Graphical calculation of half-cooling times

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Received 15 June 1989; revised 3 January 1990 *

This paper proposes a general approximate method of predicting the variattion of temperature at the thermal centre and the mean temperature during the cooling of bodies of any geometrical shape. The method is based on graphically determining the initial and exponential half-cooling times in simple geometric shapes. For this purpose, two Figures have been constructed to depict these dimensionless times, plus the prime root of the Biot Equation and other useful variables for indirect approximate calculation of thermophysical parameters.

(Keywords: graphical calculation; half-cooling times; geometric shapes)

Introduction

In refrigeration, the term half-cooling time is used to refer to the time required for a 50% reduction in the difference between the temperature, θ, of the cooling body at any point in time and the lowest attainable temperature, θ_e(equivalent to the temperature of the cooling medium)[1]. In cooling processes where the temperature of the cold source may be considered constant, after a certain time the temperature of the cooling body drops exponentially so that half-cooling time is constant thenceforth constant and its value remains constant at all points within the object, thus providing a reliable indicator of the speed of the process. If initial half-cooling time is further determined for the thermal centre and at a point in the body representative of its mass, it is possible to predict the total duration of cooling and the heat load to be dissipated.

In the cooling of bodies with simple geometric shapes, it is enough to know the first solution to the Biot Equation, which can be obtained through tables[2,3], to be able to perform an analytical calculation of the initial and exponential half-cooling times. Although there are obvious advantages to be obtained in terms of speed and accuracy by using personal computers for this calculation, graphic techniques are still valuable due to the overall view that they lend to the analysis of any process, and to their simplicity of application in either direct or reciprocal processes. For these reasons, it was decided to construct graphs which would provide a simple, direct determination of initial and exponential half-cooling times in the centre of the body and of mean temperatures for three simple geometries: an infinite slab, an infinite cylinder and a sphere. Details are also given of the use of such graphical calculations for compound geometries. The Figures are applicable to bodies of any shape through simple general Equations which allow the process to be reduced to one dimension. On the basis of half-cooling times calculated in this way, it is not very difficult, as the various examples show, to arrive with reasonable accuracy at the evolution of mean temperatures and temperatures at the centre of most solid bodies.

Development of the general model

The solution to the Equation for heat transfer by conduction in simply-shaped, homogeneous and isotropic bodies, without internal heat sources and subject to homogeneous external conditions, is given by the sum of a series of infinite terms[2-6] the general Equation of which is:

$$Y = \sum_{n=1}^{\infty} A_n \psi(\delta_n \xi) e^{-\delta_n^2 Fo} \qquad (1)$$

where:

$$A_n = \frac{2Bi}{\psi(\delta_n)\left[\delta_n^2 + Bi^2 - (\Gamma - 1)Bi\right]} \qquad (2)$$

the parameter Γ taking values 0, 1 or 2, respectively, in the cases of the infinite slab, infinite cylinder or sphere, and where:

$\psi(\delta_n \xi) = \cos(\delta_n \xi)$ for an infinite slab (3,a)

$\psi(\delta_n \xi) = J_0(\delta_n \xi)$ for an infinite cylinder (3,b)

$\psi(\delta_n \xi) = \dfrac{sin(\delta_n \xi)}{\delta_n \xi}$ for a sphere (3,c)

δ_n are the solutions to the Biot Equation for boundary conditions:

$$\frac{\partial \psi(\delta_n \xi)}{\partial \xi} = -Bi\,\psi(\delta_n \psi) \qquad (4)$$

while the remaining variables are defined as follows:

$$Y = \frac{\theta - \theta_e}{\theta_0 - \theta_e} \qquad (5)$$

$$Bi = \frac{hR}{k} \qquad (6)$$

$$Fo = \frac{at}{R^2} \qquad (7)$$

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* Publicada en: Int. J. Refrig. Vol 13 September 1990; 317-324

The roots δ_n derived from Equation (4) are discrete values increasing with the terms of the series, so that from $(Fo)_{min}$ onwards, normally exceeded for values of $Y \approx 0,8$[7], only the first term in the series is significant and Equation (1) may be replaced with sufficient accuracy[5,7] by the Equation:

$$Y = A_1 \psi(\delta_1 \xi) e^{-\delta_1^2 Fo} \qquad (8)$$

The thermal centre

If we take Equation (8) for $\xi = 0$ and replace Fo, we arrive at the dimensionless time required to attain a given Y_c value at the body's thermal centre:

$$Fo = \frac{Ln(A_1 / Y_c)}{\delta_1^2} \qquad (9)$$

Taking $Y_c = 1/2$ in Equation (9), we arrive at a simplified Equation of initial dimensionless half-cooling time:

$$Fo_{1/2} = \frac{Ln(2A_1)}{\delta_1^2} \qquad (10)$$

Then, if Equation (9) is applied for values of $Y_c = 1/4, 1/8, ...$, and Equation (10) is taken into account, the differences yield:

$$Fo_{1/4} - Fo_{1/2} = Fo_{1/8} - Fo_{1/4} = ... = \frac{Ln(2)}{\delta_1^2} \equiv Z_s \qquad (11)$$

The constant Zs is the standard dimensionless half-cooling time corresponding to the exponential zone. The difference between this and $Fo_{1/2}$ coincides with lag time[7].

The dimensionless temperature Y_c may be related to the number of half-coolings, NH, through the Equation:

$$NH = -\frac{Ln(Y_c)}{Ln(2)} \qquad (12)$$

Equations (9), (10), (11) and (12) immediately provide the dimensionless time (Fo) required for the centre of the body to reach a given temperature:

$$Fo = Fo_{1/2} + (NH - 1) Z_s \qquad (13)$$

Conversely, if real cooling times $t_{1/2}$ and $t_{1/4}$ are known for the thermal centre, by applying Equation (7) we may write:

$$\frac{t_{1/4}}{t_{1/2}} = \frac{Fo_{1/4}}{Fo_{1/2}} \equiv D \qquad (14)$$

Equations (11) and (14) yield:

$$Fo_{1/2}(D-1)=\frac{Ln(2)}{\delta_1^2}=Z_s \qquad (15)$$

By applying Equation (10) and writing $\mu=\frac{2-D}{D-1}$, we obtain:

$$A=2^{\mu} \qquad (16)$$

Once A is known, Equations (2), (3) and (4) may be used to calculate the particular value of Bi corresponding to the cooling process. Although the values $Fo_{1/2}$ and Z_s can be used as comparative indicators of the intensity of transfer between two processes, they are not in themselves suitable for showing the inherent efficacy of the operation.

We consider that this may be more suitably expressed in terms of the relative rate. For this purpose, the rate at which dimensionless temperature drops may, taking Equation (9) as the basis, be expressed by:

$$\frac{d(Y_c)}{d(Fo)}=-\delta_1^2 Y_c \qquad (17)$$

Now, given that the maximum dimensionless cooling speed for a value of Y_c will be obtained when $Bi\to\infty$, that is when δ_1^2 is maximum:

$$\left[\frac{d(Y_c)}{d(Fo)}\right]_{max}=-\left(\delta_1^2\right)_{max} Y_c \qquad (18)$$

the quotient of Equations (17) and (18) will indicate to what extent the cooling speed approaches its maximum possible value. Applying Equation (11), this may be expressed as:

$$\varepsilon=\frac{\delta_1^2}{\left(\delta_1^2\right)_{max}}=\frac{(Z_s)_{min}}{Z_s} \qquad (19)$$

Calculation Figure

The Figure in Figure 1 has been designed to determine the initial and exponential half-cooling times at the thermal centre. For simple handling, it is divided into four zones, numbered **a** to **d**.

Zones **a** and **b** serve to determine these values directly. Zone **a** shows $Fo_{1/2}$ versus Bi for the three geometries, Equations (2), (3) and (10). In zone **b** , curves A , B and C reproduce δ_1^2 versus Bi according to Equations (3) and (4), while curve D relates Z_s (upper horizontal scale) with δ_1^2 in terms of Equation (11).

Zones **c** and **d** contain auxiliary curves which facilitate the indirect determination of thermophysical parameters. Zone **c** shows A_1 versus δ_1^2 according to Equations (2), (3) and (4). Zone **d** shows A_1 versus D according to Equation (16).

The range chosen for Bi was from 0.1 to 100, as this covers all practical cases of cooling food, and is a manageable range. In any case, where Bi ≥ 200, half-cooling times are practically minimum values, so that where Bi ≥ 100, these may be taken as approximate (error ≤ 2%). Hence, the values of $(Fo_{1/2})_{min}$ are: 0,379 for the infinite slab, 0,201 for the infinite cylinder and 0,14 for the sphere, while the $(Zs)_{min}$ values are 0,281, 0,120 and 0.070 respectively.

The Figure is very simple to use, as can be seen from the solution to the following problem.

Example 1

To determine the chilling time for a 0,01 m fish fillet (k = 0,45 $W.m^{-1}.K^{-1}$, *a* = 1,22 x 10^{-7} $m^2.s^{-1}$) from an initial temperature of 26°C to a temperature of 3°C at the centre, by means of a turbulent water flow (h = 450 $W.m^{-2}.K^{-1}$) at 1°C.

Assuming that the fillet is equivalent in thermal terms to an infinite slab of R = 0.005 m, Equations (5), (6) and (12) yield respective values of Y = 0,08, Bi = 5 and NH = 3,64. A vertical line drawn in zone **a** at Bi = 5 will give the value $Fo_{1/2}$ = 0,53 at its intersection, in this case with the slab curve. If this vertical line is prolonged into zone **b** until it intersects with the relevant curve, again in this case A , and a horizontal line is drawn from this point, its intersection with the x-axis will yield $\delta_1^2 = 1{,}72$ and its intersection with the D curve projected vertically to the upper scale will yield Zs = 0,40. If Equation (13) is applied to these values, this gives:

$$Fo = 0{,}53 + (3{,}64 - 1)0{,}40 = 1{,}59$$

Through Equation (7), we obtain the real time of:

$$t = \frac{1{,}59 \times 0{,}005^2}{1{,}22 \times 10^{-7}} = 326s$$

This value differs by less than 1% from that calculated using Equation (1).

According to Equation (19), the cooling efficiency of the method would be:

$$\varepsilon = \frac{0{,}281}{0{,}40} = 0{,}70$$

This would therefore indicate 70% of maximum speed.

Conversely, if the times measured when the centre of the fillet reaches 13,5°C (Y_c = 0.5) and 7,25°C (Y_c = 0.25) are 108 and 190 s respectively, then according to Equation (14):

$$D = \frac{190}{108} = 1{,}76$$

Starting from this value on the x-axis of zone **d**, a horizontal line is drawn to intersect with the curve there. A vertical line is drawn through this point, yielding A_c = 1,24 on the upper horizontal scale. If this vertical line is prolonged into zone **c** until it meets curve A and then continued horizontally from the latter point, this will yield the value $\delta_1^2 = 1{,}72$ on the x-axis. If the intersections of this line with curves A and D in zone **b** are projected on to the upper and lower horizontal axes, this will give respective values of Bi = 5 and Zs = 0,40. A vertical line in zone **a** at Bi = 5 will intersect with curve A to give $Fo_{1/2}$ = 0,53. With the data thus derived from the Figure, when R is known, Equations (6) and (7) can be applied to determine the approximate mean value for the thermal diffusivity of the product and the k/h relationship.

Multidimensional transmission

In the case of bodies assimilable to compound shapes such as finite cylinders and rectangular prisms, only the first term of the series is considered to be significant[2,3,6]

$$Y = \left[\prod_{j=1}^{\nu} A_{ij}\right] \exp\left(-at\sum_{j=1}^{\nu} \frac{\delta_{ij}^2}{R_j^2}\right) \qquad (20)$$

where ν takes the values 1, 2 or 3 depending on the number of simple geometries comprising the body. If we choose the body's smallest half-dimension as the characteristic length R and define:

$$\alpha_j = \frac{R}{R_j} \qquad (21)$$

$$\delta^2 = \sum_{j=1}^{\nu} \left(\delta_{1j}\alpha_j\right)^2 \qquad (22)$$

$$A = \prod_{j=1}^{\nu} A_{1j} \qquad (23)$$

Equation (20) may be written in simplified form:

$$Y_c = A\exp\left(-\delta^2 Fo\right) \qquad (24)$$

Specifying Y_c = 1/2, we arrive at:

$$Fo_{1/2} = \frac{\sum_{j=1}^{\nu} \left(Fo_{1/2}\right)_j \delta_{1j}^2 - (\nu - 1) Ln(2)}{\delta^2} \qquad (25)$$

where $(Fo_{1/2})_j$ represents the half-cooling time of each component. The Expression for exponential half-cooling time will be analogous to that of the one-dimensional case:

$$Zs = \frac{Ln(2)}{\delta^2} \quad (26)$$

Application of the values derived from Equations (25) and (26) to Equation (13) will yield the dimensionless cooling time.

How the Figure in Figure 1 works for such cases is illustrated in the following example:

Example 2

This example discusses cooling semi-hard cheeses 0,20 m in diameter and 0,10 m thick. These are initially at a uniform temperature of 22°C, are piled on grid shelves and are subjected to an air current at 7°C. The thermopysical properties taken into account are: a = 1,2 x10^{-7} m^2 s^{-1} and k = 0,45 W.m^{-1} K^{-1}. The coefficient of mean surface heat transfer is estimated at h = 20 W m^{-2} K^{-1}. We wish to know the time required to reach 10°C at the centre (ie, when Y = 0.2).

The cheese may be taken to be a finite cylinder 0,10 m in height and 0,20 m in diameter. In other words, it may be considered as a compound shape formed by the intersection of an infinite slab 0,10 m thick and an infinite cylinder 0,20 m in diameter. According to formula (6), the corresponding Biot numbers will be 2,22 and 4,44 respectively.

If these Biot numbers are applied to Figure 1, they yield values of $\delta_{11}^2 = 1{,}23$ and $\left(Fo_{1/2}\right)_1 = 0{,}70$ for the slab $\delta_{12}^2 = 3{,}79$ and $\left(Fo_{1/2}\right)_2 = 0{,}29$ and for the cylinder. Application of these data to Equation (22), bearing in mind that according to Equation (21) α_1 = 1 and α_2 = 0.5, will give:

$$\delta^2 = 1{,}23 \times 1 + 3{,}79 \times (0{,}5)^2 = 2{,}18$$

As in this case there are but two components ν = 2 and, via Equations (25), (26) and (12), this results in:

$$Fo_{1/2} = \frac{0{,}70 \times 1{,}23 + 0{,}29 \times 3{,}79 - Ln(2)}{2{,}18} = 0{,}58$$

$$Zs = \frac{Ln(2)}{2{,}18} = 0{,}32$$

$$NH = -\frac{Ln(0{,}2)}{Ln(2)} = 2{,}32$$

These values applied to Equation (13) will give:

$$Fo = 0{,}58 + (2{,}32 - 1) \times 0{,}32 = 1$$

Finally, it is concluded from Equation (7) that the real cooling time sought is:

$$t = \left(\frac{1 \times (0{,}05)^2}{1{,}2 \times 10^{-7}} \right) \left(\frac{1}{3600} \right) = 5{,}79 \text{ h}$$

This result differs by less than 1% from that obtained with the complete series. Again, if the values calculated for α_1 and α_2 and the values of $\left(\delta_{1j}^2\right)_{max}$ corresponding to the infinite slab and infinite cylinder are applied in Equation (22), this yields:

$$\left(\delta^2\right)_{max} = (\pi / 2)^2 \times 1 + (2{,}405)^2 \times (0{,}5)^2 = 3{,}91$$

With an expression analogous to Equation (19), the efficiency of the process can be determined:

$$\varepsilon = \frac{\delta^2}{\left(\delta^2\right)_{max}} = \frac{2{,}18}{3{,}91} = 0{,}56$$

For more complex geometries, a number of workers[4,5,7] have proposed simple formulae, applicable in principle to any geometry and based on the use of shape factors, which give a simplified calculation with an acceptable degree of accuracy. Thus, Fikiin[8] applies a constant Φ shape factor for each geometry. Cleland and Earle[7] introduce the concept of equivalent heat transfer dimensions (EHTD) for the calculation of Zs, proposing a dependent relationship between this and the Biot number. Hereafter, similar concepts are used to propose a simple, generalized method for solving this kind of problem.

If ϕ is the number of times by which the dimensionless half-cooling time of the infinite slab is greater than that of the body in point, then

$$\phi = \frac{Fo_{1/2}(\text{infinite slab})}{Fo_{1/2}} \qquad (27)$$

It is shown mathematically that limit of this function where Bi $\to$ 0 is:

$$\phi = \Gamma + 1 = \frac{SR}{V} \qquad (28)$$

It can further be determined that beyond moderate values of the Biot number, ϕ differs from $\Gamma + 1$ to such an extent that at the other limit, when Bi $\to \infty$,

$$\phi_\infty = \frac{\left[Fo_{1/2}\left(\text{infinite slab}\right)\right]_{min}}{\left(Fo_{1/2}\right)_{min}} = \frac{0{,}379}{\left(Fo_{1/2}\right)_{min}} \qquad (29)$$

A simple expression fulfilling these conditions would be:

$$\phi = \phi_\infty + \frac{\Gamma + 1 - \phi_\infty}{\gamma Bi + 1} \qquad (30)$$

This function accounts not only for the proportion between the body's geometric dimensions ($\Gamma + 1$), but also for the dependence on the thermal conditions of the process (Bi) and the shape of the body (ϕ_∞) [A cube and its inscribed sphere have the same value ($\Gamma + 1 = 3$) and yet in identical conditions the sphere would cool faster].

The same reasoning applies to the calculation of Zs. Thus, if we say:

$$\phi_s = \frac{Zs\left(\text{infinite slab}\right)}{Zs} = \frac{\delta^2}{\delta_1^2\left(\text{infinite slab}\right)} \qquad (31)$$

whose limits for Bi $\rightarrow$ 0 and Bi $\rightarrow \infty$, respectively, are:

$$\phi_s = \Gamma + 1 \qquad (32)$$

and

$$\phi_{s\infty} = \frac{\left(\delta^2\right)_{max}}{\left(\pi/2\right)^2} \qquad (33)$$

the equivalent to Equation (30) will now be:

$$\phi_s = \phi_{s\infty} + \frac{\Gamma + 1 - \phi_{s\infty}}{\gamma_s Bi + 1} \qquad (34)$$

By determining the values of $Fo_{1/2}$ and Zs for the infinite slab in the Figure in Figure 1 and applying Equations (30) and (34), the values of Fo and Zs can be calculated for the body concerned. If these values are then applied with Equation (13), the dimensionless cooling time of the body can be calculated. The parameters γ and γ_s are adjustment coefficients which may be assigned mean values of 2,4 and 0,6 respectively.

This method has not been checked for irregularly-shaped bodies as no practical values are available for initial and exponential half-cooling times at the thermal centre, althouth the values given by Fikiin and Fikiina[4] for $\Gamma + 1$ are applicable. However, it has been possible to verify that their accuracy is acceptable (the maximum error found was less than 4%) in simple and compound figures for different dimensions and boundary conditions, for which an analytical solution does exist and whose thermal behaviour is always somewhere between that of the infinite slab and that of the sphere.

There is therefore no *a priori* reason to suppose that this method is unsuitable for all other intermediate geometries.

In the more general case, if S, V and R are known or measured, a single assay will suffice to obtain the values of ϕ_∞ and $\phi_{s\infty}$ corresponding to the body in point and to all other geometrically similar bodies. Thus, if two points obtained experimentally from the thermal centre cooling curve are applied to Equation (13), Fo and Zs can be calculated. With Figure 1 and Equations (27), (30), (31) and (34), the values can easily be calculated. With these values, thermal behaviour can then be predicted for other boundary conditions.

In the case of simple shapes and their perpendicular intersects, $\Gamma + 1$, ϕ_∞ and $\phi_{s\infty}$ can be calculated immediately. Let us look at some cases:

Sphere. $(Fo_{1/2})_{min}$ and $(Zs)_{min}$ are known (0.14 and 0.070 respectively), so that Equations (28), (29) and (30) yield: $(\Gamma + 1) = 3$, $\phi_\infty = 2{,}707$ and $\phi_{s\infty} = 4$.

Finite cylinder. If R_1 is half height and R_2 the radius of the base, Equations (21) and (28) yield:

$$(\Gamma+1)=\left(\frac{R}{R_1}\right)+\left(\frac{2R}{R_2}\right)=\alpha_1+2\alpha_2 \qquad (35)$$

It is further known that the values corresponding to Bi = ∞ are $\delta_{11}^2=(\pi/2)^2$ and $(Fo_{1/2})_{1min}$ = 0.379 for the infinite slab and $\delta_{12}^2=5{,}783$ and $(Fo_{1/2})_{2min}$ = 0.201 for the infinite cylinder, so that Equations (22) and (25) will easily yield the values of δ_{max}^2 and $(Fo_{1/2})_{min}$ for this case. If these are then applied in Equations (29) and (30)

$$\phi_\infty = 0{,}666\alpha_1^2 + 1{,}561\alpha_2^2 \qquad (36)$$

$$\phi_{s\infty} = \frac{\phi_\infty}{0{,}666} \qquad (37)$$

Rectangular prism. In this instance, the three component geometries are infinite slabs, and hence:

$$(\delta_{11}^2)_{max} = (\delta_{12}^2)_{max} = (\delta_{13}^2)_{max} = (\pi/2)^2$$

and

$(Fo_{1/2})_{1min} = (Fo_{1/2})_{2min} = (Fo_{1/2})_{3min} = 0{,}379$

Following the same procedure as before, we arrive at:

$$(\Gamma+1)=\sum_1^3 \alpha_j \qquad (38)$$

$$\phi_\infty = 0{,}659\sum_1^3 \alpha_j^2 \qquad (39)$$

$$\phi_{s\infty} = \sum_1^3 \alpha_j^2 \qquad (40)$$

To illustrate this method, example 2 above may be applied and Equations (35), (36) and (37) give: $\Gamma + 1$ = 1 + (2 x 0,5) = 2, ϕ_∞ = 1,056 and $\phi_{s\infty}$ = 1,586. If these values are applied to Equations (30) and (34) for Bi = 2,22 (which corresponds to characteristic length), this will give: ϕ = 1,230 and ϕ_s = 1,763.

At Bi = 2,22, Figure 1 can be used to determine Fo = 0,70 and Zs = 0,56 for the infinite slab. Using Equations (27) and (31)

$$Fo_{1/2} = \frac{0{,}70}{1{,}230} = 0{,}57$$

$$Z_s = \frac{0{,}56}{1{,}763} = 0{,}32$$

If these values are transferred to Equation (13) for NH = 2,3, Equation (7) will give a calculated real time of 5.74 hours, which practically coincides with the result in Example 2.

Mean temperature

To determine the evolution of mean temperature in the object, the integral value of Equation (1) must be calculated and averaged out for volume. By performing this integration and applying Equations (2) and (4), we arrive at the Equation:

$$\overline{Y} = \frac{\overline{\theta} - \theta_e}{\theta_0 - \theta_e} = \sum_{n=1}^{\infty} A_n \overline{\psi}(\delta_n) \exp\left(-\delta_n^2 Fo\right) \qquad (41)$$

where:

$$A_n \overline{\psi}(\delta_n) = \frac{2(\Gamma + 1)Bi^2}{\delta_n^2\left[\delta_n^2 + Bi^2 - (\Gamma - 1)Bi\right]} \qquad (42)$$

In infinite slabs and cylinders, the series may be reduced to its first term for the calculation of $\overline{Fo}_{1/2}$, with an error of less than 8%. In contrast, the case of the sphere requires more terms if the error is to be kept within these bounds.

Figure 2 shows the mean half-cooling times as functions of the Biot number. In this case $\left(\overline{Fo}_{1/2}\right)_{min}$ takes the values 0,196 for the infinite slab, 0,063 for the infinite cylinder and 0,031 for the sphere. Where regularly-shaped finite bodies are concerned, procedure is as in the case of the thermal centre, except that $\left(\overline{Fo}_{1/2}\right)_j$ is determined using Figure 2.

The dimensionless time which should elapse until a mean temperature $\overline{Y}$ is reached, is expressed by

$$\overline{Fo} = \overline{Fo}_{1/2} + (NH - 1)Zs \qquad (43)$$

in which

$$\overline{NH} = -\frac{Ln(\overline{Y})}{Ln(2)} \qquad (44)$$

When operating in this way, the results differ by less than 3.5% (for $\overline{Y} \leq 0{,}25$) from the exact analytical solution.

Considerations such as those for the thermal centre may be applied to irregular shapes, thus leading to similar expressions:

$$\overline{\phi} = \frac{\overline{Fo}_{1/2}(\text{infinite slab})}{\overline{Fo}_{1/2}} \qquad (45)$$

$$\overline{\phi}_{\infty} = \frac{0{,}196}{\left(\overline{Fo}_{1/2}\right)_{min}} \qquad (46)$$

and therefore

$$\overline{\phi} = \overline{\phi}_{\infty} + \frac{\Gamma + 1 - \overline{\phi}_{\infty}}{\overline{\gamma} Bi + 1} \qquad (47)$$

in which $\overline{\gamma}$ varies with geometry in the approximate relationship:

$$\overline{\gamma} = \frac{\Gamma + 1}{7{,}25\overline{\phi}_{\infty}} \qquad (48)$$

while Zs is the same as for the thermal centre.

Following exactly the same procedure as before and using Equations (43) and subsequent, we can calculate the time required to reach a given mean temperature in any body with less than 10% error (for $\overline{Y} \leq 0{,}25$). We shall set forth a practical example to illustrate this.

Example 3

With the same data as for Example 2, the aim is to find the mean temperature for the cheese once temperature has reached 10°C at the thermal centre, and the time required to attain a mean temperature of 10°C in the cheese.

Solution using the intersects method. As seen in Example 2, the following are known: $\overline{Fo} = 1$ $\delta_{11}^2 = 1{,}23$ $\delta_{12}^2 = 3{,}79$, $\alpha_1 = 1$, $\alpha_1 = 0.5$, $\delta^2 = 2.18$ and Zs = 0.32.

With Bi_1 = 2,22 and Bi_2 = 4,44, Figure 2 yields the respective values $\left(\overline{Fo}_{1/2}\right)_1 = 0{,}53$ and $\left(\overline{Fo}_{1/2}\right)_2 = 0{,}15$

Applying Equation (25):

$$\left(\overline{Fo}_{1/2}\right) = \frac{0{,}53 \times 1{,}23 + 0{,}15 \times 3{,}79 - \left[(2-1)Ln(2)\right]}{2{,}18} = 0{,}24$$

Applying Equation (43):

$$\overline{NH} = \frac{1-0{,}24}{0{,}32+1} = 3{,}38$$

Applying Equation (44):

$$\overline{Y} = (0{,}5)^{3{,}38} = 0{,}096$$

Finally, Equation (41) gives the mean temperature:

$$\overline{\theta} = 15 \times 0{,}096 + 7 = 8{,}44°C$$

which differs by less than 1% from the value obtained for the complete series.

The second part of the problem can then be solved immediately. Thus, by analogy with Equation (5):

$$\overline{Y} = \frac{\overline{\theta} - \theta_e}{\theta_0 - \theta_e} = \frac{10-7}{22-7} = 0{,}20$$

Applying Equation (44),

$$\overline{NH} = -\frac{Ln(0{,}20)}{Ln(2)} = 2{,}32$$

Applying Equation (43),

$$\overline{Fo} = 0{,}24 + (2{,}32 - 1) \times 0{,}32 = 0{,}66$$

Applying Equation (7):

$$t = \frac{0{,}66(0{,}05)^2}{1{,}2 \times 10^{-7}}\left(\frac{1}{3600}\right) = 3{,}82 \text{ h (e<1\%)}$$

Solution by shape factor method. It is already known that for the infinite slab $\left(\delta_{11}^2\right)_{max} = (\pi/2)^2$ and $\left(\overline{Fo}_{1/2}\right)_{1min} = 0{,}196$, and for the infinite cylinder $\left(\delta_{11}^2\right)_{max} = 5{,}783$ and $\left(\overline{Fo}_{1/2}\right)_{2min} = 0{,}063$.

If these values are applied to Equation (22):

$$\delta_{max}^2 = (\pi/2)^2 \times 1 + 5{,}783 \times (0{,}5)^2 = 3{,}913$$

Applying Equation (25):

$$\left(\overline{Fo}_{1/2}\right)_{min} = \frac{0{,}196(\pi/2)^2 + 0{,}063 \times 5{,}783 - Ln(2)}{3{,}91} = 0{,}040$$

Applying Equation (46):

$$\bar{\phi}_\infty = \frac{0{,}196}{0{,}040} = 4{,}955$$

Applying Equation (48),

$$\gamma = \frac{2}{7{,}25 \times 4{,}955} = 0{,}056$$

Applying Equation (47):

$$\bar{\phi} = 4{,}955 + \frac{2 - 4{,}955}{0{,}056 \times 2{,}22 + 1} = 2{,}325$$

With Bi = 2,22, Figure 2 yields $\left(Fo_{1/2}\right)_{\text{infinite slab}} = 0{,}53$

Then applying Equation (45):

$$\left(\overline{Fo}_{1/2}\right) = \frac{0{,}53}{2{,}325} = 0{,}23$$

Equation (43) gives:

$$\overline{NH} = \frac{1 - 0{,}23}{0{,}32} + 1 = 3{,}41$$

and Equation (44) gives:

$$\bar{Y} = (0{,}5)^{3{,}41} = 0{,}094$$

Finally, Equation (41) yields:

$\bar{\theta} = 15 \times 0{,}094 + 7 = 8{,}41°C$ (e<1%)

The second part of the example would be solved in the same way as with the intersects method, giving a real cooling time of 3.7 h (e<3%).

Conclusions

This paper proposes a simple method of calculating the evolution of mean temperatures and temperatures at thermal centre during the cooling of bodies in general, where initial and exponential half-cooling times, as constant parameters inherent in the process, are known.

To this end, two charts were devised to enable direct determination of half-cooling times in bodies of simple geometric shapes (infinite slab, infinite cylinder and sphere) according to the Biot number. When these values are used as components, half- cooling times can be calculated for regular compound shapes without difficulty. To arrive at such determinations in more complex shapes, a general method (hence also valid, if less accurate, for simple and compound shapes) is proposed, based on the calculation of shape factors and relating the thermal behaviour of any body to that of an infinite slab in the same conditions. This factor is associated, through simple general Equations, with the Biot number and the geometric characteristics of the body.

The resulting half-cooling times are used to predict cooling curves for the body. In the case of the thermal centre, in none of the cases tested was an error of over 4% detected, while the error was less than 10% (for $\overline{Y} \leq 0{,}25$) in the case of mean temperature, a very acceptable degree of accuracy given the current precision normally attained in the determination of the thermophysical parameters intervening in these phenomena.

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Nomenclature

A_1 , A_n	First and nth coefficients of the series
A	The same coefficient in compound-geometry bodies
Bi	Biot number
D	Relationship between quarter- and half-cooling times
Fo, $\overline{Fo}$	Fourier Number or dimensionless time
$Fo_{1/2}$, $\overline{Fo}_{1/2}$	Fourier Numbers for half-cooling at the centre and the mean value, respectively
NH, $\overline{NH}$	Half-cooling number at centre and mean value
R	Smallest half-dimension of the body (m)
R_j	Half-dimension of the component j in compound bodies (m)
S	Heat transfer surface area (m^2)
V	Volume of body (m^3)
Y, Y_c, $\overline{Y}$	Fraction of temperature yet to drop, with reference to a generic point in the body, to its centre and to its mean value, respectively
Zs	Standard or exponential half-cooling time
a	Heat diffusivity of body ($m^2\ s^{-1}$)
h	Surface heat transfer coefficient ($Wm^{-2}K^{-1}$)
k	Heat conductivity of body ($W\ m^{-1}K^{-1}$)
t	Real cooling time (s)

Greek letters

α_j	Relationship between characteristic length R and half-dimension R_j
Γ	Parameter dependent on the geometric shape of the body
$\gamma, \gamma_s, \overline{\gamma}$	Adjustment coefficients
δ_n	nth root of the Biot Equation
δ	Weighted root for complex shapes
ξ	Dimensionless distance from the point considered, with reference to characteristic length
$\theta, \theta_0, \overline{\theta}$	Temperature at any point in the body, uniform initial temperature and mean temperatures of body (°C)
θ_c	Temperature of cooling medium (°C)
ε	Efficiency of cooling process
μ	Exponent of Equation (16)
ν(=1, 2 or 3)	Number of simple geometries composing a compound shape
$\phi, \phi_s, \overline{\phi}$	Shape factors
$\psi(\delta_n \xi)$	Shape-dependent function
$\overline{\psi}(\delta_n)$	Mean value of $\psi(\delta_n \xi)$

Subscript

c	With reference to the centre of the body
n	nth term of the complete series
j	Component j of a body of compound shape
max	Maximum value
min	Minimum value
∞	When Bi;∞
1	First term in the series

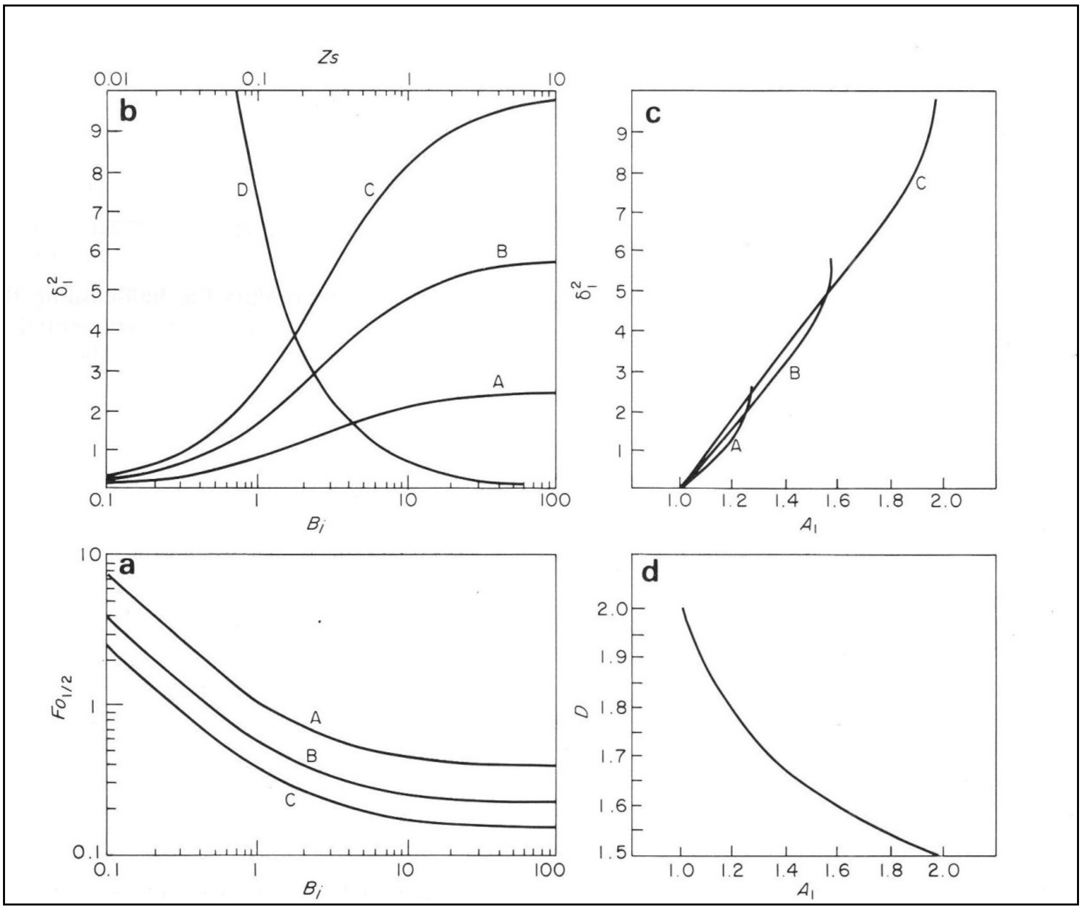

Figure 1 Half-cooling times. (**A**) Infinite slab; (**B**) infinite cylinder; (**C**) sphere. (**a**) Zone a $Fo_{1/2}$ versus Bi; (**b**) zone b δ_1^2 versus Bi (A, B and C) and Zs versus δ_1^2 (D); (**c**) zone c A_1 versus D

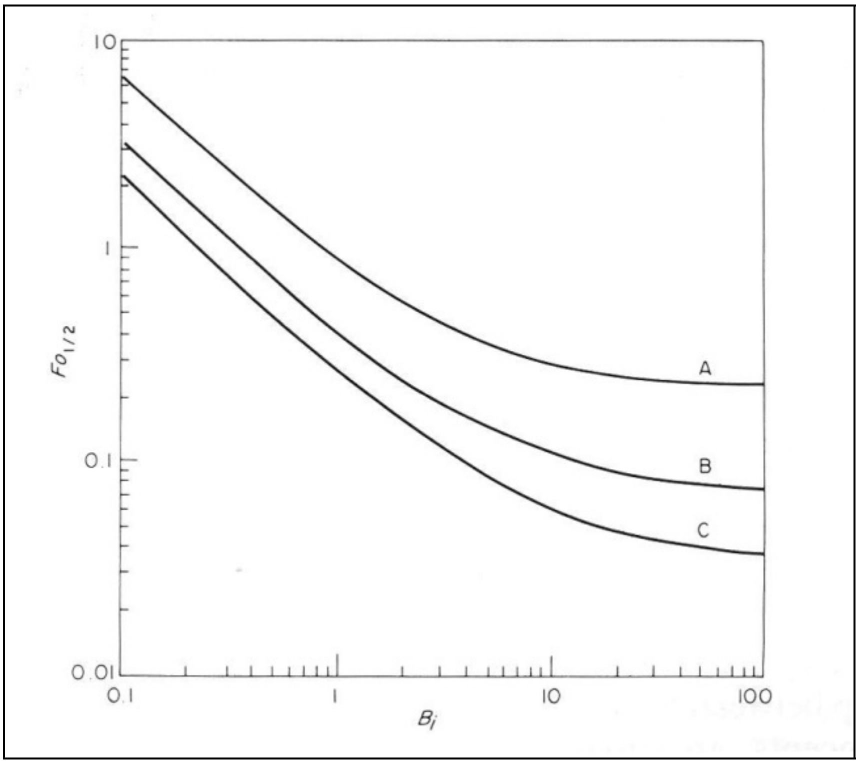

Figure 2 Mean half-cooling time. $\overline{Fo}_{1/2}$ versus Bi. (**A**) Infinite slab; (**B**) infinite cylinder; (**C**) sphere.