The most interesting case is flat spatial topology. It is also the simplest.

The effects of spatial curvature can be studied by considering, e.g., spherical topology.

We assume compact spatial sections.

We consider perturbed FRW universes filled with a massive scalar field.

The scalar field is minimally coupled.

The model can generate inflation.
The model

It's been well studied, even in LQC, though...

- Anomalies: Incorporate quantum effects, not the starting point for quantization.
- Effective dynamics: Needs a true derivation.

**Approximations**: As few as possible. Should be derived or at least checked for consistency.

In many cases these checks are only internal, within the approximated description.
Perturbations about flat FRW

- Truncation at **quadratic** order in the action.
- Includes **backreaction** at that order.
- Tests the validity of less refined truncations and provides the way to develop **approximation** methods, controlling their range of application.
Hybrid approach

Effects of quantum geometry are only accounted for in the background.

- Successfully applied in Gowdy cosmologies.
- In those cases there is no truncation. This is no drawback (think of the harmonic oscillator).
- In the present case, we only deal with the quadratically perturbed model.
Infinite *ambiguity* in selecting a Fock representation in QFT in curved space-times.

This can be restricted by appealing to *background symmetries*.

Typically this is not sufficient in non-stationarity.

Proposal: demand the **UNITARITY** of the quantum evolution.

The conventional interpretation of QM is guaranteed. This goes beyond the viewpoint of algebraic quantizations.

There is a natural ambiguity in the *separation of the background* from the field. In cosmology, this introduces time-dependent canonical field transformations.

Remarkably, symmetry invariance and dynamical unitarity select a **UNIQUE canonical pair** and a UNIQUE Fock representation for their CCR's.
Uniqueness of the Fock description
Recent works DO NOT incorporate the correct scaling (AA&N). This affects the quantum description, and in particular the effective approaches therein derived.

Moreover, one can even consider non-local canonical transformations, respecting the decoupling of field modes.

The **UNIQUENESS** of the quantization, up to unitary equivalence, is guaranteed.
Avoids the Big Bang.

Specific proposal such that:
- Evolution can be defined even without ideal clocks (massless field).
- The WdW limit is unambiguous in each superselection sector.
- It is optimal for numerical computation.

Control of changes of densitization in the scalar constraint.
The lapse function is not a function on phase space.
Massive scalar field minimally coupled to a compact, flat FRW universe.

Geometry:
\[ A_a^i = c^0 e_a^i (2\pi)^{-1}; \quad E_i^a = p \sqrt{0} e_i^a (2\pi)^{-2}. \]
\[ \{ c, p \} = 8\pi G\gamma / 3. \]
\[ a^2 = e^{2\alpha} = \left| p \right| (2\pi \sigma)^{-2}; \quad \pi_\alpha = -pc (8\pi^3 \sigma^{-1}). \]
\[ \sigma^2 = G (6\pi^2)^{-1}. \]

Matter:
\[ \varphi = (2\pi)^{3/2} \sigma \phi; \quad \pi_\varphi = (2\pi)^{-3/2} \sigma^{-1} \pi_\phi. \]

Hamiltonian constraint:
\[ C_0 = - \frac{6}{\gamma^2} \sqrt{p} c^2 + \frac{8\pi G}{V} (\pi_\phi^2 + m^2 V^2 \phi^2). \]
\[ V = \left| p \right|^{3/2}. \]
We expand inhomogeneities in a (real) Fourier basis:

\[ Q_{\vec{n},+} = \frac{1}{2\pi^{3/2}} \cos \vec{n} \cdot \vec{\theta}, \quad Q_{\vec{n},-} = \frac{1}{2\pi^{3/2}} \sin \vec{n} \cdot \vec{\theta}. \]

The basis is orthonormal, and we exclude the zero mode in the expansions.

These functions are eigenmodes of the Laplace-Beltrami operator of the standard flat metric on the three-torus, with eigenvalue

\[ -\omega_n^2 = -\vec{n} \cdot \vec{n}. \]

We only consider scalar perturbations: decoupled from vector and tensor perturbations at dominant order.
Mode expansion of the inhomogeneities:

\[ h_{ij} = (\sigma e^{\alpha})^2 \left[ 0 h_{ij} + 2 \epsilon (2\pi)^{3/2} \sum \left\{ a_{\tilde{n}, \pm}(t) Q_{\tilde{n}, \pm}^0 h_{ij} + b_{\tilde{n}, \pm}(t) \left( \frac{3}{\omega_n^2} Q_{\tilde{n}, \pm, ij} + Q_{\tilde{n}, \pm}^0 h_{ij} \right) \right\} \right], \]

\[ N = \sigma N_0(t) \left[ 1 + \epsilon (2\pi)^{3/2} \sum g_{\tilde{n}, \pm}(t) Q_{\tilde{n}, \pm} \right], \]

\[ N_i = \epsilon (2\pi)^{3/2} \sigma^2 e^\alpha \sum \frac{k_{\tilde{n}, \pm}(t)}{\omega_n} (Q_{\tilde{n}, \pm});_i, \]

\[ \Phi = \frac{1}{\sigma} \left[ \frac{\varphi(t)}{(2\pi)^{3/2}} + \epsilon \sum f_{\tilde{n}, \pm}(t) Q_{\tilde{n}, \pm} \right]. \]

The corrections include in principle higher-order perturbations.
Truncating the action at quadratic order in perturbations, one obtains:

\[
H = \frac{N_0 \sigma}{16 \pi G} C_0 + e^2 \sum \left( N_0 H^{\bar{n}, \pm}_2 + N_0 g_{\bar{n}, \pm} H^{\bar{n}, \pm}_1 + k_{\bar{n}, \pm} \bar{H}^{\bar{n}, \pm}_1 \right),
\]

\[
H^{\bar{n}, \pm}_2 e^{3\alpha} = - \pi^2 a_{\bar{n}, \pm} + \pi^2 b_{\bar{n}, \pm} + \pi^2 f_{\bar{n}, \pm} + 2 \pi \alpha \left(a_{\bar{n}, \pm} + \pi a_{\bar{n}, \pm} + 4 b_{\bar{n}, \pm} + \pi b_{\bar{n}, \pm}\right) - 6 \pi \varphi a_{\bar{n}, \pm} \pi f_{\bar{n}, \pm}
+ \pi^2 \left(\frac{1}{2} a_{\bar{n}, \pm} + 10 b_{\bar{n}, \pm}\right) + \pi \varphi \left(\frac{15}{2} a_{\bar{n}, \pm} + 6 b_{\bar{n}, \pm}\right) - \frac{e^{4\alpha}}{3} \left[\omega_n a_{\bar{n}, \pm}^2 + (\omega_n^2 - 18) b_{\bar{n}, \pm}^2\right]
+ e^{4\alpha} \omega_n^2 \left[f_{\bar{n}, \pm}^2 - \frac{2}{3} a_{\bar{n}, \pm} b_{\bar{n}, \pm}\right] + e^{6\alpha} m^2 \sigma^2 \left[\varphi^2 \left(\frac{3}{2} a_{\bar{n}, \pm}^2 + 6 b_{\bar{n}, \pm}^2\right) + 6 \varphi a_{\bar{n}, \pm} f_{\bar{n}, \pm} + f_{\bar{n}, \pm}^2\right],
\]

\[
H^{\bar{n}, \pm}_1 e^{3\alpha} = 2 \pi \varphi \pi f_{\bar{n}, \pm} - 2 \pi \alpha \pi a_{\bar{n}, \pm} - (\pi^2 + 3 \pi^2) a_{\bar{n}, \pm} - \frac{2}{3} e^{4\alpha} \omega_n^2 (a_{\bar{n}, \pm} + b_{\bar{n}, \pm})
+ e^{6\alpha} m^2 \sigma^2 \varphi (3 \varphi a_{\bar{n}, \pm} + 2 f_{\bar{n}, \pm})
\]

\[
\bar{H}^{\bar{n}, \pm}_1 e^{3\alpha} = \pi b_{\bar{n}, \pm} - \pi a_{\bar{n}, \pm} + \pi \alpha (a_{\bar{n}, \pm} + 4 b_{\bar{n}, \pm}) + 3 \pi \varphi f_{\bar{n}, \pm}.
\]
Longitudinal gauge

- We can adopt \textit{longitudinal gauge} by imposing:

\[ \pi_{a_{\tilde{n}, \pm}} - \pi_\alpha a_{\tilde{n}, \pm} - 3 \pi_\varphi f_{\tilde{n}, \pm} = 0, \quad b_{\tilde{n}, \pm} = 0. \]

- This removes the constraints \textit{linear} in perturbations.

\[ \pi_{b_{\tilde{n}, \pm}} = 0, \quad a_{\tilde{n}, \pm} = 3 \frac{\pi_\varphi \pi f_{\tilde{n}, \pm} + (e^{6\alpha} m^2 \sigma^2 \varphi - 3 \pi_\alpha \pi_\varphi) f_{\tilde{n}, \pm}}{9 \pi_\varphi^2 + \omega_n^2 e^{4\alpha}}. \]

- Together with dynamical stability, this fixes \( g_{\tilde{n}, \pm} = -a_{\tilde{n}, \pm}, \quad k_{\tilde{n}, \pm} = 0. \)

The shift vanishes, and the spatial metric is proportional to \( \,^0 h_{ij}. \)
After **REDUCTION**, a canonical set is:

\[
\bar{\varphi} = \varphi + 3 \sum a_{\bar{n}, \pm} f_{\bar{n}, \pm}, \quad \pi \varphi = \pi \varphi,
\]

\[
\bar{\alpha} = \alpha + \frac{1}{2} \sum \left( a_{\bar{n}, \pm}^2 + f_{\bar{n}, \pm}^2 \right), \quad \pi \bar{\alpha} = \pi \alpha - \sum f_{\bar{n}, \pm}\left(\pi f_{\bar{n}, \pm} - 3 \pi \varphi a_{\bar{n}, \pm} - \pi \alpha f_{\bar{n}, \pm}\right),
\]

\[
\bar{f}_{\bar{n}, \pm} = e^{\alpha} f_{\bar{n}, \pm}, \quad \pi \bar{f}_{\bar{n}, \pm} = e^{-\alpha}\left(\pi f_{\bar{n}, \pm} - 3 \pi \varphi a_{\bar{n}, \pm} - \pi \alpha f_{\bar{n}, \pm}\right).
\]

The genuine background variables are corrected with **quadratic** perturbations.

We have already **scaled** the matter field variables.
The modes of the scaled matter field satisfy a quasi-KG equation with time-dependent mass:

\[ \dddot{f}_{n, \pm} + r_n \dot{f}_{n, \pm} + (\omega_n^2 + s + s_n) \ddot{f}_{n, \pm} = 0, \]
\[ \pi_{f_{n, \pm}} = (1 + p_n) \dot{f}_{n, \pm} + q_n f_{n, \pm}, \]
\[ s = m^2 \sigma^2 e^{2\bar{\alpha}} - \frac{e^{-4\bar{\alpha}}}{2} \left( \pi_{\bar{\alpha}}^2 + 21 \pi_{\bar{\phi}}^2 + 3 e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi}^2 \right). \]

\[ r_n, s_n, p_n, q_n \text{ are of order } \omega_n^{-2}. \]

For any given background, there exists a **UNIQUE** Fock quantization with the symmetry of the three-torus and unitary dynamics.

The system can be put in the form of a KG field with time-dependent mass by means of a **mode-dependent** canonical quantization, varying in time.

This transformation is **unitarily** implementable in the privileged quantization.
The remaining **Hamiltonian constraint** reads:

\[
H = \frac{N_0 \sigma}{16 \pi G} C_0 + e^2 N_0 \sum H_{2}^{\tilde{n}, \pm}, \quad H_{2}^{\tilde{n}, \pm} 2e^{\tilde{\alpha}} = \bar{E}_{f \bar{f}} \bar{f}_{\tilde{n}, \pm}^{2} + \bar{E}_{f \pi} \bar{f}_{\tilde{n}, \pm} \pi_{f_{\tilde{n}, \pm}} + \bar{E}_{\pi \pi} \pi_{f_{\tilde{n}, \pm}}^{2},
\]

\[
\bar{E}_{f \bar{f}}^{n} = \omega_{n}^{2} + e^{2\tilde{\alpha}} m^{2} \sigma^{2} - \frac{e^{-4\tilde{\alpha}}}{2} \left( \pi_{\tilde{\alpha}}^{2} + 15 \pi_{\phi}^{2} + 3 e^{6\tilde{\alpha}} m^{2} \sigma^{2} \bar{\varphi}^{2} \right) - \frac{3}{\omega_{n}^{2}} e^{-8\tilde{\alpha}} \left( e^{6\tilde{\alpha}} m^{2} \sigma^{2} \bar{\varphi} - 2 \pi_{\tilde{\alpha}} \pi_{\phi} \right)^{2}.
\]

\[
\bar{E}_{f \pi}^{n} = -\frac{3}{\omega_{n}^{2}} e^{-6\tilde{\alpha}} \pi_{\phi} \left( e^{6\tilde{\alpha}} m^{2} \sigma^{2} \bar{\varphi} - 2 \pi_{\tilde{\alpha}} \pi_{\phi} \right), \quad \bar{E}_{\pi \pi}^{n} = 1 - \frac{3}{\omega_{n}^{2}} e^{-4\tilde{\alpha}} \pi_{\phi}^{2}.
\]

The corrections in cyan are of order \( \omega_{n}^{-2} \).
Longitudinal gauge: Metric (at linear order)

\[ h_{ij} = \left( \sigma e^{\bar{\alpha}} \right)^0 h_{ij} \left[ 1 + \epsilon 2 (2 \pi)^{3/2} \sum a_{\vec{n},\pm} Q_{\vec{n},\pm} \right], \]

\[ N = \sigma N_0 \left( 1 - \epsilon (2 \pi)^{3/2} \sum a_{\vec{n},\pm} Q_{\vec{n},\pm} \right), \quad N_i = 0, \]

\[ a_{\vec{n},\pm} = \frac{3}{\omega_n^2} e^{-3\bar{\alpha}} \left[ \pi \varphi \pi \tilde{f}_{\vec{n},\pm} + e^{-2\bar{\alpha}} \left( e^{6\bar{\alpha}} m^2 \sigma^2 \varphi - 2 \pi \bar{\alpha} \pi \varphi \right) f_{\vec{n},\pm} \right], \]

\[ \Phi = \frac{1}{\sigma} \left( \frac{\bar{\varphi}}{(2 \pi)^{3/2}} + \epsilon e^{-\bar{\alpha}} \sum f_{\vec{n},\pm} Q_{\vec{n},\pm} \right). \]
The Mukhanov-Sasaki modes and their momenta have the expression:

\[ \nu_{\pm, \pm} = A_n \tilde{f}_{\pm, \pm} + B_n \pi \tilde{f}_{\pm, \pm}, \quad \pi_{\nu_{\pm, \pm}} = \dot{\nu}_{\pm, \pm} = F_n \tilde{f}_{\pm, \pm} + G_n \pi \tilde{f}_{\pm, \pm}, \]

\[
A_n = 1 + \frac{3 e^{-4\bar{\alpha}} \pi \bar{\phi}}{\omega_n^2 \pi \bar{\alpha}} \left( e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi} - 2 \pi \bar{\alpha} \pi \bar{\phi} \right), \quad B_n = \frac{3 e^{-2\bar{\alpha}} \pi \bar{\phi}}{\omega_n^2 \pi \bar{\alpha}},
\]

\[
F_n = -\frac{3 e^{-2\bar{\alpha}} \pi \bar{\phi}^2}{\pi \bar{\alpha}} - \frac{3 e^{-6\bar{\alpha}}}{\omega_n^2 \pi \bar{\alpha}} \left[ e^{12\bar{\alpha}} m^4 \sigma^4 \bar{\phi}^2 - \frac{e^{6\bar{\alpha}} \pi \bar{\phi} m^2 \sigma^2 \bar{\phi}}{2 \pi \bar{\alpha}} \left( 5 \pi \bar{\alpha}^2 - 3 \pi \bar{\phi}^2 + 3 e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi}^2 \right) \right]
\]

\[
- \frac{3 e^{-6\bar{\alpha}} \pi \bar{\phi}^2}{2 \omega_n^2 \pi \bar{\alpha}} \left( 11 \pi \bar{\alpha} - 15 \pi \bar{\phi}^2 - 3 e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi}^2 \right),
\]

\[
G_n = 1 + \frac{3 e^{-4\bar{\alpha}} \pi \bar{\phi}}{2 \omega_n^2 \pi \bar{\alpha}} \left[ -2 e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi} + \frac{\pi \bar{\phi}}{\pi \bar{\alpha}} \left( \pi \bar{\alpha}^2 - 3 \pi \bar{\phi}^2 + 3 e^{6\bar{\alpha}} m^2 \sigma^2 \bar{\phi}^2 \right) \right].
\]

If we construct annihilation and creation variables with these invariants (for zero mass), the Bogoliubov transformation, which is mode dependent, is UNITARY in the privileged Fock quantization.
Robustness under gauge fixing

Similar results are obtained in the gauge of flat spatial sections $a_{\tilde{n}, \pm} = b_{\tilde{n}, \pm} = 0$.

Moreover, the same symplectic structure for gauge invariants is obtained.
We quantize the homogeneous sector with standard loop techniques, using improved dynamics and the MMO proposal.

In the volume basis \( \{|v\}; v \in \mathbb{R} \} \), with \( \hat{V} = |\hat{p}|^{3/2} \),

\[
\hat{N}_\mu |v\rangle = |v + 1\rangle, \quad \hat{p} |v\rangle = \text{sgn}(v) (2 \pi \gamma G \hbar \sqrt{\Delta} |v|)^{2/3} |v\rangle.
\]

The kinematic Hilbert space is \( H_{\text{kin}}^{\text{FRW-LQC}} \otimes H_{\text{kin}}^{\text{matt}} \).

The inverse volume is regularized as usual.

\[
\left[ \frac{1}{V} \right] = \left[ \frac{1}{\sqrt{p}} \right]^3, \quad \left[ \frac{1}{\sqrt{|p|}} \right] = \frac{3}{4 \pi \gamma G \hbar \sqrt{\Delta}} \text{sgn}(p) \sqrt{\hat{p}} (\hat{N}_{-\mu} \sqrt{\hat{p}} \hat{N}_\mu - \hat{N}_\mu \sqrt{\hat{p}} \hat{N}_{-\mu}).
\]
After **decoupling the zero-volume** state, we change densitization for the **FRW** constraint:

\[
\hat{C}_0 = \left[ \frac{1}{V} \right]^{1/2} \hat{C} \left[ \frac{1}{V} \right]^{1/2}
\]

\[
\hat{C}_0 = -\frac{6}{\gamma^2} \hat{\Omega}_0^2 + 8 \pi G \left( \hat{\pi}_\phi^2 + m^2 \hat{\phi}^2 \hat{V}^2 \right).
\]

The gravitational part, with the **MMO proposal**, is:

\[
\hat{\Omega}_0 = \frac{1}{4i \sqrt{\Delta}} \hat{V}^{1/2} \left[ \text{sgn}(p) \left( \hat{N}_{2\hat{\mu}} - \hat{N}_{-2\hat{\mu}} \right) + \left( \hat{N}_{2\hat{\mu}} - \hat{N}_{-2\hat{\mu}} \right) \text{sgn}(p) \right] \hat{V}^{1/2}.
\]

Takes into account the triad orientation (manifest in anisotropic scenarios).

This operator has the generic form

\[
\hat{\Omega}_0^2 |v\rangle = f_+ (v) |v+4\rangle + f_0 (v) |v\rangle + f_- (v) |v-4\rangle.
\]
Quantization: Superselection

- \( \hat{\Omega}_0^2 \) can be seen as a difference operator.
  \[
  \hat{\Omega}_0^2 |\nu\rangle = f_+(\nu) |\nu + 4\rangle + f(\nu) |\nu\rangle + f_-(\nu) |\nu - 4\rangle.
  \]

- The real function \( f_+(\nu) \) \( (f_-(\nu)) \) vanishes in the interval \([-4,0] \) \([0,4]\).

- The operator preserves the superselection sectors \( \mathcal{L}^{(4)}_{\pm \epsilon} = \{ \pm (\epsilon + 4n), \; n \in \mathbb{N} \} \)

- This operator is selfadjoint in those sectors. Its eigenfunctions are real, and determined by their value at the minimum volume \( \epsilon \in (0,4] \).
Solutions to the constraint are determined, e.g., by their initial values at minimum volume.

If the scalar field serves as a clock, an alternate possibility is to give the value at a section of constant field. This is not always possible.

The space of physical states can be identified, e.g., with $L^2(\mathbb{R}, d\phi)$. 
We quantize the rescaled inhomogeneous modes using annihilation and creation variables constructed from our canonical variables and zero mass.

We obtain a **Fock space** $\mathcal{F}$, with basis of $n$-particle states:

$$\left\{ | N \rangle = | N_{(1,0,0)}, +, N_{(1,0,0)}, -, \ldots \rangle; \quad N_{\tilde{n}, \pm} \in \mathbb{N}, \quad \sum N_{\tilde{n}, \pm} < \infty \right\}.$$ 

We proceed to a hybrid quantization, with Hilbert space $H^{FRW-LQC}_{kin} \otimes H^{\text{matt}}_{kin} \otimes \mathcal{F}$.

The Hamiltonian constraint is **not trivial**.
Quantum Hamiltonian of the perturbations

- We quantize the quadratic contribution of the perturbations to the Hamiltonian adapting the quantization proposals of the homogeneous sector and using a symmetric factor ordering:
  - We symmetrize products of the type $\hat{\Phi} \hat{\Pi}_\Phi$.
  - We take a symmetric geometric factor ordering $V^k A \rightarrow \hat{V}^{k/2} \hat{A} \hat{V}^{k/2}$.
  - We adopt the LQC representation $(cp)^{2m} \rightarrow [\hat{\Omega}_0^2]^m$.
  - In order to preserve the FRW superselection sectors, we adopt the prescription $(cp)^{2m+1} \rightarrow [\hat{\Omega}_0^2]^{m/2} \hat{\Lambda}_0 [\hat{\Omega}_0^2]^{m/2}$, where

$$\hat{\Lambda}_0 = -\frac{i}{8\sqrt{\Delta}} \hat{V}^{1/2} \left[ \text{sgn}(p)\left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}}\right) + \left(\hat{N}_{4\bar{\mu}} - \hat{N}_{-4\bar{\mu}}\right)\text{sgn}(p) \right] \hat{V}^{1/2}.$$ 

The situation is similar to that found with the Hubble parameter in LQC.
Quantum Hamiltonian of the perturbations

With the FRW densitization:

\[ \hat{H}_2^{n,\pm} = \frac{\sigma}{16 \pi G} \left\{ \frac{1}{V} \right\}^{1/2} \hat{C}_2^{n,\pm} \left\{ \frac{1}{V} \right\}^{1/2} . \]

\[ \hat{C}_2^{n,\pm} = 6 (2 \pi)^4 \sigma^2 \left[ 2 \omega_n \left\{ \frac{1}{V} \right\}^{-2/3} + \frac{\hat{Y}^-}{\omega_n} + \frac{\hat{Z}}{\omega_n^3} \right] \hat{N}_{n,\pm} + 4 \pi G \left[ \left( \frac{\hat{Y}^+}{\omega_n} + \frac{\hat{Z}}{\omega_n^3} \right) \hat{X}^+_{n,\pm} + \frac{3 i \sigma^2 \hat{W}}{\omega_n^2} \hat{X}^-_{n,\pm} \right] , \]

\[ \hat{N}_{n,\pm} = \hat{a}^\dagger_{f_{n,\pm}} \hat{a}_{f_{n,\pm}} , \quad \hat{X}^{\pm}_{n,\pm} = \left( \hat{a}^\dagger_{f_{n,\pm}} \right)^2 \pm \left( \hat{a}_{f_{n,\pm}} \right)^2 , \]

\[ \hat{Y}^{\pm} = \frac{m^2}{(2 \pi)^2} - \pi \sigma^2 \left\{ \frac{1}{V} \right\}^{1/3} \left( \frac{1}{\gamma^2 (2 \pi)^3 \sigma^2} \hat{\Omega}_0^2 + 3 \left( 5 \pm 2 \right) \hat{\pi}_\phi^2 + 3 m^2 \hat{V}^2 \hat{\phi}^2 \right) \left\{ \frac{1}{V} \right\}^{1/3} , \]

\[ \hat{Z} = - \frac{3 \sigma^2}{2 \pi} \left\{ \frac{1}{V} \right\} \left( \frac{2}{\gamma} \hat{\Lambda}_0 \hat{\pi}_\phi + m^2 \hat{V}^2 \hat{\phi} \right) \left\{ \frac{1}{V} \right\}^{2/3} , \]

\[ \hat{W} = - \left\{ \frac{1}{V} \right\}^{2/3} \left( \frac{4}{\gamma} \hat{\Lambda}_0 \hat{\pi}_\phi^2 + m^2 \hat{V}^2 \left( \hat{\phi} \hat{\pi}_\phi + \hat{\pi}_\phi \hat{\phi} \right) \right) \left\{ \frac{1}{V} \right\}^{2/3} . \]
If the matter field serves as a **clock**:

\[ \hat{C}_0 + e^2 \left( \sum \hat{C}^{\vec{n}, \pm}_2 \right) = 0. \]

\[
\langle \Psi | \hat{\pi}_\phi = \frac{1}{\sqrt{8 \pi G}} \langle \Psi | \left[ \hat{\Theta}_0^2 - e^2 \left( \sum \hat{C}^{\vec{n}, \pm}_2 \right) \right]^{1/2} \approx \frac{1}{\sqrt{8 \pi G}} \langle \Psi | \left[ \hat{\Theta}_0 - \frac{e^2}{2} \hat{\Theta}_0^{-1} \left( \sum \hat{C}^{\vec{n}, \pm}_2 \right) \right],
\]

\[ \hat{\Theta}_0^2 = P \left( 8 \pi G \hat{\pi}_\phi^2 - \hat{C}_0 \right). \]

We can pass to an **interaction picture** and use a Born-Oppenheimer-like approximation.

This can be done even without the above **perturbative expansion**.

This leads to a sort of **effective** QFT for the inhomogeneities.
• An alternate **perturbative** scheme:

\[
(\Psi| = (\Psi|^{(0)} + \epsilon^2 (\Psi|^{(2)}) \ldots
\]

• FRW solution: \((\Psi|^{(0)} \hat{C}_0 = 0,\)

\[
\hat{C}_0 = -\frac{6}{\gamma^2} \hat{\Omega}_0^2 + 8\pi G \left( \hat{\pi}^2 + m^2 \hat{\phi}^2 \hat{V}^2 \right).
\]

• **Evolution** of the perturbations:

\[
(\Psi|^{(2)} \hat{C}_0 = -(\Psi|^{(0)} (\sum\hat{C}_2^{n,\pm})^\dagger.
\]

• Solutions are characterized by their initial data at **minimum volume**.

• From these data we arrive, e.g., at the **physical Hilbert space** \(H_{\text{kin \, matt}}^{\text{mat}} \otimes \mathcal{F} \).
We have considered a perturbed FRW universe with a massive scalar field.

Two approximations:

- The action has been truncated to second order in the perturbations.
- A hybrid quantization scheme has been adopted.

First complete quantization of a model with inflation within LQC \((k=1)\).

Backreaction has been included.
For quantum simulations, the FRW prescription is **optimal**.

Opposite to the situation in other analyses, the inhomogeneities have **UNITARY** dynamics in an *(effective)* QFT approximation.

- **No internal time** (matter clock) is needed. If a matter clock is available, one can obtain the inhomogeneities evolution adopting an **interaction picture**.

- Generally, one can construct quantum states perturbatively from data at **minimum volume**. This allows one to get a **physical Hilbert space**.