRW solutions
The Fock quantization of the deparameterized system does not solve the singularity problem. Instead of deparameterizing, we are left with a global Hamiltonian constraint.

Strategy: we split the reduced phase space into homogeneous and inhomogeneous sectors and carry out a hybrid quantization.

- **Loop quantization** for the homogeneous sector (with the hope to solve the cosmological singularity)
- **Fock quantization** for the inhomogeneous sector (to deal with the field complexity)

This approach assumes a regime where the quantum geometry effects associated to the homogeneous d.o.f (global modes) are the relevant ones.

Approximation methods to construct solutions

Solutions that effectively behave as those of FRW with a perfect fluid.
1. Classical model
2. Homogeneous sector
3. Inhomogeneous sector
4. Quantum Hamiltonian constraint
5. Approximate solutions
6. Mimicking perfect fluid behaviour
7. Conclusions
Classical Model

- Gowdy cosmologies: Globally hyperbolic spacetimes with two axial, commuting Killing vectors
  

- We consider the linearly polarized Gowdy $T^3$ model with a minimally coupled massless scalar field $\Phi$ with the same symmetries

- Coordinates adapted to the symmetries $(t, \theta, \sigma, \delta)$
  
  Killing fields $(\partial_{\sigma}, \partial_{\delta})$
  
  Metric components: functions of $(t, \theta) \rightarrow$ Fourier series

- Partial gauge fixing: all the gauge freedom fixed except for
  
  ✦ The zero-mode of the $\theta$-diffeos constraint: $C_\theta$
  
  ✦ The zero-mode of the Hamiltonian constraint: $C_G$
Reduced Phase Space

• **Homogeneous sector:**
  - Geometry: phase space of the Bianchi I model with three-torus topology
  - Matter: zero mode of the matter field, $\Phi_0 \equiv \phi$ and its momentum $P_\phi$

• **Inhomogeneous sector**
  - Geometry: Non-zero Fourier modes of a gravitational field $\xi(\theta)$ and those of its momentum
  - Matter: Non-zero Fourier modes of the matter field $\Phi(\theta)$ and those of its momentum
Homogeneous Sector

- Spatially flat anisotropic geometry:
  - three (Ashtekar-Barbero) connection coefficients $c_i$ ($i, j, k = \theta, \sigma, \delta$)
  - three densitized triad coefficients $p_i$ ($|p_i| = a_j a_k$)

- We consider local rotational symmetry (LRS): $p_\sigma = p_\delta \equiv p_\perp$

\[
\{c_\theta, p_\theta\} = 2\{c_\perp, p_\perp\} = 8\pi G \gamma
\]

- Zero-mode of the scalar field: $\{\phi, P_\phi\} = 1$

\[
C_{BI} = -\frac{1}{8\pi G \gamma^2} \left[ 2c_\theta p_\theta c_\perp p_\perp + (c_\perp p_\perp)^2 \right] + \frac{P_\phi^2}{2}
\]
Loop representation

[ A. Ashtekar, E. Wilson-Ewing, 2009, PRD 79]

• Quantum states:

\[ \lambda_i(p_i) \propto \text{sign}(p_i) \sqrt{|p_i|} \]

\[ v = 2\lambda_\theta \lambda^2_\perp \]

\[ |v, \lambda_\theta \rangle ; \quad \langle v, \lambda_\theta | \tilde{v}, \tilde{\lambda}_\theta \rangle = \delta_{v,\tilde{v}} \delta_{\lambda_\theta,\tilde{\lambda}_\theta} \]

• “Polymerization” + (suitable) symmetric factor ordering:

\[ c_\perp p_\perp \to \kappa \gamma \tilde{\Omega} \]

\[ c_\theta p_\theta \to \kappa \gamma \hat{\Theta}_\theta \]

\[ ( \kappa = \pi G \hbar ) \]

\[ \hat{\Theta} \equiv \hat{\Theta}_\theta - \hat{\Omega} \]

\[ \hat{C}_{BI} = \hat{C}_{FRW} + \hat{C}_{ani} \]

\[ \hat{C}_{FRW} = -\frac{3\kappa \hbar}{8} \hat{\Omega}^2 - \frac{\hbar^2}{2} \hat{\partial}_\phi^2 \]

\[ \hat{C}_{ani} = -\frac{\kappa \hbar}{8} \left( \hat{\Theta} \hat{\Omega} + \hat{\Omega} \hat{\Theta} \right) \]
Loop representation

\[ \hat{C}_{BI} = \hat{C}_{FRW} + \hat{C}_{ani} \; ; \; \hat{C}_{FRW} = -\frac{3\kappa\hbar}{8}\hat{\Omega}^2 - \frac{\hbar^2}{2}\partial_\phi^2 \; ; \; \hat{C}_{ani} = -\frac{\kappa\hbar}{8}(\hat{\Theta}\hat{\Omega} + \hat{\Omega}\hat{\Theta}) \]

- \(\hat{\Omega}^2\) difference operator that shifts \(v\) in 4 units
- \(\hat{C}_{ani}\) also shifts \(v\) in 4 units and produces \(v\)-dependent dilatations in \(\lambda_\theta\)
- **Defined on separable sectors:** \(|v, \lambda_\theta\rangle\) with labels
  
  \[v \in \{\varepsilon + 4n, \ n \in \mathbb{N}\}, \ \varepsilon \in (0, 4]\]
  
  \[\lambda_\theta \in \{\text{countable dense set}\} \subset \mathbb{R}^+ \quad (\Lambda = \ln \lambda_\theta)\]


Zero-volume states decoupled
Inhomogeneous Sector

- **Remark:** in the deparameterized model (only $C_\theta$ remains) there exists a privileged description

  ✦ $\xi(\theta)$ and $\varphi(\theta) \equiv \frac{\Phi(\theta)}{|p_\theta|}$ such that they verify the same e.o.m

  (scalar field with time-dependent mass in a 1+1 static space-time)

  ✦ Then one chooses as annihilation and creation variables those associated to a massless scalar field

→ This description leads to a Fock quantization with unitary dynamics and vacuum state invariant under $S_1$. It is the unique one with these properties (up to unitary equivalence)


- We choose the same variables to describe our inhomogeneous sector

\[ \{a^\alpha_m, a^{\alpha*}_{\tilde{m}}\} = -i\delta_{m\tilde{m}}, \quad m \in \mathbb{Z} - \{0\}, \quad \alpha = \xi, \varphi \]
Remaining Global Constraints

- **Generator of $S_1$ translations:**
  it only affects the inhomogeneous sector

  \[ C_\theta = \sum_\alpha \sum_{m \neq 0} m a_{m}^{\alpha} a_{m}^{\alpha*} = 0 \]

- **Hamiltonian constraint:**

  \[ C_G = C_{BI} + \frac{(c_\perp p_\perp)^2}{4\pi \gamma^2 |p_\theta|} H_{\text{int}} + 2\pi |p_\theta| H_0 \]

  - **Free field contribution** \( H_0 = \sum_\alpha \sum_{m \neq 0} |m| a_{m}^{\alpha*} a_{m}^{\alpha} \)
  
  - **Self-interaction** \( H_{\text{int}} = \sum_\alpha \sum_{m \neq 0} \frac{1}{2|m|} \left[ 2a_{m}^{\alpha*} a_{m}^{\alpha} + a_{m}^{\alpha} a_{-m}^{\alpha} + a_{m}^{\alpha*} a_{-m}^{\alpha*} \right] \)

Hamiltonian Constraint Operator

\[
\hat{\mathcal{C}}_G = \frac{-3\kappa\hbar}{8} \hat{\Omega}^2 - \frac{\hbar^2}{2} \partial^2_{\phi} - \frac{\kappa\hbar}{8} (\hat{\Theta}\hat{\Omega} + \hat{\Omega}\hat{\Theta}) + \frac{2\kappa\hbar}{\beta} e^{2\Lambda} \hat{H}_0 + \frac{\kappa\hbar\beta}{4} e^{-2\Lambda} \hat{D} \hat{\Omega}^2 \hat{D} \hat{H}_I
\]

(\beta = \text{const.})

\[\hat{\mathcal{C}}_{\text{FRW}}\] \[\hat{\mathcal{C}}_{\text{ani}}\] \[\hat{\mathcal{C}}_0\] \[\hat{\mathcal{C}}_I\]


[MM-B, D. Martín-de Blas, G.A. Mena Marugán, 2011, PRD 83]

- \(\hat{H}_I\) creates and annihilates a pair of particles in each mode
- \(\hat{D}\) is a regularization of \(\hat{v}[1/\hat{v}] \neq \hat{1}\), and does not commute with \(\hat{\Omega}^2\)
- The operator \(\hat{\Omega}\hat{\Theta} + \hat{\Theta}\hat{\Omega}\) has a pretty involved action
  - It does not commute with \(\hat{\Omega}^2\)
  - It has a complicated action upon \(\Lambda\) (shifts depend on the \(v\) label)
Constructing approximate solutions of the Gowdy model

[M. M-B, D. Martín-de Blas, G.A. Mena Marugán, 2014 Class. Quantum Grav. 31]

Approximating the Anisotropy Term

• Consider the action of \( \hat{\Omega} \hat{\Theta} + \hat{\Theta} \hat{\Omega} \) on states \( |G\rangle = \sum_v \sum_\Lambda g(v, \Lambda) |v, \Lambda\rangle \)

• Shifts in \( \Lambda \) are \( \ln \left( 1 \pm \frac{2}{v} \right) \) or \( \ln \left( 1 \pm \frac{2}{v \pm 2} \right) \)

• Let us assume states with \( g(v, \Lambda) \) highly suppressed for \( v \leq v_m \gg 10 \)

\[ \rightarrow \text{ contributing shifts are never bigger than } \quad q_\varepsilon = \ln \left( 1 + \frac{2}{v_m} \right) \]

• Smooth \( g(v, \Lambda) : g(v, \Lambda + \Lambda_0) \approx g(v, \Lambda) + \Lambda_0 \partial_\Lambda g(v, \Lambda) \) for \( \Lambda_0 \leq q_\varepsilon \)

\[ \rightarrow \quad \langle v, \Lambda | \hat{\Omega} \hat{\Theta} + \hat{\Theta} \hat{\Omega} | G \rangle \approx -\langle v, \Lambda | 2 \hat{\Omega} \hat{\Theta}' | G \rangle \]

\[ \hat{\Theta}' | \Lambda \rangle = i \frac{2}{q_\varepsilon} \left( | \Lambda + q_\varepsilon \rangle - | \Lambda - q_\varepsilon \rangle \right) \] (defined on lattices of step \( q_\varepsilon \))

\( \hat{\Omega} \) difference operator that shifts \( v \) in 4 units
Approximating the Anisotropy Term

- **Anisotropy Gaussian-like profiles** peaked at $\bar{\Lambda}(v)$

\[ g(v, \Lambda) = N(v)e^{-\frac{\sigma_s^2}{2\sigma_e^2}[\Lambda - \bar{\Lambda}(v)]^2} \]

- If we choose $\sigma_s \ll \pi/2$

\[ |\langle v, \Lambda|2\hat{\Omega}\hat{\Theta}'|\mathcal{G}\rangle| \ll |\langle v, \Lambda|\hat{\Omega}^2|\mathcal{G}\rangle| \quad \Rightarrow \quad \hat{C}_{\text{ani}} \approx 0 \]

provided that $N(v)$ highly suppressed for $v \leq v_m \gg 10$
Approximating the Interaction Term

\[ \hat{C}_I = e^{-2\Lambda} \hat{D}\hat{\Omega}^2 \hat{D} \hat{H}_I \]

\[ |G\rangle = \sum_v \sum_\Lambda g(v, \Lambda) |v, \Lambda\rangle \]

\[ g(v, \Lambda) = N(v)e^{-\frac{\sigma_s^2}{2q_e^2} [\Lambda - \bar{\Lambda}(v)]^2} \]

- \( \langle v|\hat{D}\hat{\Omega}^2 \hat{D}|G\rangle \simeq \langle v|\Omega^2|G\rangle \) for \( v \geq v_m \gg 10 \), as \( D(v) \simeq 1 \)

- If we choose \( \bar{\Lambda}(v) \gg \frac{q_e^2}{\sigma_s^2} \)

\[ |\langle v, \Lambda|e^{-2\Lambda}\hat{\Omega}^2|G\rangle| \ll |\langle v, \Lambda|\hat{\Omega}^2|G\rangle| \quad \iff \quad \hat{C}_I \simeq 0 \]

provided that \( \bar{\Lambda}(v) \simeq \bar{\Lambda}(v \pm 4) \) when \( v \geq v_m \)

\( N(v) \) highly suppressed for \( v \leq v_m \gg 10 \)

(and that the content of inhomogeneities is reasonable)
Approximate Hamiltonian Constraint

\[ \hat{C}_{\text{app}} = -\frac{3\kappa \hbar}{8} \hat{\Omega}^2 - \frac{\hbar^2}{2} \partial^2 \phi + \frac{2\kappa \hbar}{\beta} e^{2\Lambda} \hat{H}_0 \quad ; \quad \langle \Psi | \hat{C}^\dagger_{\text{app}} | \phi, v, \Lambda, n^\xi, n^\phi \rangle = 0 \]

- Solutions

\[ \langle \Psi | = \int_{-\infty}^{\infty} d\phi \sum_v \sum_{\Lambda} \sum_{n^\xi, n^\varphi} \Psi(\phi, v, \Lambda, n^\xi, n^\varphi) \langle \phi, v, \Lambda, n^\xi, n^\varphi | \]

with profiles given by

\[ \Psi(\phi, v, \Lambda, n^\xi, n^\varphi) = \int_{-\infty}^{\infty} dp_\phi \psi(p_\phi) \chi(n^\xi, n^\varphi) g(v, \Lambda) e^{p_\phi (\phi)} \]

\[ g(v, \Lambda) = N(v) e^{-\frac{\sigma^2}{2q^2} [\Lambda - \bar{\Lambda}(v)]^2} \]

\[ \hat{\Omega}^2 g(v, \Lambda) = \left[ \frac{4p_\phi^2}{3\pi G\hbar^2} + \frac{16}{3\beta} e^{2\Lambda} H_0(n^\xi, n^\varphi) \right] g(v, \Lambda) \]
Approximate Hamiltonian Constraint

\[
\hat{C}_{\text{app}} = -\frac{3\kappa\hbar}{8} \hat{\Omega}^2 - \frac{\hbar^2}{2} \partial_\phi^2 + \frac{2\kappa\hbar}{\beta} e^{2\Lambda} \hat{H}_0 ; \quad (\Psi|\hat{C}_{\text{app}}|\phi, v, \Lambda, n^\xi, n^\varphi) = 0
\]

- **Solutions**

\[
(\Psi| = \int_{-\infty}^{\infty} d\phi \sum_v \sum_\Lambda \sum_{n^\xi, n^\varphi} \Psi(\phi, v, \Lambda, n^\xi, n^\varphi) \langle \phi, v, \Lambda, n^\xi, n^\varphi |}
\]

with profiles given by

\[
\Psi(\phi, v, \Lambda, n^\xi, n^\varphi) = \int_{-\infty}^{\infty} dp_\phi \psi(p_\phi) \chi(n^\xi, n^\varphi) g(v, \Lambda) e^{p_\phi(\phi)}
\]

\[
g(v, \Lambda) = N(v) e^{-\frac{\sigma_\phi^2}{2\sigma_\epsilon^2} [\Lambda - \bar{\Lambda}(v)]^2}
\]

\[
\hat{\Omega}^2 N(v) = \left[ \frac{4p_\phi^2}{3\pi G\hbar^2} + \frac{16}{3\beta} e^{2\Lambda} H_0(n^\xi, n^\varphi) \right] N(v) \quad \text{particular case} \quad \bar{\Lambda}(v) = \bar{\Lambda}
\]
Eigenfunctions FRW operator

\[ \hat{\Omega}^2 e_\rho(v) = \rho^2 e_\rho(v) \quad \text{ (positive non-degenerate continuous spectrum) } \]

- For \( \rho \geq \rho^* \gg 10 \)

- **Real eigenfunctions with analytical continuous limit for** \( v \gg \rho/2 \)

- \( e_\rho(v) \) **are exponentially suppressed for** \( v \leq v_m \approx \rho/2 \)
Approximate Solutions for $\tilde{\Lambda}(v) = \tilde{\Lambda}$

[M. M-B, D. Martín-de Blas, G.A. Mena Marugán, 2014, Class. Quant. Grav. 31]

$$
\Psi(\phi, v, \Lambda, n^\xi, n^\varphi) = \int_{-\infty}^{\infty} dp_\phi \psi(p_\phi) \chi(n^\xi, n^\varphi) f(\Lambda) e^{\rho(p_\phi, \Lambda, n^\xi, n^\varphi)(v)} e_{p_\phi}(\phi)
$$

$$
\rho(p_\phi \Lambda, n) = \sqrt{\frac{4}{3\kappa \hbar} p_\phi^2 + \frac{16}{3\beta} e^{2\Lambda} H_0(n)}
$$

- These are approximate solutions of the Gowdy model, provided that
  - $f(\Lambda) \propto e^{-\frac{\sigma_s^2}{2q_s^2} [\Lambda - \tilde{\Lambda}(v)]^2}$ with $\sigma_s \ll \pi/2$ ; $\tilde{\Lambda}(v) \gg q_s^2 / \sigma_s^2$
  - $v \gg 10 \rightarrow \rho \gg 10 \rightarrow p_\phi^2 \gg 75\kappa \hbar \approx 200G\hbar^2$ (H$_0 > 0$)
  - Small content of inhomogeneities and $\hat{C}_\theta \chi(n^\xi, n^\varphi) = 0$

- Effective constraint:

$$
\rho = \frac{2|\tilde{p}_\phi|}{\sqrt{3\pi G\hbar^2}}, \quad |\tilde{p}_\phi| \equiv \sqrt{p_\phi^2 + \frac{4\pi G\hbar^2}{\beta} e^{2\Lambda} H_0} \quad \text{FRW + massless scalar}
$$
Approximate Solutions for $\bar{\Lambda}(v)$


\[
\Psi(\phi, v, \Lambda, n^\xi, n^\varphi) = \int_{-\infty}^{\infty} dp_\phi \psi(p_\phi) \chi(n^\xi, n^\varphi) g(v, \Lambda) e^{p_\phi(\phi)}
\]

\[
g(v, \Lambda) = N(v) e^{-\frac{\sigma_s^2}{2q_\varepsilon^2} [\Lambda - \bar{\Lambda}(v)]^2}
\]

- For each value of $p_\phi$, $H_0$, and $\Lambda$, we have a difference equation relating $N(v + 4)$, $N(v)$, and $N(v - 4)$

- Approximate solutions of the Gowdy model must satisfy
  - $\sigma_s \ll \pi/2$
  - $N(v)$ highly suppressed for $v \leq v_m \gg 10$
  - $\bar{\Lambda}(v \pm 4) \simeq \bar{\Lambda}(v) \gg q_\varepsilon^2/\sigma_s^2$
Approximate Solutions for $\bar{\Lambda}(v)$


\[
\Psi(\phi, v, \Lambda, n^{\xi}, n^{\phi}) = \int_{-\infty}^{\infty} dp_\phi \psi(p_\phi) \chi(n^{\xi}, n^{\phi}) g(v, \Lambda) e_{p_\phi}(\phi)
\]

\[
g(v, \Lambda) = N(v) e^{-\frac{\sigma_s^2}{2q_\xi^2}[\Lambda-\bar{\Lambda}(v)]^2}
\]

- For each value of $p_\phi$, $H_0$, and $\Lambda$, we have a difference equation relating $N(v + 4)$, $N(v)$, and $N(v - 4)$

- Approximate solutions of the Gowdy model
  - $\sigma_s \ll \pi/2$
  - \[
  \bar{\Lambda}(v) = \begin{cases} 
  h(v_0), & \text{if } v \leq v_0, \\
  h(v), & \text{if } v > v_0,
  \end{cases} \quad h(v \pm 4) \simeq h(v) \gg q_\xi^2/\sigma_s^2 \text{ for } v_0 \geq v_m
  \]
  - $N(v_0 - 4) = e_{\rho(p_\phi, \Lambda, n^{\xi}, n^{\phi})}(v_0 - 4)$ and $N(v_0) = e_{\rho(p_\phi, \Lambda, n^{\xi}, n^{\phi})}(v_0)$
Mimicking Isotropic Matter Behaviour

- **FRW + perfect fluid:**
  \[ \hat{C}_{\text{FRW+PF}} = -\frac{3\pi G \bar{h}^2}{8} \hat{\Omega}^2 + \frac{\dot{\hat{p}}_\phi^2}{2} + \alpha(1 - w)\hat{v}^{1-w} \]

- Previous states are solutions of
  \[ \hat{C}'_{\text{app}} = -\frac{3\pi G \bar{h}^2}{8} \hat{\Omega}^2 + \frac{\dot{\hat{p}}_\phi^2}{2} + \frac{2\pi G \bar{h}^2}{\beta} e^{2\bar{\Lambda}(\bar{v})} \hat{H}_0 \]

- The analyzed states mimic a perfect fluid behavior if we choose
  \[ \bar{\Lambda}(v) = \begin{cases} \ln \left[ v_0^{(1-w)/2} \right], & \text{if } v \leq v_0 \\ \ln \left[ v^{(1-w)/2} \right], & \text{if } v > v_0 \end{cases} \]

- **Dust:** \( w = 0 \)
- **Radiation:** \( w = 1/3 \)
- **Cosmological constant:** \( w = -1 \)
Conclusions

• We have developed approximation methods in the context of LQC to construct quantum solutions of inhomogeneous and anisotropic cosmological models

• We have constructed states with peaked anisotropy profiles (on which anisotropy and self-interaction terms can be disregarded) that provide approximate quantum solutions for the Gowdy model

• Our solutions effectively behave as those of a flat FRW model with a perfect fluid (massless scalar, dust, radiation, cosmological constant, …)

• This analysis materializes an example where quantum states that are genuinely inhomogeneous lead to isotropic descriptions, at least with respect to certain physical properties
Outlook

• Apply the same analysis for more realistic scenario: FRW with cosmological perturbations

• Origin of cosmological constant? Fields producing inflation?
Outlook

• Apply the same analysis for more realistic scenario: FRW with cosmological perturbations

• Origin of cosmological constant? Fields producing inflation?

Thanks for your attention!